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CONTRIBUTIONS FROM THE CRYPTOGAMIC LABORATORIES
OF HARVARD UNIVERSITY. No. LXXVIII.

# *NEW OR CRITICAL SPECIES OF CHITONOMYCES AND RICKIA.*

By Roland Thaxter.

CONTRIBUTIONS FROM THE CRYPTOGAMIC LABORATORIES
OF HARVARD UNIVERSITY. No. LXXVIII.

# NEW OR CRITICAL SPECIES OF CHITONOMYCES AND RICKIA.

By Roland Thaxter.

Presented May 10, 1916. Received March 28, 1916.

The new forms herein described are, with few exceptions, parasites of tropical hosts which have been collected for me through the kindness of the Rev. George Schwab in Kamerun, Mr. J. C. Moulton in Borneo, and Mr. Oakes Ames and Mr. C. S. Banks in the Philippines. The West Indian forms were for the most part collected by myself; a few, together with certain Central American hosts, having been examined in the collections of the Museum of Comparative Zoölogy at Cambridge, through the courtesy of Mr. Samuel Henshaw.

Since the paper deals with two genera, only, it is hardly necessary to remark that it includes only a portion of the novelties thus obtained, further additions to which will be published later. I desire again in this connection to acknowledge my obligations to the gentlemen above mentioned, as well as to Mr. Gilbert A. Arrow, of the British Museum, who has been so kind as to determine for me numerous coleopterous hosts.

## Chitonomyces.

Since the cell-number and arrangement in a majority of the species in this genus are more or less definite, it has seemed best in the following diagnoses to letter the successive cells as follows: The basal cell *a*, the subbasal *b*. The latter is usually followed by three cells which normally form a series horizontally disposed, the anterior *c*, the median *d*, and the posterior *e*. The relative position of these cells may, however, vary considerably, and the anterior may become much reduced in size and lie above cells *d* and *e*. Of the posterior cells which lie normally beside the perithecium, the lowest, and usually the longest, is distinguished as cell *f*, the distal end of which is separated by a septum to form the small appendiculate cell *h*, external to which cell *g* bears distally the terminal free appendiculate cell *i*, from which the primary appendage arises terminally. On hosts belonging to genera other than *Laccophilus* several species occur in which no septum separates cell *f* from cell *h*, so that the pointed end of the former itself bears the appendage, which arises next the perithecium; while the latter is not distinguished as a distinct cell.

In addition to the new forms on *Laccophili* from the West Indies enumerated below the following were also met with.

*Chitonomyces hyalinus* Th. A form hardly separable from this species was found at Sangre Grande, Trinidad. Its general habit is distinctly more slender than that of the type, especially the basal portion of the receptacle and cell *i*. The elevation which subtends the apex of the perithecium, and projects somewhat inward on the right side, is also distinctly more prominent. Otherwise it shows no essential differences. A form smaller and somewhat stouter than the type was also found at the Grand Etang, Grenada. Both these forms occur on the posterior legs.

*C. psittacopsis* Th., resembling the type in all respects was found commonly both in Trinidad and at the Grand Etang on a single species of *Laccophilus*, Nos. 2687 and 2684.

*C. appendiculatus* Th., agreeing essentially with the type, was found at Sangre Grande, near the base of the left elytron of a single individual of *Laccophilus*, No. 2680.

*C. simplex* Th., exactly resembling the type and growing in the same position, was found rarely at Sangre Grande, No. 2684.

*C. paradoxus* Peyr., showing a considerable degree of variation both in color and in the conformation of the tip of the perithecium, was found commonly on three species of *Laccophilus* in Grenada and Trinidad.

*C. rhyncostoma* Th., was found on a peculiar *Laccophilus* in the Arepo Savanna, at Cumuto, Trinidad, and showed no essential differences from the type.

*C. distortus* Th., growing in the usual position on the front legs, was found rarely at the Grand Etang and varies slightly from the type.

*C. uncinatus* Th., somewhat larger and more slender, was found rarely at the Grand Etang and also at Sangre Grande. In the Trinidad material the basal cell of this species tends to become conspicuously blackened, and the projection of cell *g* is appressed against the adjacent tip of the perithecium. The longest individual measures 175 $\mu$.

*C. marginatus* Th. A form agreeing in all respects with the type was found on two individuals of *Laccophilus* at the Grand Etang.

*C. dentiferus* Th. A form closely resembling this species, and in which the base, only, of the tooth-like perithecial appendage is developed, was found on several specimens of *Laccophilus,* No. 2680, from Sangre Grande. The slender termination of the perithecium is usually rather irregularly bent outward and then inward at the apex, the whole distal portion sometimes erect, and sometimes bent partly across the distal cells of the receptacle. On the right side of cell *g,* are two superposed dark tubercular patches which occupy most of the surface on this side. These patches are not always very conspicuous, but are present in the type of *dentiferus,* although they were overlooked in the original description and figures.

*C. Hydropori* Th. A form not distinguishable from the types of this species was found on a species of *Hydroporus?* from Manila, P. I.

## **Chitonomyces cerviculatus** nov. sp.

Rather short, subsigmoid when viewed sidewise, uniformly tinged with dirty yellowish brown, with certain deeper amber-brown suffusions. Foot large; basal cell short and stout, somewhat longer than broad; cell *b* about twice as broad as long, horizontal, more strongly convex on one side; cells *c, d* and *e* small and subequal, subtriangular, dissimilar; cell *d* somewhat larger; cell *f* tapering to a very narrow base, lying almost wholly above cell *e,* and becoming much broader distally; where it is slightly and symmetrically intruded between the small narrowly triangular cell *h,* and cell *i,* which is distorted by an umbonate prominence, on the right side, that throws the insertion of cell *i* asymmetrically to the left, so that it may be sublateral in position; cell *i* symmetrical, broadly subhemispherical, the insertion of the primary appendage terminal. A short stout spiral appendage is present, arising on the right side, within and near the apex of cell *h.* Perithecium relatively large, irregularly subsigmoid; the outer margin of the venter concave, amber-brown, its distal end bulging abruptly on the left side below the stout, neck-like, strongly incurved terminal portion; which is of nearly uniform width to the apex, slightly broader below; the subterminal wall-cell, on the outer side, forming an amber-brown, somewhat appressed and distally outcurved appendage; which is sometimes subsymmetrical with the thick, dark amber margin of the opposite side; the apex broad, somewhat asymmetrically rounded, bent inward; the hyaline lips slightly prominent. Spores about 36 × 2.5 $\mu$. Perithecia, body 45–50 × 18 $\mu$, the neck

35 × 12 μ. Cell *i* 5.5 × 9 μ. Total length to tip of perithecium 90–100 × 23–27 μ.

On the outer edge of the right elytron of *Laccophilus* sp., near the middle. Grand Etang, Grenada, No. 2687, and Sangre Grande, Trinidad, B. W. I., No. 2684.

A peculiarly distorted form, well distinguished by its two appendages and the stout neck-like termination of its perithecium.

### **Chitonomyces introversus** nov. sp.

Distally flat and broad, becoming roughly triangular in outline, rather evenly suffused with dull amber brown; basal cell relatively large, curved below, much broader distally; cell *b* horizontal, greatly flattened; cell *c* very small; cell *d* broad and flattened; cell *e* somewhat larger, subtriangular, obliquely separated from cell *f*, which is long and narrow; its lumen, which is abruptly broader distally, otherwise nearly obliterated by the unusual thickness of the outer wall; the rest of the receptacle turned abruptly inward so that it lies sometimes almost at right angles to the general vertical axis; cell *g* relatively long, its base posterior, its upper (outer) margin more or less distinctly concave; cell *h* more than half as long, also lying obliquely or subhorizontally; cell *i* paler, slightly longer than broad, its inner (lower) side flattened and lying just above a shelf-like prominence that subtends the tip on the left side of the perithecium. The latter broad and short, its tip short, bent abruptly to the right away from the subtending prominence; so that, when the perithecium lies at the right, it is viewed end on; the apex irregularly rather coarsely lobed. Spores lying nearly horizontally in the upper part of the perithecium about 36 × 3.6 μ. Body of the perithecium, exclusive of tip, 54–60 × 27–32 μ. Total length 80–95 μ; the three lateral dimensions 40 × 75 × 82 – 50 × 86 × 90 μ.

On the posterior legs of *Laccophilus* sp. No. 2687, Grand Etang, Grenada, B. W. I.

A rare species easily distinguished by its triangular form and the horizontal position of cells *g–i*.

### **Chitonomyces oedipus** nov. sp.

More or less deeply and uniformly tinged with dull amber-brown; short and stout. Basal cell inflated below; the inflated portion somewhat resembling the human ear in outline, convex below, its axis horizontal, bearing the foot on its flat upper margin, near one end of which the short distal portion of the cell, narrow below and abruptly broader above, arises; cell *b* small, flattened, horizontal, somewhat irregular, producing a more or less well developed outgrowth from its anterior margin, which may project outward horizontally or grow downward against the termination of the basal cell, from which it may appear to arise; cells *c*, *d* and *e* small, flattened, irregular and confused with one of the persistent basal cells of the perithecium; cell *e* smaller, subtriangular, separated from cell *b* by the whole width of cell *d;* marginal region evenly convex; cell *f* long, narrowing below to its oblique base, distally somewhat overlapped by cell *g*, which is larger than cell *h;* cell *i* longer than broad, tapering to its rounded extremity, strongly curved inward so as slightly to overlap the tip of the perithecium, thus making the primary appendage horizontal in position. Perithecium relatively large and stout; the outer margin nearly straight; a variably developed rounded elevation projecting externally near its base; the short abruptly distinguished tip bent to the right, then outward; subtended externally by a variably developed erect outgrowth, which tapers to a blunt point and may exceed the somewhat irregular apex, which is slightly compressed about the pore; a long, straight, curved or subsigmoid, more deeply colored, divergent, then erect appendage, tapering from a broad base to a sharply pointed or somewhat blunt, attenuated apex, arises from one of the wall-cells close beside the end of cell *h* on the right side, and extends some distance above the apex of the perithecium. Spores about 40 × 4 μ. Perithecia 72 × 30 μ; the outer spine 21–26 μ, the inner 45–55 μ; inflated part of basal cell 36 × 23 μ. Total length to tip of perithecium 100–120 × 35 μ.

Growing singly between the terminal claws of the posterior legs of *Laccophilus* sp.; Sangre Grande, Trinidad, B. W. I., No. 2684, and the Grand Etang, Grenada, No. 2687.

A very peculiar species, at once distinguished by its swollen base, which is evidently an adjustment to its peculiar habitat, and by its perithecial appendages.

### **Chitonomyces Grenadae** nov. sp.

Tinged with dirty brownish, rather irregularly developed. Basal cell longer than broad; cell *b* somewhat flattened, horizontal, more or less clearly distinguished above and below by constrictions; cells *c*, *d* and *e* irregularly disposed, small; cell *f* long and narrow, its base but slightly oblique, its distal end but slightly overlapped by cell *g* which projects a short distance free above cell *h;* cell *i* more deeply suffused, sometimes amber-brown, long and stout, nearly straight or usually variously and slightly bent, its basal septum horizontal, its tip asymmetrical, the apex turned inward, of nearly the same diameter throughout, extending for about half its length above the extremity of the perithecium. Perithecium somewhat misshapen; tapering to the base, where it is subtended by a distinct constriction; the second external wall-cell forming a thick amber-brown margin, becoming broader and ending in a more or less pronounced elevation, distally, which subtends a second slight elevation followed by a constriction, above which the tip is irregularly bent inward and to one side; the apex blunt, with large slightly prominent lip-cells. A variably developed appendage arises from the wall-cell on the left side, next the end of cell *h,* sometimes quite short, normally longer, somewhat inflated, distally recurved and pointed. Spores about 36 × 2 μ. Perithecia 55–60 × 14 μ. Cell *i* 28–30 × 7 μ. Total length to tip of perithecium 70–75 × 20–22 μ.

On the margin of the right elytron of *Laccophilus* sp. near the tip. Grand Etang, Grenada, B. W. I., No. 2687a.

Most nearly related to *C. rhyncostoma,* from which it differs in many respects, but more particularly in the relatively much shorter free portion of cell *g,* and by the distorted tip of its perithecium. It was not observed on any of the very numerous individuals of the same host collected in Trinidad.

### **Chitonomyces uncinulatus** nov. sp.

Uniform pale dirty yellowish brown, regularly curved throughout, almost crescent shaped. Basal cell rather stout, tapering considerably below; cell *b* about twice as broad as long, horizontal, the anterior margin slightly concave, the posterior convex; cells *c, d* and *e* subequal, the base of cell *c* in contact with cell *b;* cell *f* very long and narrow, its base only slightly oblique, distally slightly and symmetrically intruded between cells *g* and *h,* cell *h* extending free some distance above the apex of cell *g,* and bent so as partly to overlap the perithecium; cell *i* longer than broad, distally rounded, and terminated by the dark insertion of the appendage. Perithecium long, curved outward, and tapering evenly to a blunt apex; the outer margin slightly indented at the junction of the wall-cells; a long irregularly curved, slender, darker appendage, ending in an abrupt bluntly pointed hook and developed from one of the wall-cells, diverges irregularly from a point beside the apex of cell *h*. Spores about 35 × 2 μ. Perithecium 72 × 18 μ; its appendage about ·24–30 × 3.5 μ. Total length to tip of perithecium 100–120 × 27 μ.

On the margin of the right elytron of *Laccophilus* sp. near the middle; Sangre Grande, Trinidad, B. W. I., No. 2680.

This species is well distinguished by its falcate habit, and the long, hooked or subhelicoid appendage which arises from a point near the apex of cell *h*. It is a singular fact that the two other species which possess a somewhat similar appendage similarly placed, namely *C. rhyncostoma* and *C. cerviculatus*, although otherwise quite different, occupy a similar position on the host.

### **Chitonomyces manubriolatus** nov. sp.

Rather short and stout, subsigmoid in habit, amber-yellow, becoming rather deeply tinged with amber-brown. Basal cell short, subtriangular, abruptly bent, obliquely separated from cell *b*, which overlaps it for nearly half its length on its posterior, convex, side; cell *b* similar or somewhat smaller, together with the three cells above it, more deeply tinged with clear amber-brown: cell *c* irregularly triangular, broader than cells *d* or *e;* cell *d* reaching to the posterior margin for a short distance below cell *e*, which is thus wholly separated from cell *b;* cell *e* flat, oblique, its transverse axis uniform throughout; cell *f* but slightly narrower below, externally concave, obliquely separated from cells *e* and *g*, sometimes hardly twice as long as broad; cell *g* externally convex, prolonged distally and externally to form a large, free, deep amber-brown appendage, which may exceed the tip of the perithecium, is suberect or variably divergent, projecting free from the posterior margin, straight or irregularly bent below, narrowed at the base, above which it is rather abruptly swollen, and tapering thence to its blunt apex; cell *h* normal in form and position; cell *i* nearly hyaline, tapering somewhat to its rounded extremity, subsymmetrical, erect, about twice as long as broad, or less. Perithecium strongly convex externally, the outline of its margin not quite even, owing to slight indentations at the junction of the wall-cells; the tip somewhat distinguished, curved inward, short, stout, the outer lip-cell forming a rounded prominence, the inner broad and flat, the lateral ones symmetrically prominent. Spores about 40 × 3.5 μ. Perithecium 75–85 × 27 μ, its appendage about 45 × 11 μ. Total length to tip of perithecium 125–145 × 40–45 μ.

On the posterior legs of *Laccophilus* sp., Sangre Grande, Trinidad, B. W. I., No. 2684; and Grand Etang, Grenada, No. 2687.

A peculiar species resembling *C. spinosus* and *C. Italicus* in having a well developed appendage arising from cell *g*.

### **Chitonomyces helicoferus** nov. sp.

General color clear pale straw-yellow; the axis nearly straight. Basal cell hardly twice as long as broad, tapering slightly to the relatively small foot; cell *b* about as long as cell *a*, its anterior margin slightly concave; cells *c*. *d* and *e* relatively long, subequal, the base of cell *d* occupying the whole distal margin of cell *b;* cell *c* usually prominently convex above cell *b;* cell *f* rather long and narrow, its base somewhat oblique; cells *g* and *h* subequal, the former dark amber-brown distally; cell *i* prolonged far beyond the insertion of the appendage, which becomes lateral in position on the inner side, where the base of the proliferation forms a slight angle; the cell-insertion blackish; the cell proper hyaline; the proliferation divergent, tapering, blackish olive, its sharp apex recurved in an abrupt helix of one turn. Perithecium narrower below, its outer margin evenly convex; the flaring tip abruptly distinguished by a dark amber-brown, erect, finger-like projection, appressed and extending to the small hyaline blunt apex, which it partly conceals; the inner subterminal wall-cell subtriangular, projecting on the inner side to form a divergent, bluntly subconical process; the lower margin of which lies close beside cell *i*, and part of its proliferation. Spores about 25 × 2 μ. Perithecia 54–60 × 12.6 μ; anterior projection 14 × 5 μ, the posterior 18 μ, by 18 μ at base. Total length to tip of perithecium 100 × 19 μ.

On the margin of the left elytron near the middle, of *Laccophilus* sp.; No. 2680, Sangre Grande, Trinidad, B. W. I.

A very characteristic species, most nearly related to *C. melanurus*, from which it is most readily distinguished by the modification of cell *i*, the proliferation of which is quite different in shape and color, while the cell itself is hyaline instead of opaque. A dozen well developed and perfect specimens have been examined.

### **Chitonomyces bicolor** nov. sp.

Hyaline or becoming faintly tinged with dirty yellowish brown, except the large opaque terminal cell. Foot sharply pointed and basally swollen. Basal cell two or three times longer than broad, tapering somewhat to its base; cell *b* small, flattened, horizontal; cells *c*, *d* and *e* subequal; cell *d* an isosceles triangle, pointed above; cells *c* and *e* subtriangular and similar, cell *e* somewhat larger; cell *f* long and narrow, its base extending downward external to cell *e* for less than half the latter's length; cell *g* somewhat broader than cell *h*, its base slightly overlapping the termination of cell *f*; cell *i* extending to, or slightly higher than, the apex of the perithecium, free, blackish, somewhat translucent, opaque externally and distally, abruptly and more or less strongly curved outward distally, the convex margin roughened or tuberculate. Perithecium externally rather strongly convex below, tapering somewhat, distally, to the broadly spreading asymmetrical termination; which is outwardly prominent and rounded, and may be suffused; a much more prominent free rounded projection directed inward, and often wholly or partly overlapping cell *i*; a slight median papillate projection also arises near the pore. Spores about 24 × 2 μ. Perithecia 60–80 × 12–14 μ. Cell *i* about 30–7 μ. Total length to tip of perithecium 100–125 μ.

On the outer margin of the left elytron of *Laccophilus* sp., Sangre Grande, Trinidad, No. 2684; and on the same species at the Grand Etang, Grenada, B. W. I., No. 2687.

A species most nearly allied to *C. Javanicus*, but distinguished especially by the different modification of cell *i*, the opacity of which does not involve cell *g*, as in the Javan species.

### **Chitonomyces seticolus** nov. sp.

Basal cell about twice as long as broad, slightly broader distally; cell *b* horizontal, broader than long, distinguished above and below by well defined constrictions: cells *c* and *d* small, subequal, the latter an isosceles triangle, distally pointed; cell *e* somewhat larger and obliquely adjusted to the base of cell *f*, which extends more than half way to its base: the remaining cells all becoming opaque, or nearly so, forming a black-brown margin to the perithecium, externally symmetrically convex, somewhat translucent along its inner edge, and rather closely rugose; the rugosity hardly distinguishable at maturity, but indicated by faint transverse darker lines; the whole dark area continuous with cell *i*, which curves slightly against the tip of the perithecium which it more or less completely conceals; its basal septum indicated by a darker transverse line, its apex abruptly narrowed to form a short curved finger-like projection which rises above the base of the laterally projecting primary appendage. Perithecium rather narrow, becoming yellowish, slightly concave near the base; the tip becoming abruptly narrowed to a hyaline, short, curved, neck-like termination; the apex bent, usually, toward cell *i;* the lip-cells rather prominent and asymmetrical. Spores about 25 × 2 $\mu$. Perithecia 50–60 × 6 $\mu$. Cell *i* 12.5 × 5.5 $\mu$ not including its distal projection, which is 5 × 3 $\mu$.

Growing among the bristles just within the margin of the distal half of the left elytron of *Laccophilus* sp., No. 2687, Grand Etang, Grenada, B. W. I.

This species is closely allied to *C. marginatus*, which was found on one or two individuals from the same locality growing in its usual position on the wing-margin nearer the base. The present species differs in the presence of a series of closely set transverse ridges on cells *f* to *h*, which, although they are clearly defined in young individuals, might readily escape notice in more mature specimens; and also in the form of cell *i*, which is much longer, and is terminated by a short, narrow, finger-like process; which, in *C. marginatus*, is replaced by a broad truncate extension, as long or longer than the body of the cell itself, from which it is also distinguished by a clearly defined transverse hyaline area.

### **Chitonomyces striatus** nov. sp.

Minute and slender. Basal cell hardly larger than the relatively large foot, somewhat suffused with brown below, about the same diameter throughout, hardly twice as long as broad; cell *b* horizontal, of about the same width, distinguished by slight indentations, about two thirds as long as broad; cells *c*, *d* and *e* subequal, the base of cell *c* in contact with cell *b*; cell *d*, only, triangular; cell *e* slightly larger, obliquely separated from cell *f*, and not in contact with cell *b;* cell *f* relatively short, its rounded extremity obliquely and subsymmetrically separated from cells *g* and *h*, which are nearly equal and as long or longer than cell *f;* the region included by cells *g* and *h*, and the upper half of cell *f*, tinged with brownish and crossed, on the right side only, by seven or eight blackish horizontal parallel ridges, which extend beyond the middle of the adjacent venter of the perithecium, which is otherwise nearly hyaline; cell *i* continuing the axis of cells *g* and *h* directly, erect, about twice as long as broad, distally hyaline and tapering slightly, suffused below, with several blackish flat tubercular patches, the blackened base of the appendage terminal. Perithecium relatively long, narrow and subsigmoid; tapering distally; the strongly convex outer margin uneven, owing to slight elevations which mark the junction of the lower wall-cells; the third external wall-cell producing from its base, a rather slender, very slightly curved, blunt appendage, which projects upward at an angle of more than 45°; the slender tip curved abruptly outward, and ending in a hood-like apex. Spore about 20 × 1.5 $\mu$. Perithecia 46–50 × 10 $\mu$, the appendage 9–10 × 2–2.5 $\mu$. Total length to tip of perithecium 75–80 × 16–17 $\mu$.

On the superior prothorax of *Laccophilus* sp.; No. 2687, Grand Etang, Grenada, B. W. I.

This species is most nearly related to *C. dentiferus* among described species. It is distinguished from all other forms, however, by its conspicuous transverse blackish striations on the right side.

## **Chitonomyces elongatus** nov. sp.

Long, slender, of nearly the same diameter from the basal cell to the base of the perithecial tip: evenly suffused with dirty yellowish or amber-brown; except the basal cell which is more deeply suffused, often blackened on one side and at the base, and also a deeply suffused region along the anterior margin of the tip of the perithecium which merges into a more or less distinct transversely mottled area extending half way across the tip on the right side. Basal cell rather long, convex along its suffused margin, obliquely separated from cell *b* which it overlaps on one side almost completely, and by which it is overlapped for less than half its length on the opposite side; cells *d* and *e* relatively long, lying side by side, similar; cell *c* small and overlapping the distal end, only, of cell *d*, for the most part on the left side; cell *f* very long and narrow, its base somewhat oblique and but slightly overlapped by cell *g*, which is relatively long and narrow, slightly broader distally, and hardly longer than cell *h;* cell *i* more than twice as long as broad, bent against the margin of the perithecium some distance below its apex. Perithecium very long and narrow, the tip hardly distinguished, except by its suffusion; the apex well distinguished on the inner side, somewhat longer than broad, of the same diameter throughout, distally flattened or bluntly rounded, with slightly projecting lips around the median terminal pore. Spores 40 × 2.5 μ. Perithecia 86–100 × 10–12 μ. Cell *i* 12 × 5.5 μ. Total length to tip of perithecium 120–155 × 12–16 μ.

On the tip of the right elytron of *Laccophilus* sp.; Sangre Grande, Trinidad, B. W. I., No. 2680 b.

Although this species is not marked by any of the bizarre characteristics so frequently met with in the genus, it does not seem closely allied to any species known to me. It is most clearly distinguished by its slender form, the obliquity of cell *b*, and by the mottled suffusion on one side of the tip of the perithecium.

### **Chitonomyces longirostratus** nov. sp.

Pale straw-yellow. Foot small; basal cell more or less, often greatly elongated, becoming opaque; cell *b* squarish, or longer than broad; cell *c* similar to, or somewhat smaller than cell *b*; cells *d* and *e* separated from cell *b* by the whole length of cell *c*, both small; cell *f* long and narrow, its base hardly oblique, slightly overlapped by cell *g*; cell *i* hyaline, four times longer than broad, the apex slightly asymmetrical. Perithecium subclavate, strongly curved sidewise, the tip abruptly distinguished, enormously elongated, opaque on one side, translucent purplish brown on the other, of about the same diameter throughout, perfectly straight and rigid; the apex short, distinguished on one side; the lips hyaline and somewhat prominent. Spores about 30 × 2 μ. Perithecia, body 65–78 × 18–20 μ, the tip 245–260 × 8–11 μ. Total length, to base of perithecial tip, 140–156 μ.

On the outer margin, at the tip, of the right elytron of *Laccophilus* sp., Sangre Grande, Trinidad, B. W. I., No. 2680a.

A most peculiar species, clearly distinguished by the extraordinary development of its black and remarkably elongated, stiff, black perithecial tip; which is bent to one side in such a fashion that the body of the perithecium and the receptacle are turned edgewise in preparations under a cover glass, and the arrangement of cells *f* to *g* is thus not determinable in fully developed individuals after they are mounted. Four mature and several younger individuals have been examined.

### **Chitonomyces inflatus** nov. sp.

Form rather stout; hyaline, becoming very faintly yellowish. Basal cell suffused with blackish brown, deeper below, conspicuously inflated above; cell *b* small, somewhat broader than long, but forming an abrupt constriction between cell *a* and the cells above, usually slightly distorted from the fact that the receptacle is often more or less geniculate in this region; cell *b* followed by two relatively large cells, apparently cells *d* and *e*, which are subequal, broader, and distinguished by a slight indentation above; a third cell, apparently cell *c*, small, subtriangular, lying somewhat obliquely above them is visible only on the left side; cell *f* relatively short, ending in the insertion of the secondary appendage, the appendiculate cell *h* not being separated from it; cell *h* overlapping cell *f* for about half its total length; cell *i* normally longer than broad, tapering slightly to its nearly symmetrical rounded tip. Perithecium relatively large and stout, curved outward distally, its upper half free; the tip slightly distinguished and bent outward; the apex hardly distinguished, rather narrow, blunt, subsymmetrical; one or more of the lip-cells slightly prominent. Spores about 40×2.5 μ. Perithecium 75×25 μ. Total length to tip of perithecium 100–120 × 26–30 μ.

On the anterior legs of a small dark dityscid. Manila, P. I. (Banks), No. 2409.

A species readily distinguished by its large inflated and suffused basal, and its constricted subbasal cells. It corresponds to *C. Bidessarius* and a few species among those which inhabit hosts other than *Laccophili*, from the fact that no appendiculate cell *h* is separated from cell *f*. The identity of cells *c* and *d* is also obscure, it being uncertain whether the small cell mentioned in the description, which lies above and partly overlaps the other two, should be regarded as cell *c* or cell *d*.

### **Chitonomyces excavatus** nov. sp.

Wholly hyaline. Basal cell rather short and stout, not twice as long as broad: cell *b* flattened, horizontal; cells *d* and *e* similar, greatly elongated, becoming slightly broader distally; cell *e* slightly longer, cell *d* surmounted by cell *c*, which is relatively small and subtriangular; cell *h* not separated from cell *f*, which thus tapers to its appendiculate point, and is about half overlapped by cell *g*, its whole length being about equal to that of cell *e*; cell *g* rather strongly convex externally toward its distal end, rather large; cell *i* normal, slightly longer than broad, subsymmetrical. Perithecium relatively short, about one fourth free; the tip curved abruptly over a rounded concavity formed by a large, broad, blunt, erect, tooth-like external process: the small somewhat compressed apex subtended on the inner side by a flattened elevation resulting from a thickening of the wall. Spores about 45 × 2.5 μ. Perithecia about 70 × 20 μ, exclusive of the tooth-like process which is 11 × 18 × 22 μ. Total length to tip of perithecium 12–150 × 30–35 μ.

On the margin of the right elytron of a small dark dityscid, Manila, P. I. (Banks), No. 2409.

This species, like the preceding which occurs on the same host, lacks the usually secondary appendiculate cell, which is not separated from the end of cell *f*. It is well distinguished by the abrupt concavity formed between the tip of the perithecium and the prominence which subtends it externally. It is further peculiar from the unusual development of cells *d* and *e* which are relatively much more elongate than in any other described species.

## RICKIA.

Owing to the considerable variety and diversity of the hosts attacked by species of this genus it promises to become one of the largest among the Laboulbeniales. I have referred in a previous paper, (These Proceedings, **47**, 10, 1912), to the diversity presented by the different forms which have thus far come under my observation, and to their considerable variations from the type form as it is illustrated by *R. Wasmanni* of Europe. A study of very copious material has led me to believe, however, that in this as in other groups of which our knowledge is still fragmentary, it is far better to interpret generic types with great liberality. I have, therefore, preferred to make the genus Rickia a rather comprehensive one, the variations as to antheridia, character of the axis, whether branched or simple, triseriate or biseriate, which are the more important matters in which divergence is observable, being so combined or transitional that a subdivision has seemed to me distinctly undesirable.

As in the paper cited above, I have called the appendage which always terminates the axis of young individuals and which, together with its two-celled base, may be carried up by the developing receptacle or left near the base as a lateral appendage, the *primary* appendage.

In triseriate forms the middle series is called *median*, the lateral series on the perithecial side *anterior*; while the other is called *posterior*. It should be mentioned also, that the cell-numbers in the different cell-series may vary, even in the so-called "determinate" receptacles; being often less numerous in small and depauperate individuals or more numerous in those which are more luxuriantly developed, although the average number is usually uniform.

## **Rickia Passalina** nov. sp.

Wholly hyaline. Axis indeterminate, slender, elongate, simple or once to several times variably branched; the cells of the receptacle biseriate above the single basal cell, mostly several times longer than broad, shorter and somewhat broader immediately below the perithecium; the cells of both rows, more often every second cell, cutting off a small appendiculate cell distally and externally, which is separated from the appendage by the usual, though not very conspicuous, blackish septum; the appendages four or five times as long as broad, for the most part appressed or but slightly divergent, those near the perithecia usually somewhat stouter and longer: the slightly tapering base of the primary appendage projecting distally and externally from the second cell of the anterior series, and consisting of a large basal and much smaller distal cell which is separated from the short appendage by a blackened constricted septum. Perithecia terminating the primary and secondary axes, rather long and narrow; the anterior margin free nearly to the base; the posterior united throughout to the last five cells of the posterior cell-series of the receptacle, all of which are without appendages, with the exception of the last, which bears one terminally just below the free apex of the perithecium, the body of which is slightly and subsymmetrically inflated; the tip externally or laterally geniculate, being bent abruptly sidewise or inward; the apex slightly recurved, tongue like, subtended by a prominent elevation. Spores about 20 × 2.5 μ. Perithecia 40 × 10 μ. Total length of axis varying from about 150 to 900 μ, its width 8–12 μ. Appendages 9–12 × 2.5 μ.

On *Passalus cornutus* Fabr. (Type), M. C. Z., No. 2172, Ganard Co., Kentucky (Hyatt). On passaline beetles of various genera and species from: Para and Manaos, Brazil, Nos. 2215 and 2235, 2223, M. C. Z. (Mann); Grande Etang, Grenada, B. W. I., No. 2069; Dominica (Laudet), No. 2170, M. C. Z.; No 2169, M. C. Z., Polvon, Nicaragua; No. 2163, and 2161, Guatemala, both in M. C. Z.

This species appears to be very common and widely distributed. It belongs to the type formerly distinguished as 'Distichomyces' the very slender receptacle being biseriate instead of triseriate as in a majority of the Rickiae. The antheridia are scanty and the antheridial cells appear to become free in rather irregular groups. The axis though often simple may be divided into several more or less elongate secondary axes, or a main axis may persist from which as many as ten short secondary axes may arise on either side, and since each axis is eventually terminated by a perithecium there may sometimes, though rarely, be as many as a dozen of the latter on a single individual. The unbranched condition appears, however, to be the normal one, although, whenever the termination of an axis is injured, branching invariably follows. The peculiar contour seen in the side view of the tip of the perithecium is usually not visible owing to the fact that the latter is more often bent abruptly sidewise so that it is viewed end on in most preparations. Its outline is somewhat variable, the tongue-like projection being stouter or more slender in different cases and more or less distinctly recurved, while the subtending rounded projection which, together with the tongue, remotely suggests the head of a tufted fowl, is also variable in its prominence. The species is most nearly allied to *R. nutans* of Ceylon.

## **Rickia apiculifera** nov. sp.

Axis typically simple, not infrequently rather copiously branched, especially when injured; sometimes rather broad and short, but often greatly elongated; not infrequently slightly broader just above the single basal cell, but sometimes quite slender throughout; all the axes finally terminating in perithecia; the axis biseriate, the cell-number indeterminate; usually not much longer, often shorter, than broad. The basal cells of the primary appendage arising, usually, from the third or fourth cell above the basal cell, on either the anterior or posterior side. The appendages in general rather large, usually more or less persistent, numerous, divergent, especially near the base, irregularly distributed. Antheridia scanty, the cells becoming free. The terminal cell of the anterior cell-series larger and broader, and distally in oblique contact with the base of the perithecium; or extending upward beside it, and sometimes cutting off a terminal appendiculate cell which may reach even to the middle of the anterior margin of the perithecium; the posterior series of axis-cells continuous along the corresponding margin of the perithecium, which is thus united on this side to a series of usually five or six cells, some or all of which may each cut off an appendiculate cell; the terminal one either bearing an appendage directly; or cutting off a large appendiculate cell distally, which lies just below the apex of the perithecium. Perithecium relatively long and narrow, its anterior margin one half to almost wholly free; the tip distinguished; the apex ending in a slightly bent, free, bluntly pointed projection formed by one of the inner lip-cells. Spores 25–28 × 3–4 μ. Perithecia 36–50 × 10–16 μ. Total length very variable, 100–1000 × 10–20 μ. Appendages 18–20 × 3.5 μ.

On *Passalus tlascala* Perch., (Type), No. 2068 and *Neleides antillarum* Arrow, No. 2069, Grand Etang, Grenada, B. W. I. On several passalline beetles; No. 2162, Guatemala, M. C. Z.; No. 2163 and 2165, Guatemala (Kellerman); No. 2166, Yucatan, M. C. Z.; No. 2167, Polvon, Nicaragua.

This species differs from all other known forms which possess a similarly pointed perithecium, in developing no axes which are indeterminate and sterile; every axis, whether primary or secondary, finally ending in a perithecium. It is extremely variable in habit; typically simple; but when growing luxuriantly, tending to branch copiously and to become greatly elongated.

### **Rickia bifida** nov. sp.

Wholly hyaline. Receptacle consisting of a short basal stalk-portion, including a larger basal cell and usually three smaller subequal cells, the upper two paired; the remainder of the main axis above this short stalk-portion and a branch near its base, though occasionally subject to abnormal branching, usually form two broad simple upcurved, subsymmetrical, slightly tapering, often nearly similar, elongate slender divisions with indeterminate apical growth; the cells of which are biseriate, mostly several times longer than broad, some, both of the inner and outer series, cutting off appendiculate cells; the appendages thus arising at irregular intervals, of the usual type, appressed or but slightly divergent, cylindrical or slightly tapering: the primary appendage small, borne on a large straight slightly inflated two-celled free base, arising from the second outer cell of the axis-division, the basal cell three times as long as the terminal. The perithecium arising at or near the base of the axis-division; mostly erect on a short stalk, which consists of a pair of cells; the anterior triangular, its upper angle extending above the base of the ascigerous cavity; while the posterior does not extend beyond this cavity, and forms the base of a series of four successively smaller flattened cells which lie in contact with the posterior margin of the perithecium; the series ending below the apex, the first and fourth cells appendiculate. Perithecia rather long and narrow, hardly inflated; the tip usually more or less clearly distinguished; the apex ending in a blunt, finger-like, erect projection. Spores 28 × 2.5 μ. Perithecia 35 = 40 × 10 μ; the marginal cells × 3.5 μ; the stalk 9 × 6–7 μ. Stalk portion of receptacle 7–12 × 7–12 μ; its longest divisions 75–100 × 5.5–7 μ. Free base of primary appendage 12–16 × 3.5–4 μ. Appendages 15–20 × 3 μ.

On various passaline beetles, No. 2163 (Type) and 2164, Guatemala (Kellerman). Nos. 2168 and 2169, Polvon, Nicaragua (M. C. Z.); No. 2171, Rio de Janeiro (M. C. Z.); No. 2238, Amazon (Mann), M. C. Z.

This small and delicate species is closely allied to the *R. dichotoma* and *R. apiculifera*. From the former it is distinguished by its small size, general habit, and by minor details of structure; while from forms of the latter which have produced irregular and abnornal branches, it is sometimes distinguished with difficulty. The divisions of the receptacle, however, do not produce perithecia terminally; and, in normally developed individuals a single perithecium, only, arises on a short stalk from near the base of one of the divisions.

### **Rickia dichotoma** nov. sp.

Hyaline, branched, the axes biseriate. Basal cell of the receptacle broad, pointed distally, its upper half or more intruded between two irregularly paired cells lying above it, which it may almost completely separate; this pair is followed by two cells irregularly paired, above which the receptacle becomes furcate; one or both of its two stout divisions becoming once more, often almost immediately, furcate; one or both of the inner branches terminating in a perithecium; the others indeterminate, somewhat tapering, rather straight and rigid at least below, often greatly elongated; some of the cells bearing small appressed appendages which occur scattered on both sides; all the branches consisting of biseriate cells, and diverging more or less regularly at an angle of about 45°. The perithecial axes of variable length, the cells asymmetrically placed with reference to one another, four to twelve in each series; the upper cell of the anterior series protruding abruptly beside the base of the ascigerous cavity: the posterior series continued along the posterior margin of the perithecium nearly to its apex; these marginal cells, six to seven in number, successively smaller, externally hardly convex, all of them usually cutting off distally and laterally small appendiculate cells, except the basal and also the subterminal cell, which is smaller than the terminal; the latter being an appendiculate cell separated from it distally. Perithecia relatively long and narrow; the anterior margin and the apex free, the latter terminating in a well defined, erect, short, stout prolongation of one of the inner lip-cells; the tip well distinguished. Base of the primary appendage long and slender, arising externally from the third, fourth or fifth cell above the basal cell of the receptacle. Spores about 30 × 3.5 μ. Perithecia 55–65 × 14–18 μ, the marginal cells × 7 μ. Basal part of the receptacle 45 × 30 μ; × 12 μ at base; its longest divisions 250–675 μ.

On the superior surface of a species of *Euzercon* parasitic on passaline beetles; No. 2794, Diquini, Hayti, M. C. Z. (Mann).

A species most nearly related to *R. Cornuti*, and also to *R. arachnoides* among other species parasitic on mites. From the latter it is at once distinguished by the straight hyaline projection from the apex of its perithecium.

### **Rickia Cornuti** nov. sp.

Hyaline. Receptacle branching, the axes biseriate: consisting of a short basal primary axis comprising a small basal cell, the pointed end of which is usually intruded between the lower of usually two successive pairs of cells, all the members of which may cut off one or more, often two, appendiculate cells; the slender two-celled, straight, narrowly conical, free divergent base of the primary appendage arising from, or just above, one of the upper members. Above, the receptacle is divided into two somewhat divergent branches: one of them simple or giving rise from the inner side near the base to one, rarely two, short perithecial branches; always greatly elongated, with indeterminate apical growth, the very numerous cells biseriate and for the most part cutting off appendiculate cells bearing appendages of the usual type: the other division similar, or more often shorter, and terminated by a perithecium. Perithecia rarely nearly sessile, more often borne on a variably developed axis of biseriate cells irregularly paired, from two to eight in each series, one to four if borne on a special secondary perithecial branch; the upper cell of the anterior series extending only to the base of the ascigerous cavity; the posterior series continuous with a row of usually seven appendiculate, rather prominently convex, rounded, subequal cells in contact with the posterior margin of the perithecium nearly to its apex; the uppermost of these cells broader and rounded distally, unlike the others in bearing its appendage directly, without any basal cell, on its inner distal surface. Perithecium relatively long and narrow, its anterior margin wholly free; nearly straight or rather strongly convex; the tip well distinguished; the apex terminating in a well marked, erect, terminal, blunt, short projection formed by one of the inner lip-cells. Appendages very numerous, slightly tapering, rather stout, appressed or slightly divergent. Spores about 36 × 4 μ. Perithecia 60–75 × 14–18 μ, the marginal cells × 10 μ. Basal portion of the receptacle 18–27 × 25 μ, the basal cell × 9 μ. Longest division of the receptacle 675 μ (varying to less than half this length), its width 12 μ, becoming less distally. Appendages 16–18 × 2.5–3.5 μ. Base of primary appendage 16–18 × 4.5 (base)–1.8 μ (apex).

On *Passalus cornutus* Fabr., No. 2172 (M. C. Z.), Ganard Co., Kentucky (Hyatt).

This species appears to be most nearly related to *R. dichotoma* from which, as well as from other nearly allied forms, it is at once distinguished by the presence of paired appendages on all the cells of the basal part of the receptacle, with the exception of the basal cell itself. The greatly elongated divisions of the receptacle are more or less flaccid, and tend to become spirally twisted in mounting, in contrast to the more or less rigid divisions of *R. dichotoma*. It is subject to considerable variation in size and in general habit, owing to the fact that one of the long divisions of the receptacle may be replaced by a much shorter division, terminated by a perithecium, and that one or two additional perithecia may arise on short secondary branches from either of the main divisions near the base. No antheridia have been recognized in the material examined.

## **Rickia depauperata** nov. sp.

Minute, hyaline, usually short and stout, receptacle biseriate. Basal cell abruptly distinguished, short, stalk-like, sometimes more than half included between the basal cells of the anterior and posterior cell-series, which may extend obliquely downward nearly to the foot. Receptacle biseriate; the anterior series consisting of usually four subequal cells, and extending to or above the middle of the perithecium, all but the uppermost usually cutting off an appendiculate cell: posterior series consisting of seven or eight cells, mostly broader than long, their longer axes more or less radially arranged, cutting off distally and externally appendiculate cells which are somewhat prominent; the two celled base of the primary appendage diverging from the fifth or sixth cell, the subterminal cell not associated with an appendiculate cell; the terminal cell bearing an appendage directly, the blackened base of which lies close beside the apex of the perithecium. Perithecium straight, erect, half or less free along the anterior margin, tapering distally, the tip hardly distinguished, the apex ending in a blunt free prolongation of one of the inner lip-cells. Appendages short and stout, often considerably inflated, the free dark base often cup-shaped. Spores about 18 × 2 $\mu$. Perithecia 20–28 × 7–9 $\mu$. Total lengths to tip of perithecium 35–40 × 19–23 $\mu$. Larger appendages 7 × 3.5 $\mu$.

On *Celaenopsis* sp., collected in Hayti by W. H. Mann, M. C. Z., No. 2795 (type), Diquini; No. 2792, Petion; No. 2793, Cape Haytien.

A minute species allied to *R. Euzerconalis*, but at once distinguished by the absence of any median cell-series. No antheridia were recognized in the dozen or more individuals examined.

### **Rickia Dominicensis** nov. sp.

Receptacle triseriate, the anterior series reduced to two cells; the posterior and the median, which originates near the middle of the perithecium, forming together a slender free flagellum: hyaline, except the basal cell; which is hyaline below, its distal portion prolonged upward on the posterior side and deep black, except at its tip, the opaque area extending obliquely up beside the two lower cells of the posterior series, forming a contrasting margin, its hyaline apex ending at or near the blackened base of the first posterior appendage: anterior series consisting of two cells; the lower subtriangular, its inner margin convex, cutting off distally and externally an appendiculate cell; the upper lying in somewhat oblique contact with the base of the perithecium, its attenuated distal portion curved up beside the lower fourth of the anterior margin of the latter: the four lower cells of the posterior series long, subequal; the fourth reaching nearly to the tip of the perithecium; the second united wholly, the third for half its length to the posterior margin of the perithecium; the lowest, which is broader below, separating by its whole width the lower cell of the anterior series from the basal cell of the receptacle: the posterior series, a majority of the cells of which cut off single appendiculate cells distally and externally, becomes united above the middle of its third cell to the cells of the median series; which are long and flattened, without appendages, the two lower united to the upper half of the inner margin of the perithecium nearly to its apex; the posterior and median series together forming a long, slender, tapering, free, flagellum-like prolongation; which is determinate, owing to intercalary division below the two-celled base of the primary appendage, by which it is terminated; the cells of the two series irregularly paired. Perithecia erect, long and narrow, the tip somewhat distinguished, slightly bent distally, the apex terminating in a slightly curved projection. Spores about 20 × 2 μ. Perithecia about 36 × 6 μ. Opaque portion of basal cell 26 × 3.5 μ. Total length to tip of perithecium about 50 μ; to tip of flagellum (variable) 160–190 μ. Greatest width 15 μ. Appendages 15–25 × 3 μ.

On antennae of a large species of *Passalus?;* Dominica, B. W. I., No. 2170, M. C. Z.

This peculiar and aberrant species is at once distinguished by the blackened extension of the basal cell which closely resembles in general appearance the similar modifications of the basal cells which occur in many of the stilicolous species of *Corethromyces*. It is most nearly related to the African *R. filifera*, in which the median series of cells, which in the present species form the inner half of the flagellum, is replaced by two cells lying opposite the apex of the perithecium; the free portion of the flagellum being therefore uniseriate. A reëxamination of abundant material of the African form shows, also, that the flagellum, as in the present species, is terminated by the primary appendage and its two basal cells.

## **Rickia parvula** nov. sp.

Hyaline. Basal cell short, irregularly triangular; its base, only, free; obliquely and asymmetrically adjusted to the two cells above it. Receptacle triseriate, the anterior series consisting of four superposed cells, the two lower greatly flattened and obliquely associated, the lower cutting off a relatively large appendiculate cell distally and externally; the third small and subtriangular, extending inward below the base of the ascigerous cavity; the fourth flattened beside the outer margin of the perithecium, extending upward perhaps one third of its length, and so narrow as to be hardly recognizable: median series consisting of seven obliquely superposed cells; the two lower smaller, the lowest somewhat above the base of the perithecium, the upper four externally free, the uppermost bearing a terminal appendage directly; the two lower cutting off appendiculate cells, while the subterminal is without any appendage: posterior series consisting of three or four cells, each cutting off an appendiculate cell distally and externally, the upper larger, the uppermost also bearing the large base of the primary appendage which diverges slightly from it laterally and distally. Perithecia relatively large, somewhat curved inward throughout; the tip rather short and moderately distinguished; the apex broad, blunt, bent slightly sidewise. Spores about 16 × 2 μ. Perithecia 28 × 9 μ. Total length to tip of perithecium 40 × 18 μ. Base of primary appendage 10 × 3.5 μ.

On *Celaenopsis* sp., No. 2697, St. Anns Valley, Port of Spain, Trinidad.

This minute form was found in company with *R. excavata*. It is distinguished from depauperate forms of *R. Euzerconalis* by the blunt apex of its perithecium, which lacks the finger-like projection so characteristic of this and numerous other species. Two mature specimens, only, have been examined, in which no antheridia were recognized.

### **Rickia radiata** nov. sp.

Hyaline, triseriate. General form short and compact with more or less even outline, basal cell forming a well developed abruptly differentiated stalk, the apex of which is intruded between, and nearly separates, the paired triangular basal cells of the anterior and posterior series, one or both of which may cut off marginal appendiculate cells distally and externally; one or both of the basal cells themselves grow downward to form an elongated, attenuated, hyaline, buffer-appendage which may or may not be separated by a septum at the base. Receptacle triseriate, subdeterminate; the anterior series consisting of five cells, the second and third broader; each cutting off distally and externally one, sometimes two, superposed small cells which bear prominently projecting subulate hyaline sessile antheridia, which are not distinguished by any blackened septum; the fourth cutting off an appendiculate cell; *the fifth smaller and bearing the long straight* narrow free, somewhat divergent base of the primary appendage, which consists of two nearly equal cells: the middle series consisting of four, or usually five, subequal cells, extending in contact with the inner margin of the perithecium to the point where the base of its conical free tip is faintly distinguished: the posterior series consisting of three or four cells, the second cutting off one or two superposed small cells bearing subulate antheridia; the third, or third and fourth, extending up along the anterior margin of the perithecium to about its middle. The antheridia and appendages more or less radiately disposed; the latter all, with the exception of the primary, relatively long, especially those from the basal cells, and all distinguished by a relatively long constricted blackened base. Perithecia rather short and stout, distally subconical; the tip hardly distinguished; the apex broad, truncate, or bluntly rounded. Spore 22 × 2.5 μ. Perithecia 28 × 11 μ. Antheridia 6–7 μ. Longest appendages 24 × 3.4 μ; base of primary appendage 10–12 × 3 μ. Outgrowths from basal cells 30–55 μ. Basal cell of receptacle 10–15 × 5.5 μ. Total length to tip of perithecium 45–55 × 23–28 μ.

On *Celaenopsis* sp., No. 2780, Kamerun.

A species most nearly allied to *R. Celaenopsis*, distinguished by the subradiate arrangement of its antheridia and appendages, the sessile subulate character of the former, and the long blackened bases by which the latter are distinguished. The attenuated buffer-outgrowths from the basal cells of the two outer series, though not always present, are not known in any other species of the genus.

### **Rickia Hypoaspitis** nov. sp.

Receptacle hyaline triseriate. Basal cell relatively large, not intruded distally. Receptacle triseriate; the anterior series consisting of two, rarely three, superposed cells; the lower somewhat larger, usually, but not always, cutting off an appendiculate cell distally and externally; the uppermost cutting off a similar cell which bears a sessile pointed compound antheridium, the neck of which bends upward beside the base of the perithecium: the posterior series consisting of four or five cells, one to several of which cut off appendiculate cells distally and externally; the uppermost bearing the base of the primary appendage which projects free, and is strongly divergent, rather stout, its two cells nearly equal: the middle series consisting of three, very rarely of four, usually subequal flattened cells, all of which are in contact with the inner margin of the perithecium to its tip; the uppermost extending above the corresponding cell of the posterior series. Perithecium rather short and stout, its anterior margin almost wholly free, the tip not distinguished, its apex bluntly rounded. Spores about 18 × 2.5 μ. Perithecia 35 × 12 μ the marginal cells × 3 μ. Appendages 10–12 × 3.4 μ. Basal cell 10–12 × 4.5–5 μ. Total length to tip of perithecium 55–65 × 16–25 μ.

On *Hypoaspis* sp., No. 2797, Grand Etang, Grenada, B. W. I.

This very minute species is perhaps most nearly related to *R. Celaenopsis*. As in this species, its sessile antheridium arises at the base of the perithecium and is not distinguished by any blackened base, a characteristic also found in *R. minuta*. The host is a very minute mite which was found parasitic on a termite, and which Mr. Banks, who has kindly examined it for me, regards as a new species.

### **Rickia Euxesti** nov. sp.

Hyaline, short and usually stout. Receptacle triseriate, the basal cell forming a short stalk, usually bent slightly, and distally intruded between the subsimilar basal cells of the two marginal series: the anterior series consisting of five, rarely six or four, superposed cells, broader than long, their transverse axes often subradiately arranged, the upper slightly larger, the uppermost lying below the base of the perithecium which is slightly tipped to one side; all the cells, except the one or two lowest, cutting off one, often two relatively large cells distally and externally, a majority of which usually bear typical antheridia, while a few may be appendiculate: median series consisting of four, rarely five or three cells, the two, rarely three, lower usually larger, subequal, lying below the base of the perithecium; the two, rarely three, upper successively smaller and flatter, lying in contact with its inner margin: posterior series consisting of seven to rarely nine or five cells, the upper three or four successively much smaller, the rest subequal, the uppermost bearing the basal cell of the primary appendages, the rest, except the lowest, cutting off one, or less often two, relatively large superposed cells which bear either appendages or antheridia: all the cells of the receptacle proper lying below the perithecium of about the same size, the latter subtended by a distinct stalk-cell. Antheridia relatively large and numerous, the basal ring broad, the neck strongly curved outward. Appendages scanty, usually broken and proliferous, rather short, subcyclindrical. Perithecium slightly tilted inward, its body subsymmetrical, long elliptical, the outer margin free, the broad short tip slightly distinguished, the apex broad, blunt, symmetrically rounded. Spores about 18–20 × 2.5 μ. Perithecia 26–36 × 16–18 μ. Antheridia 10 × 3.5 μ. Total length to tip of perithecium 50–90 × 20–26 μ, smaller specimens 30 × 15 μ.

On *Euxestus Parki* Woll., Nos. 2419, 2422 and 2552, Manila, Philippines.

A species somewhat similar to *R. Europsis,* but clearly distinguished by its large curved antheridia and nearly free perithecium, as well as by other points of difference. On the smaller hosts it is less well developed, smaller, less stout, with fewer cells.

### **Rickia Europsis** nov. sp.

Hyaline, short and stout, broadly elliptical but asymmetrical, the posterior margin being more or less straight. Basal cell small, short, abruptly distinguished from the body, its pointed distal end symmetrically intruded between the nearly equal and similar basal cells of the two marginal series. Receptacle triseriate: the anterior series more distinctly and evenly convex; consisting of nine, less frequently ten cells, broader than long except the distal ones, abruptly successively larger, or the three lower larger; each, except the terminal, and sometimes the subterminal, separating a cell distally and externally, sometimes two, which bear antheridia or appendages of the usual type: posterior series consisting of six or seven cells, the lower four or five large, subequal, forming a straight erect series; all broader than long and separating distally and externally one, or often two, small cells which bear antheridia or appendages; the series ending in the two relatively short and broad cells which form the base of the primary appendage: the median series consisting of normally seven successively smaller cells in contact with the posterior margin of the perithecium, and extending from just below its base somewhat beyond the base of the primary appendage to its tip. Appendages rather short and stout. Antheridia typical, short and stout, the neck rather broad and not very abruptly distinguished. Body of the perithecium symmetrical, long-elliptical, hardly more than its apex free; the latter blunt and bent abruptly sidewise. Spores about 24 × 3 μ. Perithecia 40–54 × 16–18 μ. Appendages 6–9 × 4 μ.

Antheridia 7 × 3.5 μ. Total length to tip of perithecium 60–66 × 36–43 μ.

On *Europs* sp., No. 2338, Kamerun, W. Africa.

In general appearance this species bears some resemblance to *R. elliptica*, from which it is readily distinguished by its numerous typical antheridia, and the absence of any terminal projection from the tip of the perithecium.

## **Rickia gracilis** nov. sp.

Hyaline, elongate, of nearly the same width throughout. Receptacle triseriate, the basal cell usually abruptly bent, brown next the foot but otherwise hyaline, intruded so as to separate completely the basal cells of the marginal series: anterior series consisting of about twenty cells very obliquely related and several times longer than broad; the uppermost, only, lying beside the base of the perithecium; all, including the lowest, cutting off two, to less often four, cells, which bear appendages or antheridia: posterior series similar to the anterior, and consisting of about twenty cells which extend hardly higher than those of the anterior series, and bear terminally the rather long slender base of the primary appendage: median series consisting of about eighteen cells of regular form and diameter, from four to six times as long as broad, except the uppermost; the three to four distal ones successively smaller and rounded, the last three lying beside the lower fourth of the perithecium. Antheridia numerous, hyaline, the venter rather stout, the neck slightly curved and moderately well distinguished. Appendages normal, slender, relatively long, hardly tapering. Perithecia yellowish with a tinge of purplish, erect symmetrical, except the base, which turns outward slightly, three fourths free, or slightly more, externally; the margins slightly convex, tapering symmetrically to the small truncate apex; the tip somewhat paler, and not distinguished. Spores about 35 × 3.6 μ. Perithecia 75 × 29 μ. Antheridia 14 × 4 μ. Appendages 10–20 × 3 μ. Total length to tip of perithecium 450 × 34 μ; the receptacle for the most part × 26 μ.

On *Stenotarsus Guineensis* Gerst., No. 2363b, Kamerun.

A well defined species, quite unlike the others occurring on the same host; and distinguished by its long, slender, uniform receptacle, and laterally placed, erect, symmetrical perithecium. Six specimens have been examined, only one of which is fully matured.

## **Rickia Danaëalis** nov. sp.

Hyaline; form sometimes short-triangular below the free distal half of the perithecium. Receptacle triseriate; the basal cell relatively large and broad, somewhat intruded between the two marginal cells above it, or more often adjusted to the basal cell of the anterior series; that of the posterior tapering to a narrow contact base: anterior series consisting of usually seven or eight cells, somewhat irregular in outline, the three or four distal ones smaller, lying beside the lower third or more of the perithecium; all, except the lowest, cutting off one cell or two cells irregularly superposed or lying side by side, which bear, the lower ones usually antheridia, the upper usually appendages: posterior series similar to the anterior, consisting of usually five or six cells; the lowest larger than the corresponding cell of the anterior series, the cells of both series above them asymmetrically disposed; the series terminated by the free, or almost wholly free, stout, divergent base of the primary appendage: median series consisting of five or six cells, often as large as those of the posterior series; the four or five upper, and sometimes part of the lowest, lying beside the perithecium; the uppermost distally free between the perithecium and the base of the primary appendage; the basal one intruded somewhat between the second cells of the two marginal series, or between the second posterior and the third anterior marginal cells. Appendages relatively long and divergent; hyaline, becoming tinged with brown, especially at the base; straight, nearly cylindrical, tending to form two distal symmetrically divergent tufts; the primary appendage longer, more deeply suffused. Antheridia of the normal type; hyaline, slightly curved; the necks rather short and well distinguished, usually in pairs, arising from the second and third, sometimes also the fourth, cells of the anterior series. Perithecium subtended by a well marked stalk-cell, somewhat lateral, two thirds to more than one half free on both sides, thick-walled, nearly straight, or more or less distinctly bent distally; the middle third of the free portion slightly convex along the posterior margin; the tip otherwise hardly differentiated, somewhat asymmetrically conical, tapering to a blunt point. Spores about 36 × 3.6 $\mu$. Perithecia about 75–95 × 28–32 $\mu$. Antheridia 16 × 5.5 $\mu$. Appendages, lower 20 $\mu$ the upper to 65 $\mu$ × 7 $\mu$. Primary appendage to 75 $\mu$, its base about 18 × 10 $\mu$. Total length to tip of perithecium 120–190 × 45–60 $\mu$.

On various parts of *Danaë Senegalensis* Gerst., No. 2570, Kamerun.

A clearly defined species which varies somewhat in its general form according as it is shorter, stouter and broadly triangular, or more elongate. The long, stout, basally suffused, more or less symmetrically divergent distal groups of appendages give it a characteristic appearance.

### **Rickia Scydmaeni** nov. sp.

Receptacle triseriate, straight, or bent just above the basal cell, hyaline, subsymmetrical in outline; broad distally, and tapering continuously and considerably to the sometimes rather abruptly distinguished long basal cell, the round extremity of which is slightly intruded between the two lower cells of the marginal series; which are somewhat longer and unequal, and very obliquely separated from the two cells next above: anterior series consisting of six or usually seven cells; shorter, smaller and more rounded distally; all except the lowest usually cutting off a relatively large cell, which bears an appendage, or rarely an antheridium: posterior series similar to the anterior, consisting of usually seven cells: median series consisting of five successively smaller cells, the lowest lying above the second pair of marginal cells between which it is intruded; all three series extending upward to about the same level above the base of the perithecium, the cells rather irregular in form and size. Base of the primary appendage free, symmetrically adjusted to the two distal cells of the posterior and median series; its basal cell large and broad, the distal partly hyaline, or becoming nearly opaque and indistinguishable from the blackened constricted region which subtends the short inflated hyaline appendage; secondary appendages divergent, asymmetrically clavate, the basal blackish brown suffusion involving the lower half nearly or quite to the tip, the upper margin hyaline, at least distally. Antheridia normal, scanty, large, stout, with short not abruptly differentiated necks, as large, or nearly as large, as the appendages, similarly and evenly suffused. Perithecium about four fifths free, thick-walled, deeply suffused, but slightly asymmetrical, distally subconical; the large tip rather clearly distinguished, slightly inflated and darker in the middle, whence it tapers evenly to the small hyaline truncate apex. Base subtended by a well defined stalk-cell. Spores about 30 × 3 $\mu$. Perithecia 18 × 4 $\mu$. Appendages 18–22 × 5.5 $\mu$. Antheridia 18 × 4 $\mu$. Total length to tip of perithecium 150–160 × 35–38 $\mu$.

On the inferior surface of *Scydmaenus bicolor*, No. 1422, Kittery Point, Maine.

This species does not appear to be nearly allied to any described form, and is clearly distinguished especially by its dark appendages. It seems to be decidedly rare.

### **Rickia Stenotarsi** nov. sp.

Quite hyaline, or the perithecium tinged with blue; tapering more or less symmetrically to the base, and often slightly twisted below, so that it is turned partly edgewise. Receptacle triseriate, the basal cell usually strongly bent, with a small brown patch next the foot, intruded between the two nearly equal cells above it, so as to separate them more or less completely; the anterior series extending to slightly below the middle of the perithecium; consisting of usually twelve somewhat elongated and flattened cells, the three upper smaller, lying beside the perithecium; all except the basal cell cutting off from one to three cells obliquely superposed, or lying side by side, or somewhat irregularly associated: posterior series similar to the anterior, consisting of usually eleven cells, the distal much smaller, round, and lying lower than the corresponding cell of the median series; the long, slightly tapering free base of the primary appendage diverging laterally between them: median series consisting of eleven or twelve cells, the three or four distal ones lying beside the basal third or less of the perithecium; the basal cell but slightly intruded between the second pair of marginal cells; the lower four or five cells subrectangular, twice or somewhat more than twice as long as broad. Antheridia hyaline, divergent, numerous, rather slender, often slightly curved; the neck not abruptly distinguished. Appendages hyaline, rather slender, slightly tapering, the lower shorter, the upper, including the primary appendage, much longer. Perithecium hyaline, or yellowish, or blue; the blue color, when present, extending to the well defined subtending stalk-cell or even lower; its posterior margin more than half free; nearly erect, relatively long, distally asymmetrical owing to a more or less distinct bend of the large long well distinguished tip, which is more convex on its inner side; the small narrow truncate or rounded apex turning more or less abruptly upward. Spores about 40 × 4 $\mu$. Perithecia 80–120 × 28–35 $\mu$. Antheridia 22 × 4 $\mu$. Appendages 22 × 46 × 4 $\mu$. Total length to tip of perithecium 200–350 × 40–55 $\mu$.

On the elytra of *Stenotarsus Guineensis* Gerst., No. 2363, Kamerun, W. Africa.

This well defined species is perfectly hyaline, except that in a majority of the very numerous individuals examined, the perithecia, and sometimes certain cells below it, are tinged with rather bright blue, as if stained with haematoxylin. That this color is not accidental seems to be indicated by the fact that some perithecia from the same source are quite hyaline, and that no other parts of any individuals are thus colored: yet it is not certain that this very unusual color may not have been due to the accidental presence of some staining material in the containing bottle.

### **Rickia latior** nov. sp.

Quite hyaline, short and broad. Receptacle triseriate, the basal cell brown, its broad blunt upper half hyaline and completely separating the first pair of marginal cells: anterior series reaching to about the middle of the perithecium, consisting of ten obliquely superposed cells, all of which, except the basal, cut off from one to four cells, unusually prominent with free ends; which bear appendages, or copious antheridia, arising side by side in a more or less regular transverse series: posterior series consisting of eight cells, similar to the anterior, and terminating in a very small cell; which bears the large, long, slightly tapering base of the primary appendage: median series consisting of from six to eight cells; the lower three or four obliquely superposed; the upper three or four lying beside the lower two fifths or less of the perithecium, the uppermost opposite the last cell of the posterior series. Antheridia very numerous, long, slender, tapering, slightly curved, hyaline. Appendages variable in length, the uppermost usually longer; hyaline, tapering. Perithecia very thick-walled, becoming pale straw-yellow, asymmetrical, more or less strongly and abruptly curved inward, so that the apex may project laterally beyond the posterior margin of the receptacle; the tip very abruptly distinguished, especially along its anterior margin; the apex well distinguished, longer than broad, distally truncate, or obliquely rounded. Spores 38 × 3.8 $\mu$. Perithecia 80–90 × 27–30 $\mu$. Antheridia 9 × 3.6 $\mu$. Appendages 18–60 × 3.6 $\mu$. Total length to tip of perithecium 48–70 $\mu$.

On the elytra of *Stenotarsus Guineensis* Gerst., No. 2363, Kamerun.

This form is perhaps too closely allied to *R. Stenotarsi* which occurs on the elytra of the same host. It differs in its short broad habit, obliquely superposed axial cells, simpler structure, more numerous antheridia, which often occur in fours; and the abrupt and peculiar distal modification of its perithecium; which is, however, faintly suggested by the bent tip of *R. Stenotarsi*. Were it not that the material of both is abundant and in perfect condition, and the individuals clearly distinguished in all cases, I should hesitate to separate them, and in any case it may eventually prove more desirable to separate the present form as a var. *latior* of the preceding.

### **Rickia introversa** nov. sp.

Straight, short and stout, asymmetrical. Receptacle triseriate, the cell-numbers somewhat variable; the foot and basal cell large, the latter more or less completely and deeply involved by a brownish black suffusion; its apex slightly intruded between the cells above it: anterior series consisting of about sixteen cells and extending to the tip of the perithecium, which is bent abruptly over its broad, blunt termination; the subterminal cell and the one next below, extending higher than the small terminal one, which lies against the concave side of the perithecial tip; all the cells, except four or five of the flattened terminal ones, and usually also the basal, being sharply pointed outward; owing to the separation, distally, of a narrow cell, sharply pointed inward, which, in the lower members, may be nearly as broad as the cells from which they are separated, and which bear externally antheridia or appendages: posterior series similar in general to the anterior, consisting of six or seven stouter cells and ending in the lower of the two cells which form the base of the primary appendage; all, usually including the lowest, bearing antheridia or appendages laterally, not marginally, as in the anterior series; so that, when they lie at the right, their origins are not visible: median series consisting of four or usually five cells, its much larger basal cell not at all or but slightly intruded below. Appendages closely appressed, distally hyaline, and not clearly distinguishable, except their contrasting blackish brown bases, the suffusion extending some distance above the indistinguishable basal septum, and also involving more or less distinctly the cell which bears them. Primary appendage of the normal type, unlike the secondary; its base not clearly differentiated from the cells below it. Antheridia very large, scanty, appressed, or overlapping the receptacle; the necks long and stout, not abruptly distinguished. Anterior margin of the perithecium hyaline, united to the receptacle as far as the tip, which is bent abruptly over the rounded termination of the anterior series; its posterior margin about two thirds free above the base of the primary appendage: the apex blunt and more or less evidently and coarsely bilobed, the lobes lying side by side; a minute stalk-cell below the base. Spores scanty, about 18 × 2 μ. Perithecia 35–40 × 11–13 μ. Appendages 6 × 1.5 μ. Antheridia 12 × 2 μ.

On the upper surface of the abdomen of *Coproporus latus* Motsch., No. 2380, Mindanao, P. I.

This species is closely allied to the two following forms which occur on the same host, all of which have an appearance, unusual in members of this genus, from the contrasting suffusions of the bases of the appendages and of the cells which bear them, giving the margins a transversely banded appearance. There are not more than one or two antheridia distinguishable in any of the eight individuals examined, and the hyaline portion of the appendages is inconspicuous from its appressed habit, and tendency to become disorganized.

### **Rickia nigrofimbriata** nov. sp.

Asymmetrical, straight. Receptacle triseriate, the basal cell relatively large, short, broad, subtriangular, becoming wholly suffused with brown: anterior series consisting of about sixteen cells which are subequal, except the five terminal ones which bear no appendages; the four subterminal more flattened, and obliquely superposed; the terminal forming a short, stout, rounded, free, finger-like prominence extending above the apex of the perithecium; the remaining cells, except the basal, cutting off a relatively large cell distally and externally; all of which, like the lower half of the small, stout, abruptly upcurved appendages, are nearly or quite opaque, giving the otherwise hyaline body a fringed appearance: posterior series similar to the anterior, consisting of eleven or twelve cells, the basal and distal without appendages: median series consisting of eight or nine successively smaller cells; the lowest slightly intruded between the third pair of marginal cells; the free base of the primary appendage placed between the two terminal cells of the posterior and median series, its basal cell short and broad, its distal nearly twice as long, and narrower; the appendage short, erect, normal. Perithecium relatively long and narrow; its posterior margin somewhat more than one half free, symmetrically somewhat convex; the tip not clearly distinguished, partly free above the anterior series; the apex bluntly rounded, bent sidewise; a small stalk-cell clearly distinguished. Perithecium 38 × 10 $\mu$. Appendages 6 × 2.5 $\mu$. Total length 80 × 24 $\mu$.

On the tip of the abdomen of *Coproporus latus* Motsch. No. 2381, Mindanao, P. I.

Three specimens of this species have been examined in which the spores are not fully matured in the asci. It is most nearly related to the two preceding species, especially to *R. introversa*, from which it is distinguished by the finger-like termination of its anterior cell-series, its differently shaped perithecium, more numerous cells, etc. More fully matured perithecia may show some further modification of the tip.

### **Rickia inclusa** nov. sp.

Nearly symmetrical, except for the prominence of the distal cells of the anterior series; spathulate in habit. Receptacle subhyaline, except the lower appendiculate cells, triseriate, the basal cell short and stout: anterior series consisting of about twenty-five cells all, except the three terminal ones, cutting off distally and externally single triangular cells; the lower of which are flatter, broader and more sharply pointed inward, becoming deeply suffused, contrasting, and not differentiated from the bases of the appendages; the cell just below the base of the perithecium larger than the rest; the thirteen cells above it in contact with the perithecium, subequal, the two subterminal cells equal, flattened, obliquely tilted; the terminal one pointed inward distally and externally, forming a rounded free prominence: posterior series resembling the anterior in general, consisting of twenty subequal and subtriangular cells, the last cell cutting off a relatively large basal cell which, together with its appendage, is bent inward against the base of the primary appendage, the lowest cell of which is rounded and somewhat larger than the uppermost cell of the median series which it terminates: median series consisting of twenty cells, the six lying below the perithecium larger, irregularly rounded and subequal, the rest in contact with the perithecium, much smaller, rounded and subequal. Appendages small and appressed, blackish brown externally, bent upward at the more deeply suffused base; the cells which bear them, from below up to the base of the perithecium on the anterior side and to its middle on the posterior, also opaque and contrasting. Perithecia a little anterior in position, straight, erect, nearly symmetrical, fusoid-elliptical, broadest in the middle, completely enclosed by the receptacle, except the partly free short broad, erect tip; the apex bilobed, the large lobes unequal, somewhat asymmetrically and laterally placed; a minute subtending stalk-cell distinguished. Spores copious, about 28 × 2.5 μ. Perithecium 77 × 27 μ. Total length to tip of perithecium about 140 × 50 μ.

On legs of *Coproporus latus* Motsch, No. 2380, Mindanao, P. I.

This species is most nearly related to the two species occurring on the same host (*R. introversa* and *R. nigrofimbriata*), but is at once distinguished by its almost wholly included perithecium, as well as by other differences. Two specimens, only, one of them fully mature, have been examined. No antheridia are recognizable in either individual.

### **Rickia circumdata** nov. sp.

Long-spathulate, nearly symmetrical, quite hyaline or more or less deeply and completely suffused with blackish brown, darkest at the base, and hyaline above; sometimes opaque below, and becoming gradually hyaline distally. Receptacle triseriate, subdeterminate, (in one instance biseriate), the basal cell always quite hyaline, contrasting: anterior series consisting of from about twenty to twenty-five cells, the lowest and uppermost smaller, the rest subequal when suffused, and much flattened, with straight walls, but tending to be piriform when hyaline; each cell cutting off two to three flattened triangular superposed cells, or the lower only one such cell, all of which bear appendages or antheridia: posterior series similar to the anterior and symmetrical with it, ending in the small lower cell of the base of the primary appendage, from which the free small upper narrow cell projects outward, parallel to the other appendages: median series consisting of about the same number of cells, which decrease in size from the mid-region upward and downward; those below the perithecium squarish or subrectangular; suffused, or more rounded in hyaline individuals; those in contact with the perithecium becoming much smaller upward; the lowest cell hardly intruded between the second pair of marginal cells. Perithecium slightly lateral in position, erect or slightly tilted, subfusiform, subsymmetrical, broadest just below the middle; becoming deeply suffused with brown, except in hyaline individuals; the tip distally compressed, not distinguished, usually turned sidewise so as to be free, the anterior and posterior series sometimes almost or quite meeting behind it; distally hyaline, compressed; the bluntly rounded or truncate apex always quite free. Appendages hyaline, subcylindrical, of the ordinary type, slightly longer than the numerous hyaline antheridia. The latter tapering, without abruptly defined necks. Spores about 28 × 3 μ. Perithecia 54–64 × 20 μ. Appendages 12 × 3 μ. Antheridia 10 × 3.5 μ. Total length to tip of perithecium 130–175 × 50–55 μ.

On *Episcaphula piciventris* Gorh. No. 2566, and *Episcaphula* spp. Nos. 2334, 2563 and 2566, Kamerun.

A species which in general form recalls *R. inclusa*, its perithecium being similarly surrounded by the receptacle. It varies from quite hyaline individuals to others which are more or less completely suffused and quite opaque below; the distal margin being, however, always hyaline. In one specimen examined the median series has failed to develop, producing an otherwise normal biseriate individual.

## **Rickia Papuana** nov. sp.

Broadly spathulate, nearly symmetrical, tapering to a slender basal cell. Receptacle triseriate, hyaline: anterior series extending to the tip of the perithecium; consisting of twenty-five cells, more or less, all except the lowest much flattened; those beside the perithecium radiately arranged, and becoming gradually smaller distally; the lowest one or two cells cutting off one or two, the rest five or six small cells, which lie side by side horizontally, producing very numerous antheridia and scanty appendages: posterior row similar to the anterior and subsymmetrical with it; its basal cell much larger, consisting of about the same number of cells, ending in the base of the primary appendage, the two cells of which are separated by a very oblique septum, the lower thus greatly overlapping the upper on its inner side; its base broad, convex and included: median series consisting of about eleven cells, the lowest and those beside the perithecium smaller. Antheridia very numerous, hyaline, appressed; the necks well distinguished, purplish, the hyaline tips curved outward. Appendages hyaline, subcylindrical, appressed. Perithecium completely surrounded by the receptacle, a portion, only, of the tip free, subfusiform and subsymmetrical, the tip distally compressed and subcylindrical, partly free; the apex bluntly rounded. Perithecia about 75 × 24 μ. Antheridia 12 × 4 μ. Appendages 20 × 4 μ. Total length to tip of perithecium 235 × 90 μ.

On *Catops* sp., British New Guinea, No. 1840.

The hosts on which this peculiar form occurred were obtained for me in New Guinea by Mr. Muir to whose kindness I am much indebted. Six individuals have been examined, in only one of which is the perithecium nearly mature, although no spores are visible. It is probable that the dimensions of the perithecium of more fully developed individuals will prove greater than those given above, and that its erect, free finger-like tip may become more prominent with age. The first appendiculate cell which is cut off, appears to bear an appendage; while all, or a great majority of those cut off beside it, which lie in a subhorizontal series, bear antheridia. Owing to the fact that the antheridia are thus crowded side by side, it is very difficult to estimate the number developed in connection with a given marginal cell, but there appear to be from four to six. The extremely oblique septation of the base of the primary appendage, is much more striking than in any of the several species where a slight obliquity is observable.

## **Rickia pallida** nov. sp.

Hyaline, tapering symmetrically downward from the mid-region of the perithecium to the foot. Receptacle triseriate, the basal cell rather small, subtriangular, not distinguished from and slightly if at all intruded between the two cells above it: anterior series extending to, or almost to, the base of the perithecial tip, consisting of usually eighteen cells; the upper ten usually in contact with the perithecium, the distal ones smaller, those below the perithecium subequal, irregularly three-sided except the lowest; all the cells of the series, except the distal and basal, cutting off from one to five relatively small obliquely superposed cells bearing appendages or antheridia; the bases of which, crowded in oblique rows, are visible only on the left face, that is only when the anterior series lies at the left: posterior series consisting of about sixteen cells, similar to the anterior series, the basal cell also separating an appendiculate cell: median series consisting of more often fifteen cells, the four below the perithecium larger, irregularly squarish to hexagonal; those next the perithecium extending to the base of its tip, two or three of the distal ones extending beyond the base of the primary appendage, which is inconspicuous and projects somewhat obliquely from the left face; four or five successive distal cells of the series, usually not including the terminal one, cutting off one, or the lower two, appendiculate cells; the appendages, or their bases, lying on the right face. Antheridia numerous, hyaline, tapering almost uniformly to the blunt apex; the neck but slightly distinguished, straight or slightly curved. Appendages longer and larger than the antheridia, cylindrical or somewhat swollen, hyaline, becoming faintly yellowish. Perithecia almost symmetrically subfusiform, subtended by a well defined irregularly triangular stalk-cell, the tip wholly free or its base partly enclosed on one side, conical, its margins evenly continuous with those of the distal cells of the anterior and median series; the apex rather broad, round or truncate, sometimes with minute erect lateral projection. Spores about 42 × 4 μ. Perithecia 100–130 × 30–40 μ. Antheridia 15 × 4 μ. Appendages 15–22 × 5–6 μ. Total length to tip of perithecium 200–275 × 60–85 μ.

On *Amblyscelis* sp., Nos. 2565 and 2571, Kamerun.

A rather large pale species, somewhat similar to *R. Papuana* and *R. circumdata* in general appearance, but distinguished by numerous points of difference; for example in the production of appendiculate cells from the distal cells of the median series. In a number of specimens certain appendiculate cells, irregularly scattered, may become blackened abnormally, and other cells may occasionally be similarly modified.

### **Rickia Ancylopi** nov. sp.

Nearly symmetrical, wholly suffused with dirty brown, except the hyaline contrasting appendiculate cells; darker or opaque in the region of the perithecium. Receptacle triseriate, the foot large broad and rounded, the basal cell evenly suffused with pale dirty brown, its pointed distal third or more intruded between the basal cells of the two marginal series: anterior series extending slightly further than the median series, consisting of usually fifteen obliquely superposed cells which are subequal and more or less deeply suffused, except the three uppermost; the latter nearly hyaline and small; all except one or two of the uppermost cutting off externally and distally one or two quite hyaline cells, which bear normal antheridia or appendages: posterior series similar to the anterior, its basal cell somewhat larger, and usually cutting off a single appendiculate cell; the three terminal cells smaller and hyaline; the two upper usually not associated with appendages; the well developed and wholly free base of the primary appendage, the two cells of which are subequal and separated by a slightly oblique septum, projecting outward above the terminal cell: median series consisting of usually fourteen cells, the five or six lying below the base of the perithecium subequal and each of about the same diameter throughout its length, except the uppermost, which is shorter broader and more deeply suffused; the group forming a straight jointed axis, somewhat constricted and darker at the septa; the cells beside the perithecium hardly distinguishable from it in the general suffusion of this region, the distal cell lying free beyond the base of the primary appendage, but not reaching higher than the base of the perithecial tip. Appendages mostly ovoid, shorter than the numerous somewhat divergent, hyaline, rather stout antheridia; the well distinguished, stout, purplish necks of which, are usually curved slightly outward. Perithecia nearly symmetrical, not quite median, erect or very slightly tilted, becoming deeply suffused with reddish brown; the margins nearly straight, hardly distinguished from the adjacent marginal cells; the tip well distinguished, very short and broad, distally blackened below the broad flat contrasting hyaline apex. Spores about 28 × 3 μ. Perithecia 60–66 × 12–18 μ. Appendages 54 × 3.6 μ. Antheridia 11 × 4 μ. Total length to tip of perithecium 145–160 × 32–36 μ.

On the elytra of *Ancylopus bisignatus* Gerst, No. 2562, Kamerun.

A species very readily distinguished by the peculiar appearance of the portion of the receptacle lying below the perithecium, where the median cell-series forms a clearly defined suffused bamboo-like axis, to the ends of the segments of which the suffused or blackened cells of the marginal series are sometimes, though not always, adjusted in almost perfectly symmetrical pairs, having the appearance of broad dark appendages projecting outward from them and ending in a more deeply blackened tooth-like termination which subtends the slight convexity on which the appendiculate cell rests. Like other suffused forms, this species recalls in its general appearance that of some forms of *Rhachomyces*.

### **Rickia Episcaphae** nov. sp.

Receptacle triseriate, symmetrical below the perithecium and of equal width almost to the basal pair of marginal cells, whence it tapers rather abruptly to the foot; the basal cell hyaline, somewhat broader distally, and slightly intruded between the two cells above it; which are similar, symmetrically paired, and subtriangular, more or less completely and deeply suffused with reddish brown, the suffusion often involving to some extent the cells immediately above: median series consisting of from twelve to sixteen cells; the fourteen to sixteen lying below the perithecium very regular in form and size, somewhat longer than broad, subrectangular with rounded angles, their walls, especially, more or less faintly suffused with dirty brownish; the five or six distal ones, lying beside the lower third of the perithecium, much smaller and subequal: anterior series consisting of about twenty-five to twenty-nine cells, those below the perithecium subequal, subtriangular, hyaline or faintly and irregularly suffused inwardly with dirty yellowish brown; the six to eight distal cells evenly suffused with yellowish brown; small, rounded; the upper somewhat smaller and extending to about two thirds or three quarters of the total length of the perithecium; all the cells, except two or three of the terminal ones, cutting off one to three superposed cells, which bear copious appendages and scanty antheridia: posterior series similar to the anterior, ending beside the distal cell of the median series, the basal cell of the primary appendage lying just above their terminations, in contact with the perithecium, and not distinguished from the cells below, its distal cell short, broad, subtriangular, free, diverging laterally. Antheridia very scanty, usually only one or two on the anterior series, opposite the base of the perithecium; about as long as the hyaline, stout, rather irregular appendages, purplish; the neck more deeply colored, long, slender, and abruptly distinguished. Perithecia usually erect and straight, the insertion slightly lateral, rich purplish brown, the surface more or less distinctly mottled; one half or more of the posterior, and one fourth of the anterior margin free; the enclosed portion narrower; the tip broadly conical, the margins slightly convex, short, abruptly distinguished, subtended by a paler or almost hyaline line; the apex flat and broad, hyaline, the lip-cells projecting slightly. Spore about $50 \times 5\ \mu$. Perithecia $80–100 \times 27–30\ \mu$. Antheridia $16 \times 4\ \mu$. Appendages $10–20 \times 2\ \mu$. Total length to tip of perithecium $310–425\ \mu$.

On *Episcapha antennata*, Nos. 2391 and 2392, Mindanao, Philippines.

Numerous individuals have been examined of this large and handsome species which is most nearly related to *R. Coptengalis*, from which it differs in the more highly developed receptacle, the relation of its perithecium to the latter, and the form of its tip, as well as in the very scantily developed antheridia.

### **Rickia Eumorphi** nov. sp.

Straight or slightly curved, hyaline and rich brown, contrasting; large, long, broader in the region of the perithecium, below which it tapers very gradually, or hardly at all, to the basal cells of the marginal series. Receptacle indeterminate, triseriate; the basal cell hyaline, abruptly narrower and intruded somewhat between the two nearly opaque cells above it, the deep suffusion of which may extend downward beside its extremity in hooked prolongations: anterior series wholly or partly involved by the opacity which makes the structure of the median region quite indistinguishable, the distal cells beside the perithecium usually hyaline or translucent; consisting of from thirty-four to forty-eight similar cells, and extending to, or somewhat above, the base of the perithecial tip; all except the small terminal one cutting off from one to usually not more than three small hyaline cells, bearing antheridia or appendages: posterior series similar to the anterior and consisting of about the same number of cells, terminating somewhat lower down than the anterior; its distal cell lying beside that of the median series, the two bearing distally between them the free relatively long and narrow base of the primary appendage, the somewhat larger lower cell of which is separated from the upper by a slightly oblique septum: median series wholly opaque and indistinguishable, except four or five of its distal cells; the three or four terminal ones being hyaline. Antheridia numerous, conspicuous, the cylindrical, purplish brown, usually slightly curved necks often longer than the slightly inflated venter. Appendages often shorter, rarely longer, than the antheridia, hyaline, subcylindrical. Perithecia rather long and narrow, sometimes slightly tilted, the basal portion suffused with purplish brown to somewhat above the middle, deeper on the posterior side, the distal portion tapering slightly and subsymmetrically to the coarse blunt apex; its lower half, or less, abruptly hyaline or nearly so; its upper half or more abruptly and deeply suffused, the suffusion extending lower than the short broad tip proper, which is not distinguished except by its lateral deeper suffusion; a rounded translucent brownish area below and including the median pore. Spores about 40 × 3.6 $\mu$. Perithecia 90–100 × 20–24 $\mu$. Antheridia 20 × 4 $\mu$. Appendages 80–20 × 4 $\mu$. Total length to tip of perithecium 230–400 × 40–55 $\mu$.

On *Eumorphus cyanescens* Gerst., No. 2390, Mindanao, Philippines.

A fine and very striking species, most nearly allied to *R. Coptengalis* and *R. Episcaphae*, and resembling *R. Berlesiana* in its axial suffusion, which is much more extensive than in this species.

### **Rickia nigrescens** nov. sp.

Nearly symmetrical, tapering slightly below to a rather slender base, more or less deeply suffused with blackish brown. Receptacle triseriate, the basal cell nearly hyaline below, distally somewhat broader, deeply suffused, and hardly distinguishable from the two short paired basal cells of the marginal series, which become quite opaque: anterior series consisting of about fourteen suffused cells, not differing greatly in size, those in contact with the perithecium slightly smaller, all but the opaque basal one cutting off distally and externally one, or two, small nearly hyaline cells, in the latter case lying side by side and not superposed, which bear appendages or antheridia; the latter mostly from the mid-region: posterior series similar to the anterior, extending about as far, and ending in the base of the primary appendage: median series hardly intruded between the second pair of marginal cells, consisting of usually twelve cells, the upper somewhat smaller; the cells of all the series more or less similar, rather small, subisodiametric, or but slightly longer than broad, arranged nearly symmetrically in tiers, of which there are usually nine or ten between the basal cell and the base of the perithecium which is subtended by a well defined stalk-cell. Perithecium slightly more than half free on both sides, deeply suffused with olive brown; straight, erect, nearly symmetrical; the distal half conical; the tip hardly distinguished; the apex hyaline small, rounded. Antheridia somewhat appressed, straight, or the purplish neck slightly curved outward. Appendages hyaline, stout, cylindrical, becoming somewhat gelatinous and coherent, soon disappearing. Spores about 35 × 2.5 μ. Perithecia 70–78 × 24 μ. Appendages 20 × 4.5 μ. Antheridia 12 × 4 μ. Total length to tip of perithecium 150–190 × 32–35 μ.

On the elytra of *Coproporus hypocyploides* Bernh., No. 1830, Sarawak, Borneo; No. 2416, Manila, Philippines.

A deeply suffused species which at first sight suggests a form of *Rhachomyces,* the receptacle above the basal cell being rather slender and about the same width nearly to the base of the perithecium. It is closely allied to the following species from which it differs in color, in the greater number of tiers below its perithecium, the more regular arrangement of its cells, and also in the almost sharply conical free portion of the perithecium.

### **Rickia pallescens** nov. sp.

Erect, straight, somewhat soiled or stained with brownish above. Receptacle triseriate, basal cell rather large, distally slightly broader and not intruded between the basal cells of the marginal series; which are similar to the basal cell, or the posterior slightly longer, and subsymmetrically paired as are the two shorter broader marginal cells immediately above them: anterior series consisting of usually twelve superposed cells becoming smaller distally, and in contact with somewhat more than half the anterior margin of the perithecium; all the cells, except the lowest cutting off, the subbasal one, the rest several small appendiculate cells distally and externally; which, in the latter case, lie side by side; those of the mid-region bearing for the most part antheridia, the rest mostly minute appendages: posterior series similar to the anterior and extending nearly as far upward, terminating in the primary appendage which, with its base, is hardly distinguishable from the secondary ones associated with it: median series consisting of eight or nine cells, the three lowest larger, the rest becoming gradually smaller; the distal or subdistal usually cutting off an appendiculate cell or bearing an appendage directly; the series not extending quite as far upward as the other two. Antheridia numerous and typical, rather large, appressed or somewhat divergent, asymmetrical; the necks somewhat tapering and tinged with dirty purplish brown, more or less distinctly curved outward. Perithecium erect, straight, deeply suffused with dirty olive brown; rather long and narrow, free on both sides from just below the tip; which is abruptly distinguished, straight and stout, the apex slightly distinguished, subhyaline, broad, flat, symmetrically rounded. Spores about 30 × 2.8 μ. Perithecia 68–75 × 18–22 μ. Antheridia 15 × 4 μ. Total length to tip of perithecium 150–200 × 35–45 μ.

On the legs of a small species of *Coproporus*, No. 2543, Manila, P. I.

Closely allied to *R. nigrescens* of which it may prove only a variety. It differs in its nearly hyaline receptacle and pale perithecium which is more completely surrounded by the former: the tip and apex broader and differently shaped.

### **Rickia Circopis** nov. sp.

Straight, erect, slightly asymmetrical. Receptacle triseriate, hyaline; basal cell rather short, distally broader, hardly intruded between the somewhat larger and longer basal cells of the two marginal series: anterior series extending slightly higher than the two others, consisting of usually ten successively smaller cells, or the four lower subequal, the lowest smaller than that of the posterior series, all the rest cutting off one, sometimes two superposed small cells distally and externally, which bear either appendages or antheridia: posterior series similar to the anterior, consisting usually of nine cells surmounted by the small base of the primary appendage, the upper cell of which is free: median series consisting normally of eight cells, the lowest lying wholly above the second pair of marginal cells, the second somewhat larger. Appendages very small, rounded, or but slightly longer than broad, antheridia normal, hyaline, curved outward, the necks rather stout and not abruptly distinguished. Perithecia pale straw-colored, nearly symmetrical, straight, erect, somewhat less than half free on both sides; the tip stout, abruptly distinguished, tapering to the minute papillate apex; the base subtended by a well defined stalk-cell. Perithecia 60–80 × 20–24 $\mu$. Appendages 3.5–5.5 × 3.5 $\mu$. Antheridia 12 × 3.5 $\mu$. Total length to tip of perithecium 135–155 × 40–45 $\mu$.

On the inferior thorax of *Circopes Philippinensis* Grouv., No. 2274, Kamerun.

Somewhat similar in general appearance to *R. pallescens* but distinguished by the different relation of its appendiculate cells, the form color and relations of it perithecia, as well as in other respects.

### **Rickia Episcaphulae** nov. sp.

Hyaline, asymmetrical, subsigmoid. Receptacle triseriate; basal cell large, longer than the cells next above it, between which it is but slightly intruded: anterior series consisting of about thirteen cells of irregular size and outline; the lower longer; the middle broader and shorter; the six or seven distal cells which lie beside the perithecium, becoming smaller, irregular and broader in proportion to their length; the lower three usually without appendages; the rest cutting off single relatively large cells, which bear antheridia or appendages: posterior series similar, consisting of nine or ten cells; the basal longer than that of the anterior, and separating an appendiculate cell, as do all the others, except the second, third and last; which, with the corresponding cell of the median series, subtends the base of the primary appendage: median series consisting of nine or ten cells, similar in general to those of the other series; the lowest not intruded between the second pair of marginal cells; the four or five distal ones smaller, and lying beside the perithecium. Distal cell of the base of the primary appendage free, divergent, short and broad; longer than the somewhat flattened basal cell. Appendages short and stout, somewhat inflated. Antheridia large and slightly curved, with relatively short stout necks. Perithecia rather narrow, curved toward the posterior side and of nearly uniform diameter; the tip, only, free, except at its abruptly spreading base; the apex not distinguished, broadly rounded. Perithecium 50 × 12–14 $\mu$. Appendages 10 × 4 $\mu$. Antheridia 12–16 × 4–5 $\mu$. Total length to tip of perithecium 160–180 × 36–40 $\mu$.

On *Episcaphula* sp., No. 2446, Kamerun.

Two specimens, only, have been examined, neither of them in very good condition. The species seems well distinguished, however, from its subsigmoid habit, and the protruding finger-like tip of its perithecium.

## **Rickia Saulae** nov. sp.

Receptacle triseriate, hyaline, becoming suffused with dirty brownish distally; basal cell abruptly bent, short and stout: anterior series consisting of eight cells extending to or slightly beyond the middle of the perithecium, the second and third more than twice as long as broad, all somewhat rounded, becoming smaller distally, the upper three in contact with the perithecium and overlapping it so as to be hardly visible externally, all cutting off small cells distally and externally, which bear appendages or normal antheridia; one to three of the middle members cutting off two such cells, which are asymmetrically related, lying side by side or partly superposed: posterior series similar to the anterior in number, and extending to about the same point on the perithecium; the distal cells so placed that they do not appear externally, lying obliquely opposite the corresponding cells of the anterior series; the primary appendage and its base not clearly distinguishable: median series consisting of only four visible cells, three lying below the perithecium, the two lower much longer, the lowest intruded between the second pair of marginal cells nearly to their bases. Perithecium relatively large, nearly as broad as the receptacle, becoming rather deeply and somewhat unevenly suffused with brown, the tip and base paler; median, slightly tilted inward, nearly symmetrical; subtended by a distinct stalk-cell; stout, the margins more or less continuously curved to the broad blunt apex;

neither the tip nor the apex definitely distinguished. Antheridia scanty, with relatively long straight slender necks. Spores about 28 × 3 μ. Perithecia 60–65 × 25–28 μ. Total length to tip of perithecium about 150 μ. Antheridia 16 × 4 μ.

On *Saula* sp., No. 2387, Mindanao, P. I.

A species distinguished by its large, broad, deep yellowish brown perithecium, which is almost as broad as the receptacle below it; its posterior margin appearing to be wholly free, owing to the fact that the median and posterior series extend upward on the lateral (left) surface, and lie wholly within its margin, when it is viewed sidewise.

## **Rickia Phalacri** nov. sp.

Long, straight, erect. Receptacle triseriate, tapering very gradually to the base, with a faint axial brownish suffusion; the basal cell small and more or less suffused with purplish brown; the two lower cells of the marginal series small, somewhat unequal; all the series consisting of usually twelve cells each, those of the median series mostly subrectangular; the cells between the basal pair and the base of the perithecium arranged in oblique tiers of three cells each: the cells of the marginal series, with the exception, usually, of the anterior one, cutting off relatively large cells bearing antheridia or appendages; the two small uppermost cells of the anterior, and of the median series, in lateral contact with the very base of the perithecium; the posterior series terminating in the rather conspicuous slightly divergent base of the primary appendage. Antheridia and appendages somewhat appressed, the former rather scanty, convex on the inner side, and slightly curved outward. Perithecium laterally placed above the posterior and median cell-series, more or less evenly suffused with rich purplish brown; straight, erect, relatively long and narrow, nearly symmetrical, subtended by a well defined stalk-cell; the tip well distinguished, short broad, tapering to a rather broad hyaline, subtruncate, symmetrical apex. Spores about 28 × 2.5 μ. Perithecia 58–65 × 16–18 μ. Appendages 9–11 × 3.5 μ. Total length to tip of perithecium 135–175 × 18–20 μ.

On the elytra of *Phalacrus* sp., No. 2495, Manila, P. I.

A pretty species well distinguished by its usually straight, rather elongate receptacle, on which the rich brown perithecium is inserted a little at one side.

## **Rickia Sarawakensis** nov. sp.

Asymmetrical. Receptacle hyaline, triseriate, both the foot and the basal cell relatively very large, the latter geniculate below, distally broader and rounded: anterior series consisting of two superposed cells, the upper smaller and separated from the base of the perithecium by a small well defined stalk-cell: both cutting off two cells distally and externally, which bear antheridia or appendages of the normal type: posterior series consisting of four cells, which may be subequal; all cutting off two cells which bear antheridia or appendages; the uppermost corresponding to the lower cell of the base of the primary appendage, larger, almost wholly free beside the base of the perithecium, cutting off an appendiculate cell externally and followed distally by the upper cell which is bent inward toward the perithecium: median series consisting of two superposed cells, the lower larger, but slightly intruded between the second pair of marginal cells; the upper very small, lying just beside the base of the ascigerus cavity. Perithecium wholly free, its insertion slightly oblique, rather short and stout, rosy brown, usually subsigmoid; the tip, which is not at all distinguished, bent outward slightly; the small hyaline truncate apex bent abruptly upward or inward. Spores about 28 × 2.8 $\mu$. Perithecia 45 × 19 $\mu$. Appendages 10 × 4 $\mu$. Antheridia 10 × 3.5 $\mu$. Total length to tip of perithecium 95 × 10–12 $\mu$.

On elytra of *Phalacrus* (?) sp., Sarawak, Borneo, No. 2371.

A small and simple species clearly distinguished by its free rosy brown perithecium. It is one of the few species in which the lower cell of the base of the primary appendage cuts off an appendiculate cell. The body of the receptacle consists of an unusually small number of cells, for a species of this normal type, which do not in general differ greatly in size, being mostly somewhat longer than broad and rather irregularly rounded in outline.

## **Rickia Parasiti** nov. sp.

Hyaline, relatively long and narrow in general habit. Receptacle triseriate, the basal cell relatively long and slender, but slightly broader at the apex which is only slightly if at all intruded between the cells above it: anterior series consisting of three, or usually four, superposed cells; the lowest larger, twice as long as broad; the others successively smaller or subequal; each, or the two upper, only, cutting off distally and externally one, rarely two, superposed small cells which bear either short appendages, which are subtended by a blackened septum; or subulate sessile antheridia, not distinguished either by a constriction or a blackened septum at the base; the upper cell and its antheridia in oblique contact with the base of the perithecium: median series consisting of eight or more, often nine, cells; the lowest larger, lying below the base of the perithecium, the rest united to its inner margin; the distal one often somewhat larger, the rest rather small, becoming subequal and rounded: posterior series nearly straight or but slightly convex externally, consisting of eight or more, often nine, superposed cells, some or all of which may cut off distally and externally small appendiculate cells bearing subcylindrical usually appressed appendages; the lowest cell somewhat longer than the corresponding cell of the anterior series, and subtriangular in form; the rest subequal or slightly smaller distally, the series ending in the two-celled base of the primary appendage which may be obliquely or almost horizontally placed, and is thus variably divergent; its upper cell very small, its lower nearly uniform with the other cells of the series, its base opposite the distal cell of the median series. Perithecium erect, tapering from a rather broad base, the outer margin free, slightly concave, the tip slightly distinguished; the apex, only, free on the inner side, slightly bent inward, distally broad, slightly sulcate. Spores about 25 × 2.5 μ. Perithecia 45–60 × 18–20 μ. Appendages 9–11 × 3 μ. Basal cell 35 × 9 μ. Total length to tip of perithecium 90–125 × 26–36 μ.

On *Parasitus* sp., No. 2796, Mexico, M. C. Z. (Mann).

Although very different in general appearance, this species is most nearly allied to *R. minuta;* which is, however, very readily distinguished by the crest-like curvature of its posterior cell-series and its bladder-like appendages, as well as by other points of difference.

### **Rickia Gryllotalpae** nov. sp.

Hyaline, rather elongate, strongly or slightly sinuous, tapering gradually below to the foot. Receptacle triseriate, the basal cell relatively long, slightly intruded distally, slightly and abruptly swollen above the foot: anterior series consisting of ten to eleven subequal cells, somewhat longer than broad, which cut off one or more, often two, almost vertically superposed cells bearing appendages; while the terminal, and sometimes the subterminal, bear single pointed antheridia, the upper more or less appressed externally beside the base of the perithecium: posterior series similar to the anterior, consisting of eleven or twelve cells all cutting off cells which bear appendages only; the series surmounted by the free base of the primary appendage which diverges almost at right angles, its basal cell cutting off a secondary appendiculate cell distally on the lower, or sometimes on the upper side: median series consisting of usually twelve cells, the upper four or five extending beside the base of the perithecium and beyond the insertion of the base of the primary appendage; the basal cell longest, and intruded between the third pair of marginal cells. Appendages strictly marginal, of the usual type; becoming faintly tinged with yellowish brown, stout, rather short, usually broader above the constricted base: the primary appendage usually more or less persistent and smaller. Antheridia usually single, quite hyaline, spine-like, without differentiation between the neck and venter, or the usual constricted and suffused base. Perithecium almost free; its inner margin, only, in contact for a short distance with the terminal cells of the median series; erect, or tilted slightly inward, straight, somewhat asymmetrical; the outer margin more convex; the tip more or less clearly distinguished, tapering subsymmetrically to the rather blunt truncate apex. Spores about 35 × 3 μ. Perithecia 75–85 × 24–27 μ. Antheridia 15 μ. Appendages 12–20 × 7.5 μ. Total length to tip of perithecium 230–325 × 28 μ.

On the wing tips of *Gryllotalpa* sp., No. 2155, U. S. Nat. Mus.; Africa, (Mearns).

Several specimens have been examined all in good condition. The antheridia are similar in type to those of *R. Lycopodinae* and *R. minutus* and occur in the same position as in the last mentioned species. The production of one or more secondary appendiculate cells from the base of the primary appendage, is unusual.

### **Rickia Lycopodinae** nov. sp.

Rather broad distally, tapering below; the hyaline margins contrasting with the deep black-brown or opaque axis and perithecium. Receptacle triseriate; the foot round, and as large as the thick-walled basal cell, which is distally intruded so as nearly to separate the two basal cells of the marginal series: anterior series consisting of about twelve to fifteen cells, mostly short broad and obliquely superposed;

all, with a few irregularly distributed exceptions, cutting off distally and externally single cells which may be even larger in size, and bear either antheridia or appendages; the terminal cell broad, flattened, often suffused, and forming a straight, very oblique, clearly defined base to the perithecium: posterior series consisting of from about sixteen to twenty cells; similar to the anterior, and terminating in the basal cell of the base of the primary appendage, which lies about opposite the middle of the perithecium; the cells of both marginal series often tending to become vertically divided: median series consisting of about fourteen to eighteen cells which become deeply suffused with blackish brown, the suffusion becoming opaque, involving to some extent the adjacent cells and continuous with that of the perithecium; the cells rather regular and longer than broad, except the six or seven distal ones which lie beside the perithecium, extending to the base of its tip; about three of the terminal cells externally free beyond the base of the primary appendage; the free upper subconical cell of which is larger than the basal cell and diverges laterally. Appendages thick-walled, inflated, straight and stout, somewhat irregular. The antheridia numerous, sometimes more so than the appendages. Perithecia more or less opaque, asymmetrical, short and stout; the base straight and oblique; the tip well distinguished, hyaline or paler, bent abruptly sidewise so that it is partly concealed in side view, tapering to the broadly rounded undifferentiated apex. Perithecia 55 × 24–27 μ. Free portion of antheridia 15–20 μ. Appendages mostly 7–10 × 5.5 μ. Total length to tip of perithecium 125–156 × 40–45 μ.

On legs of *Lycopodina* sp.; M. C. Z., No. 2801, Madagascar (Wulsin).

A striking species, remarkable for the fact that its numerous antheridia belong to the type present in *R. minuta* and several other species, being partly immersed and without the usual characteristic blackened insertion. The general habit, however, is rather that of the more typical forms of the genus, although it is somewhat anomalous in its tendency to show abnormal vertical divisions in its marginal cells, which may be smaller than the appendiculate cells separated from them above.

### **Rickia Zirophori** nov. sp.

Receptacle triseriate, of about the same diameter from the base of the perithecium to that of the median cell; uniformly hyaline to pale yellowish, usually elongate, straight or sometimes variously bent. Basal cell very small, sometimes almost obliterated by the intrusion of the basal cell of the anterior series: anterior series consisting of two elongate cells; the upper shorter and separated below by a septum running obliquely inward and downward, cutting off distally a marginal series of four to six closely associated appendiculate cells; the lower slightly prominent distally, just below the septum, and sometimes cutting off one or two appendiculate cells at this point: posterior series similar, the upper cell shorter than that of the anterior, its margin often entirely occupied by a marginal row of small closely associated appendiculate cells, the uppermost usually lying between the perithecium and the base of the primary appendage, which is small and short, its basal cell slightly intruded and much smaller than the somewhat rounded distal cell: median series consisting of a very small cell lying beside the base of the perithecium and a greatly elongated cell which extends to within a short distance of the base of the marginal cells which are in contact below it and are of about the same uniform diameter throughout. Appendages of the usual type, cylindrical or slightly tapering. Antheridia rarely developed, the antheridial cells apparently becoming free in small groups. Perithecia almost wholly free, long, narrower toward the base, the margins slightly convex and nearly symmetrical, nearly erect or slightly divergent outward, the base oblique, the tip more or less distinguished, tapering considerably, and usually symmetrically, to the rather narrow bluntly rounded apex. Spores about 35 × 3.5 $\mu$. Perithecia 116–136 × 27 $\mu$. Appendages 40 × 4 $\mu$. Total length to tip of perithecium 300–400 × 34–40 $\mu$.

On the legs and inferior surface of *Zirophorus* sp., No. 2802, vicinity of Port of Spain, and at Arima, Trinidad, B. W. I.

This species is a third well marked representative of the section to which *R. marginata* and *R. Lispini* belong, and is very nearly allied to the latter, from which it differs in its much greater size and elongate habit, and differently shaped practically free perithecium. In a few individuals, in which the perithecium has aborted, the appendages appear to be in some instances replaced by groups of three or four short free antheridia, as in *R. Leptochiri* and some others. Although the host is not uncommon in Trinidad, the parasite appears to be decidedly rare.

### Rickia Lispini Th.

This species, which was first recorded from the Argentine, has now been obtained from the Amazon region, No. 2231; from Guatemala, No. 1625; from California, No. 2757 and from Kamerun, No. 2606. It shows little variation, from these diverse regions, except in size, the largest, measuring 150 $\mu$ in length, being included in the material from the Amazon. The host is in all cases a *Lispinus*, or belongs to a closely allied genus.

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# *THE VELOCITY OF POLYMORPHIC CHANGES BETWEEN SOLIDS.*

BY P. W. BRIDGMAN.

INVESTIGATIONS ON LIGHT AND HEAT MADE AND PUBLISHED WITH AID FROM THE RUMFORD FUND.

# THE VELOCITY OF POLYMORPHIC CHANGES BETWEEN SOLIDS.

By P. W. Bridgman.

Received, October 29, 1915.

In preceding papers [1] I have presented complete data for a number of polymorphic transitions between solids under high pressures. It is well known, however, that data which may be complete from the point of view of thermodynamics may not be at all complete from other points of view, and therefore cannot be complete enough to determine the mechanism of a process. For example, a complete description of the thermodynamic behavior of a perfect gas gives no hold on the viscosity or the thermal conductivity. The kinetic theory of gases, however, which describes the mechanism, does account for viscosity and thermal conductivity as well as for the thermodynamic properties. A knowledge of other properties than those of thermodynamics is important, therefore, because of the additional light it may throw on the complete mechanism. In this paper additional data of this kind, data for the reaction velocity from one phase to another, are given for many of the substances for which the thermodynamic data have been given in preceding papers.

The plan of presentation is as follows. First, is given such description as may be necessary of experimental methods, the method of computation from the data, and the abbreviated graphical method by which an entire curve is represented by a single point. This will involve emphasis of two important facts, namely that there is a distinct region on both sides of the equilibrium point within which the reaction will not run, and that the reaction velocity varies according to the direction in which it is running, being almost always slower in that direction which is accompanied by rising pressure. The data for all the individual substances for which the data are now at hand will then be presented, and finally, the significance of the data and their suggestions as to a possible mechanism will be discussed.

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[1] P. W. Bridgman. (A) Proc. Amer. Acad., 439–558 (1912).
(B) Phys. Rev., 126–141 and 153–202 (1914).
(C) Jour. Amer. Chem. Soc., **36**, 1344–1363 (1914).
(D) Phys. Rev., **6**, 1–33 and 94–112 (1915).
(E) Proc. Amer. Acad., **51**, 55–124 (1915).
(F) Proc. Nat. Acad., **1**, 513–516 (1915).
(G) Proc. Amer. Acad., **51**, 576–625 (1916).

Throughout the rest of this paper reference will be made to these papers by letter.

## METHODS.

The apparatus was the same as that which has already been described and most of the measurements were made at the same time as those of preceding papers. The only measurements made with this subject alone in view were a few on potassium chlorate. To every equilibrium point of the preceding papers there may correspond two velocity curves, giving the rate of reaction with rising and falling pressure. But in many cases it was not practicable to obtain these two curves, especially if the reaction were a rapid one or one accompanied by a small change of volume, so that the data given here are by no means so complete as the equilibrium data.

The following brief description of method applies to readings all at the same temperature. For definiteness we will suppose that we are measuring the rate at which the low pressure phase (I) changes to the phase stable at higher pressures (II). Pressure on I is first slowly increased until it has been carried so far into the region of stability of II that nuclei of II are formed spontaneously. After the formation of nuclei, the phase II grows at the expense of I by an advance of the surface of separation. The formation of II is accompanied by dropping of the pressure back toward the equilibrium line. The reason for this fall of pressure is, of course, that the phase stable at the higher pressure necessarily has the smaller volume. Since the fall of pressure is proportional to the change of volume, and therefore to the amount of I changed into II, the reaction velocity may be simply measured by measuring the rate at which pressure drops back. It is obvious that a precisely similar procedure gives the rate at which II changes to I, the pressure now rising as the reaction runs. These two sets of readings with rising and falling pressure are to be repeated at different temperatures.

For the measurements of time an ordinary watch was used, and the pressure was determined in the usual way by measuring with a Carey Foster bridge the resistance of a coil of manganin wire subjected to the pressure. The resistance was given by the position of the slider on the bridge wire, the setting being made for no galvanometer deflection. It can be well understood that this measurement of pressure might consume a little time. I could not make a measurement of pressure, if all the adjustments had to be made, in much less than 15 seconds, but if I knew approximately what the pressure was, the measurement could be made in considerably less time. It was not possible, therefore, to make accurate measurements when the pressure was rapidly changing, and if the transition were very rapid, as that between ice I and III near the triple point with the liquid,[2] it was not possible to make any measurements at all. On the other hand, if the reaction is slow, it is possible to make the readings as accurately as the time can be read from the second hand of an ordinary watch, that is, to within perhaps one second. This is accomplished by setting the slider at a definite mark and keeping the key pressed down. As the pressure changes with time the galvanometer swings, and the null position is indicated when the cross hair passes the zero mark. Furthermore, by setting the slider at a mark and reading when the galvanometer passes zero, the accuracy of the readings is greatly improved. With a magnifying glass the slider may be set on a division within perhaps 1/50 mm., whereas the position of the slider, if between divisions, cannot be estimated to better than 1/10 mm. The sensitiveness of the galvanometer was such that the mirror magnified the motion of the slider four times. Of course the battery current was chosen so small that the circuit could be closed indefinitely with no perceptible alteration of resistance due to heating by the current.

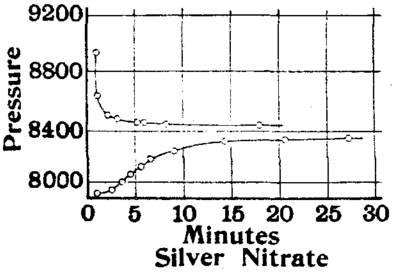

FIGURE 1. The rate at which pressure rises or falls toward equilibrium for Silver Nitrate at 75°.

Such measurements give immediately data for two curves at each of several temperatures, showing pressure as a function of time as it rises from below or drops from above. An example of such a curve is shown in Figure 1, taken from the data for silver nitrate. It is not however, these curves in which we are primarily interested, but the rate at which the transition runs at constant temperature as a function of the distance measured in kilograms from the equilibrium point. The rate at which pressure changes may evidently be found directly from the tangents to curves like those of Figure 1. In practise the readings were made at frequent enough intervals so that the tangents may be replaced by the secants connecting successive observations. From the rate of change of pressure we can obtain immediately the rate of transition, that is, the fractional part of the complete transition per unit time, if we know the total change of pressure corresponding to the complete transition from one phase to the other. The total change of pressure is given directly by the curves from which the change of volume has been determined. Such a curve of reaction velocity from

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[2] A, p. 534.

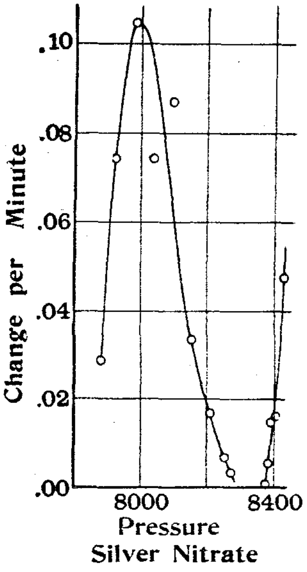

Fig. 2.

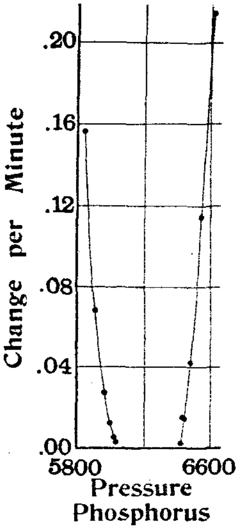

Fig. 3.

FIGURE 2. The slope of the curves of Figure 1, expressed as fractions of the total transition per minute, plotted against pressure. Two of the points on the high pressure branch are beyond the scale of the diagram. The total range of velocity on the high pressure branch is 5000 fold.

FIGURE 3. Transition velocity of White Phosporus as a function of pressure at 0°.

above and below is shown in Figure 2; this was obtained directly from the original curve shown in Figure 1.

Curves like these giving the time rate of transformation as a function of pressure were computed from the data and plotted for all the substances for which the time rate could be measured. A few examples of these for different substances are shown in Figures 3, 4 and 5.

To completely present all the data would mean to give all the curves like these, a pair of curves corresponding to every observed pressure and temperature point of the transition curves. Such a complete presentation would demand a prohibitive amount of space, and would be of little significance, because the precise form of the curves depends on the dimensions of the particular apparatus. There are, however, certain features common to all the curves which will be discussed here. By making abstraction of these essential features, each curve may be represented by a single point. The various representative points for a single substance may then be collected into a single diagram, and curves drawn through them. To every transition curve there corresponds, therefore, a pair of curves giving the main

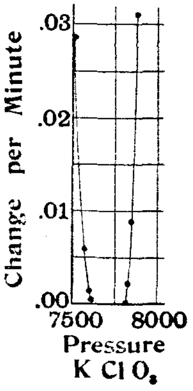

Fig. 4.

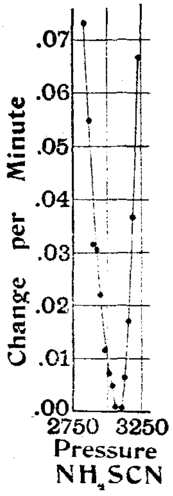

Fig. 5.

FIGURE 4. Transition velocity of Potassium Chlorate as a function of pressure at 200°.

FIGURE 5. Transition velocity of Ammonium Sulfocyanide as a function of pressure at 0°.

features of the transition velocity curves at each point of the transition curve. The following discussion of the transition velocity curve will make plain what the essential features for comparison are, and how it is possible to represent each curve by a single point.

The four transition velocity curves of Figures 2, 3, 4, and 5 are typical of all. These curves are all similar in appearance except for the hook on the rising pressure branch for $AgNO_3$. This hook is connected with the formation of nuclei of the new phase; it is obvious that immediately after formation of the nuclei the growth of the new phase is less rapid than it is after the surface of separation has had a chance to become fully developed. It is probable that all the velocity curves have this hook in the initial stages, but in most cases it was not possible to observe it. In this discussion we confine our attention to the parts of the curve beyond the hook. With this restriction, all four curves are essentially similar. It is in the first place evident that the speed of reaction becomes rapidly greater at pressures increasingly remote from the equilibrium pressure. One could not of course, expect otherwise. This means that some special convention is necessary to give any meaning to the term "*the* velocity of a transition." Such a convention is suggested by the curves themselves. What we mean by a rapid transition is one which increases greatly in speed for a slight shift of pressure away from the pressure of equilibrium. In this sense, therefore, a transition is more rapid if it is represented in Figures 2–5 by a steeper curve. Or stated conversely, a "rapid reaction" in this sense is, paradoxically, one that stops rapidly. This so-called "speed" is really the pressure acceleration of speed. It is natural, therefore, to take the slope of the reaction velocity curve as the measure of the speed of the transition. Throughout the rest of this paper, the acceleration, measured in this way, will be taken as the "speed." This would give a perfectly definite result if the curves were straight lines, but this is not the case. What is more, it does not seem possible to set up a single type of equation which shall be satisfied by all the curves. The most interesting feature of the curves, however, and that least affected by accidental properties of the rest of the apparatus, is the limiting slope when the velocity of transition becomes zero. This limiting slope is evidently to be obtained by extrapolating the curves until they cross the axis, and drawing the tangent at the point of crossing. The numerical value of the limiting slope of the tangent (the acceleration or "speed") expressed as fractional change per minute per kgm. per cm.$^2$, is one of the essential features referred to above. Corresponding to curves like those of Figures 2–5, there are two values, therefore, for the limiting transition acceleration, those with rising and falling pressure, and these two values may be plotted as points in another diagram.

It is evident from the diagrams that there is very little possible ambiguity as to the way in which the curves should be extrapolated to the axis. The range of velocities covered by the readings is very wide, and it was possible to approach very close to the axis in nearly every case. In several cases the lowest observed velocity was 5000 times less than the greatest. That the character of the velocity curves is such as to allow this extrapolation is an important fact that will be referred to again in the discussion. One consequence of this is, however, to be insisted on here. The points at which the extrapolated curves for rising and falling pressure at the same temperature cross the axis are not the same. This is particularly evident in Figures 3 and 4 for phosphorus and $KClO_3$. This means that there is a pressure range within which the reaction will not run at all, even when the two phases are in contact with each other. This range may be called the "region of indifference." Now as a matter of fact the probability is that this statement is not rigorously true; in at least one case it was possible to just detect the progress of the reaction within the limits of the "region of indifference." In another case just as careful search failed to detect any progress of the transition within the region. This was on the I–II curve of $TlNO_3$ at 3100 kgm. In any event the velocity within this region must be so small as to belong to phenomena of quite a different order. Thus in the best marked case, that of carbamid, the curve plotted for a range of transition velocities of 1000 fold was of exactly the character of those shown above, indicating unambiguously an extrapolated pressure of zero velocity, but below the lowest measurable point the curve apparently turns and runs along nearly parallel to the axis. This means that if Figures 2–5 were drawn on a scale one meter long, the curves might turn abruptly and run along the axis at less than 1 mm. distance. Although it was not possible to measure the rate of transition within the region it was possible to state that it was of the order of 0.00002 parts per minute. The reaction would not, by its own progress, carry the pressure into this region in practical limits of time. Pressure had to be artificially shifted into this region and the subsequent reaction observed. Various special precautions were necessary in these readings because of the extreme minuteness of the effect. In view of the great difference in order of magnitude, it seems justifiable to suppose that the mechanism involved on the two parts of the curve is different. In the later discussion I shall make the suggestion that the velocities as ordinarily measured are the velocities of a surface of separation, whereas the minute velocities which are just detectible in some cases are due to the transition running at corners or edges separating one phase from the other. For our present purposes we shall neglect these very small velocities, and discuss without reserve the "region of indifference" as if the reaction velocity were mathematically zero within this region. The width of this region is evidently another datum of significance to be obtained from the velocity curves. It varies greatly from substance to substance, and also varies greatly with pressure and temperature along the transition curve of the same substance. By collecting into a single diagram the widths of the indifferent region for the same substance, we obtain curves showing the width of the region as a function of pressure or temperature along a transition line. In the following these curves are given.

The data for individual substances follow. These comprise curves for the limiting acceleration from above and below, and width of the band of indifference, plotted against equilibrium pressures on the transition line, together with such comment on individual peculiarities as may be necessary. In a number of cases, fragmentary data have been collected, not sufficient to collect into curves. These isolated values are also given.

## Data for Individual Substances.

Phosphorus.— The transition curves for the two varieties of white phosphorus were given in an earlier paper (C) of this series. The transition velocity data were determined at that time, but were not published. Figure 6 shows the transition accelerations both from above and below, as a function of pressure on the equilibrium line (the temperature range is from 0° to 68°), and in Figure 7 the width of the band of indifference is shown. The observed reaction velocities cover a range of 150 fold. It is to be noticed that the acceleration is greater with falling than with rising pressure, that it is greater at the lower pressures and becomes nearly constant at the higher pressures, that the band of indifference passes through a minimum and becomes rapidly greater at the higher pressures, and that it does not run parallel to the velocity curve.

Ammonium Sulfocyanide.[3]— The limiting accelerations from above and below are given in Figure 8. The results are somewhat irregular;

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[3] E, p. 72.

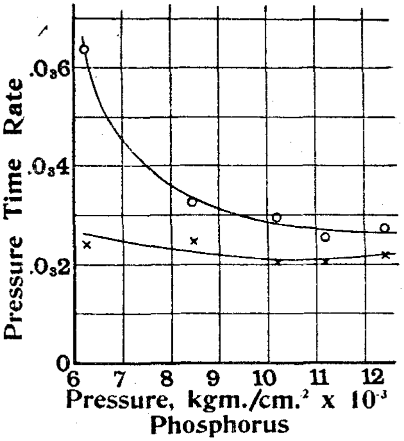

FIGURE 6. The pressure acceleration of the velocity of transition at zero velocity as a function of pressure along the transition curve of the two modifications of White Phosphorus.

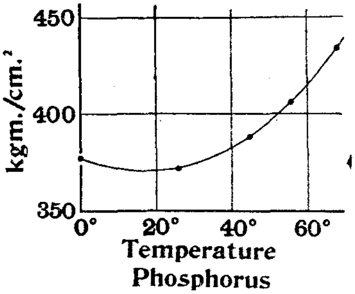

FIGURE 7. The breadth of the band of indifference between the two modifications of White Phosphorus as a function of temperature along the transition line.

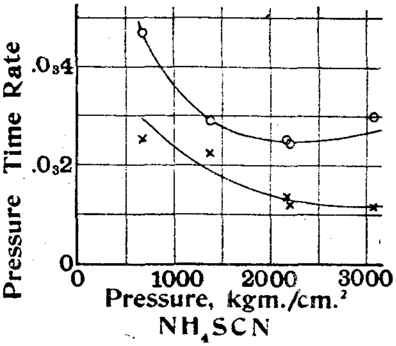

FIGURE 8. Ammonium Sulfocyanide. The pressure acceleration of the velocity of transition at zero velocity as a function of pressure along the transition line.

it is certain, however, that the reaction with falling pressure is the more rapid, and that in general the reaction becomes slower at the higher pressures. This seems natural when we recall that this transition is of the ice type, higher pressures corresponding to lower temperatures. The band of indifference is very narrow, too narrow to accurately measure. It seems, however, to increase regularly in width with increasing pressure, from about 15 kgm. at a pressure of 680 kgm. to 40 kgm. wide at 3080 kgm.

CARBON TETRABROMIDE.[4]— Observations at only two temperatures were made, on the II–III curve. At 176.6° the accelerations from above and below were 0.00076 and 0.00038 parts per minute per kgm. respectively, and at 152°, 0.00052 and 0.00069. The reversal in order of magnitude is probably not genuine; because of decomposition these results can at best give only the order of magnitude. The band of indifference was 50 and 90 kgm. wide at the two temperatures respectively; as is normal, the width is greater at the lower temperature and lower pressure.

URETHAN.[5]— Measurements were made at only one point, at 0° for the transition I–III. The accelerations were 0.000088 from above and 0.000060 from below, rather slower than the usual. In the preceding paper mention has already been made of the curious behavior of the band of indifference, that it is wider on the I–II and III–II curves at temperatures above the triple point than it is on the I–III curve below the triple point.

CAMPHOR.— The transition data of this substance has not yet been published[6]; it is fairly complicated and contains six modifications. The only one of the transitions with a change of volume large enough so that the time rate could be measured is II–III. The transition line runs from approximately 23° and 2800 kgm. to 115° and 11900 kgm. with little curvature. The accelerations on this curve are shown in Figure 9. The reaction is more rapid with falling than with rising pressure (the reversal at 6000 is probably not genuine) and grows more rapid at the higher pressures. The breadth of the band of indifference is shown in Figure 10; the band becomes rapidly wider as the pressure increases. It is remarkable that the reaction velocity should increase at the same time that the limits of indifference become wider.

CAESIUM NITRATE.[7]— There are two modifications, the transition line running nearly linearly from 153.7° at 1 kgm. to 207.1° at 6000 kgm. This substance seems to be normal in every way, and the measurements on it are as satisfactory as any. The transition accelerations are shown in Figure 11; the velocity is always greater from above and

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[4] E, p. 90.
[5] E, p. 118.
[6] See, however, F for the general nature of the phase diagram.
[7] G, p. 587.

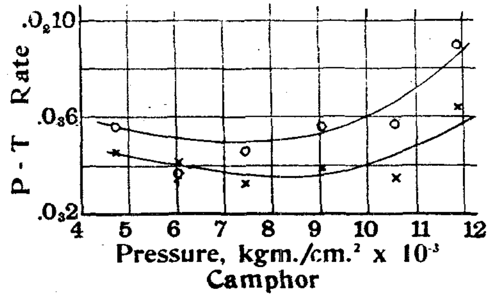

FIGURE 9. Camphor. The pressure acceleration of the velocity of transition at zero velocity as a function of pressure along the transition line between the modifications II and III.

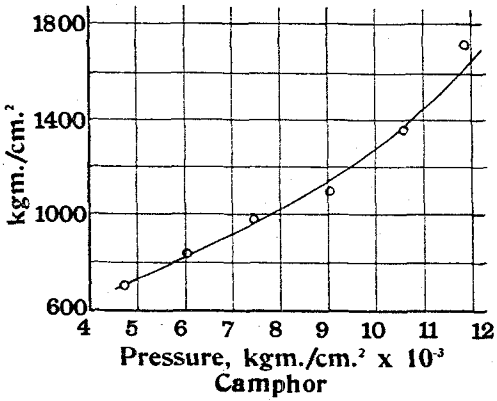

FIGURE 10. Camphor. The breadth of the band of indifference between the modifications II and III as a function of pressure along the transition line.

decreases with rising pressure. The band of indifference was inappreciable in width at the two lower points, but at 5500 suddenly jumped to 90 kgm.

SILVER NITRATE.[8]— There are two modifications, the transition is of the ice type, and there is a sudden and remarkable increase of curvature of the transition line near 7000 kgm. The transition acceleration curves are shown in Figure 12 and the breadth of the region of indifference in Figure 13. The acceleration with falling pressure is greater throughout; the region of increase of curvature of the transition line is mirrored by a remarkable increase of reaction velocity.

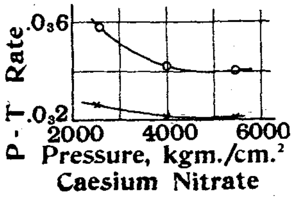

FIGURE 11. Caesium Nitrate. The pressure acceleration of the velocity of transition at zero velocity as a function of pressure along the transition line between the two low temperature modifications.

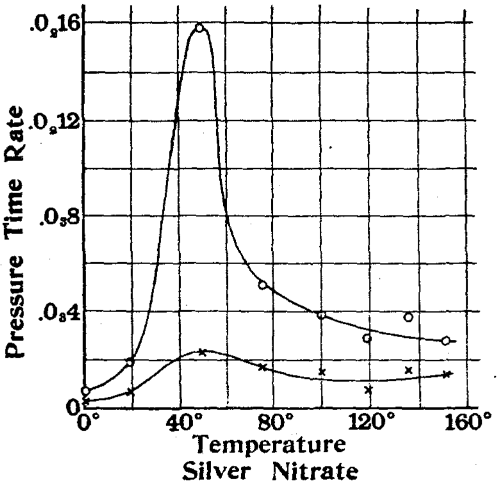

FIGURE 12. Silver Nitrate. The pressure acceleration of the velocity of transition at zero velocity as a function of temperature along the transition line. The position of the maximum corresponds to a region of rapid change of direction of the transition line.

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[8] G, p. 582.

Ammonium Nitrate.[9]— It was possible to make measurements on several of the transition curves. The limiting accelerations are shown in Figure 14 and the width of the bands in Figure 15. A single observation on the II–III curve at 67° gave a band width of 115 kgm. and limiting accelerations of 0.0013 and 0.00026 with falling and rising pressure respectively. The reversal of the acceleration,

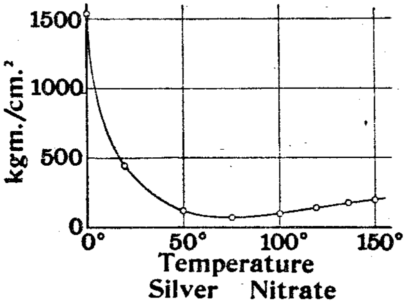

Figure 13. Silver Nitrate. The breadth of the band of indifference as a function of temperature along the transition line.

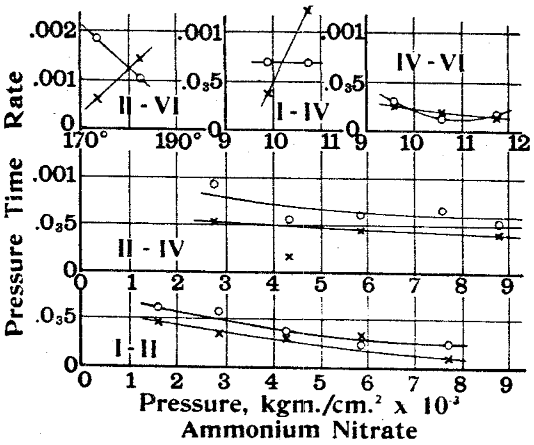

Figure 14. Ammonium Nitrate. The pressure acceleration of the velocity of transition at zero velocity as a function of pressure or temperature along the transition lines of the several modifications. Observe the change of abscissa for II–VI.

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[9] G, p. 605.

that with rising pressure in some cases being greater, is an interesting feature of these curves, but may be due to experimental error. The sudden increase of the width of the band of indifference for II–IV from zero to a fairly large value at the higher pressures is an interesting effect, but is not unique to this substance; $CsNO_3$ affords another example.

Carbamide.— This has three modifications, but only two transition curves available for measurement.[6] On the upper of these two curves above 100°, the transition is extraordinarily rapid, much too rapid to measure, and the width of the band is sensibly zero. In no case was the difference of pressures reached from above and below more than 5 kgm., and in many cases there was no appreciable difference. On the lower curve between 0° and 100° for the transition I–III the reaction was slow enough to measure. Three different experimental arrangements were used; the carbamide was either rammed tightly into a steel cup under mercury or submerged in loose pieces under mercury, or rammed into a light steel shell and pressure transmitted by kerosene. There is a very slight dissolving action of the kerosene, not sufficient to vitiate the pressure-temperature values, but sufficient to slightly round the corners, and so vitiate the measurements of the time rate. This was one of the few cases in which a restraining action of the steel shell was noticeable; the reaction accelerations were less and the width of the band of indifference greater

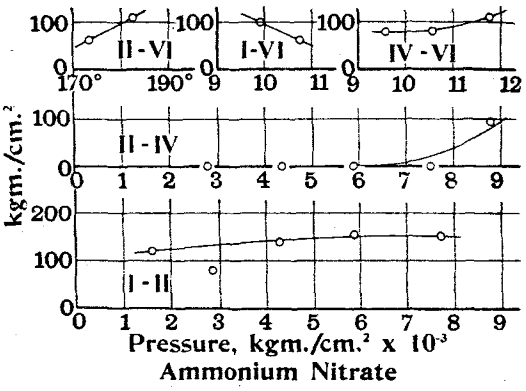

Figure 15. Ammonium Nitrate. The breadth of the band of indifference between the several modifications as a function of pressure or temperature along the transition lines.

when the carbamide was restrained under mercury than when it was free. This is merely another manifestation of some peculiarity of behavior during the transition that will be further commented on in the paper containing the equilibrium data.

There are measurements at only three points with the carbamide free under mercury; the results for the transition acceleration are shown in Figure 16 and for breadth of the band of indifference in Figure 17. The acceleration with falling pressure is greater than normal, and both accelerations are greater at higher pressures. These data suggest, however, and the suggestion is much strengthened by the other discarded data, that at temperatures between 65° and 100° the acceleration with rising pressure becomes greater, and that the acceleration with falling pressure may even pass through a minimum, to rise again. This may have some connection with the third phase. The band of indifference becomes rapidly less in width at the higher

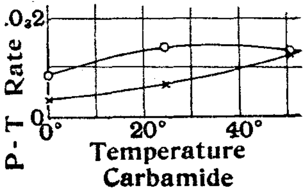

FIGURE 16. Carbamide. The pressure acceleration of the velocity of transition at zero velocity as a function of temperature along the transition line between I and III.

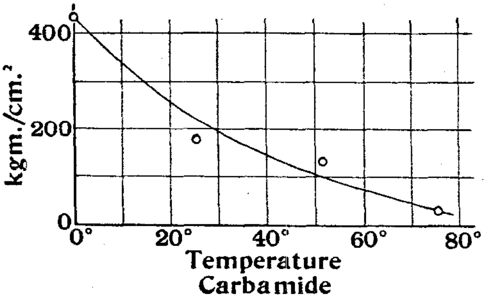

FIGURE 17. Carbamide. The breadth of the band of indifference between the modifications I and III as a function of temperature along the transition line.

temperatures, and becomes inappreciable above 80°. In Figure 16 the point obtained at 75° with the restrained carbamide has been included; this is legitimate if this point is regarded merely as setting an upper limit.

It has already been mentioned that carbamide was the substance which shows a reaction velocity of a smaller order of magnitude at points within the extrapolated region of indifference. This phenomenon was observed at 0°. It was possible to detect motion at points only 65 kgm. apart, whereas the extrapolated width of the indifferent band was 435 kgm.

Acetamide.— The results are to be published in the following paper.[6] Measurements of the rate were made only with a specimen which was afterwards found to be impure. The results are too irregular for graphical representation. The acceleration with falling pressure is throughout about twice as great as that with rising pressure, and increases markedly with rising temperature along the transition line. The width of the indifferent band falls greatly with increasing temperature from about 700 kgm. at 25° to about 70 kgm. at 100°. Above 100°, although no measurements were made on the pure specimen the band of indifference remained of appreciable width, perhaps of the order of 50 kgm.

Potassium Chlorate.— A large number of measurements were made on this substance,[10] but it does not pay to reproduce them in detail because the effects were different in character than for other substances. It has already been stated that at the lower temperature the region of nucleus formation was so extensive as to cover the entire region of observation, so that the pure surface growth could not be observed. At higher temperatures, however, the phenomena become like those for normal substances. The acceleration with falling pressure is greater than with rising, and both accelerations become greater at higher temperatures. There is a well defined band of indifference which becomes narrower at the highest temperature. That there should be a band of indifference at all shows that the effect of rising temperature is much less in increasing the facility of nucleus formation than in increasing surface transition velocity.

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[10] E, p. 78.

## DISCUSSION.

A simple interpretation of the results is complicated by several obscuring factors, which we will first discuss. There is, in the first place, the effect of the heat set free during the transition. The effect of this is that the temperature at which the transition is running is not the observed temperature of the bath, but differs from it by an unknown amount depending on the magnitude of the heat of transition, the transition velocity, and the thermal conductivity of the surrounding envelope. If the transition is very rapid, the measured rate may be entirely controlled by the rate of dissipation of the latent heat, the two phases being always under equilibrium conditions at the momentary actual temperature of the interior. As heat is conducted away from the interior, the pressure so changes as to assume the value appropriate to the temperature of the interior, and it is this rate of change of temperature which is really measured. This is preëminently the case on a melting curve. Of course it is not possible to carry the solid any distance at all into the domain of the liquid, and the apparent rate of change of pressure is only the rate at which pressure follows the return of temperature along the equilibrium line. The reverse displacement, that of the liquid into the domain of the solid, is possible to carry out, because liquids may be subcooled. But even in the reverse case, the latent heat of freezing is so large, and its rate of dissipation is so slow compared with the rate of crystallization of the liquid, that the apparent rate is governed by the thermal conductivity. The result is that the apparent rate of melting of most solids is exactly the same as the apparent rate of freezing. Figure 18 for mercury shows an example of this. In some cases, however, the velocity of crystallization is unusually slow, so that an appreciable fraction of the latent heat is conducted away during the freezing, and it may be possible to detect the difference of velocity between melting and freezing. Figure 19 shows such results for benzophenone. If the conductivity were perfect, results of this character would be shown by all liquids; the rate of melting would be infinite and the rate of freezing would be a different characteristic rate for every substance. As a consequence of this large disturbing factor, it has not been worth while to try to collect any data for the rate of melting or freezing under pressure.

Slow thermal dissipation is evidently going to be a disturbing factor also in the case of solid transitions, but to a much less degree than for melting. There are two reasons; the latent heat of transition is usually very much less than that of melting, and the velocity of a solid transition is usually less than that of melting or freezing. The thermal effect is, however, entirely absent only when there is no heat of transition; that is, when the transition curve is vertical. Several substances approach rather close to this ideal.

The unavoidable result of slow rate of dissipation of latent heat is that only a relative significance can be attached to these results. There are other factors also which rule out the possibility of an absolute interpretation. The effect of the manner of the mechanical constraint of the specimen is to be considered. Most of the specimens were originally in the form of dry powder or small crystals; these were

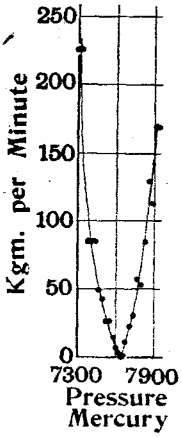

Fig. 18.

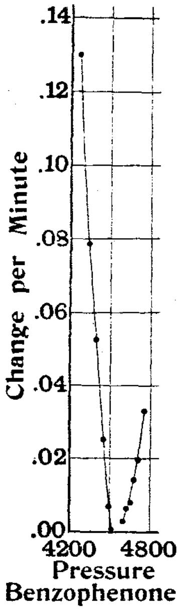

Fig. 19.

FIGURE 18. The time rate of change of pressure as a function of pressure during the melting and freezing of mercury under pressure at 0°. The symmetry of the curves is due to slow thermal conduction.

FIGURE 19. The rate of freezing or melting of benzophenone, expressed as fractional parts of the total change per minute, as a function of pressure. The steeper curve at the left shows that melting is more rapid than freezing.

pounded dry into a thin steel shell about 6 inches long and 11/16 of an inch in diameter, open at both ends, and perforated on the sides with small holes to allow ready access of the liquid transmitting pressure throughout the entire mass. Several measurements were made to detect a possible restraining action of this shell, and in nearly every case none could be detected, but the fact that some slight effect could be detected in one or two extreme cases where the lateral holes were discarded and the shell left closed on the bottom end leaves open the possibility of a very slight effect in all cases. The state of subdivision of the substance must also have an effect. If it is in the shape of a fine powder compacted together by the hammer, the new phase must spread somewhat less rapidly from grain to grain than it would in a homogeneous fused mass. Some of the substances were fused into place, but even these always developed cracks during use, thus introducing the same element of uncertainty. Furthermore, there may be a slight specific effect exerted by the liquid transmitting pressure. This liquid penetrates through all the crevices, and might even in some cases exert a catalytic effect, although this extreme is unlikely. Usually pressure was transmitted by kerosene, but mercury was used for most organic substances. In the case of $KClO_3$ a greater reaction acceleration was found with mercury for the medium than with kerosene. This may be in part a thermal conduction effect. It is also likely that in some cases slight impurity may affect the transition velocity, although its effect on other properties might be inappreciable.

These considerations suggest the use to which these observations can legitimately be put. They must not be used in calculating the absolute rate of growth of any one phase at the expense of another, they may be used cautiously and with great reserve in comparing the velocity under similar conditions of different materials, since the form of apparatus was nearly constant for all these measurements, and they may be advantageously employed in finding the variation of rate of the same substance under different conditions of pressure and temperature and direction of transition. This last is the use to be made of them in this paper.

We now have to inquire just what it is that we have been measuring. It is common knowledge that there are two distinct processes involved in either crystallization from a melt or in the formation of new polymorphic phases. These are: first the formation of nuclei of the new phase, and secondly a surface growth of the new phase, starting from the nuclei. In the case of two solids, nuclei of the new phase are formed only when temperature or pressure is raised or lowered a suitable amount away from the equilibrium point. Of course no hard and fast bounds can be set to this region within which nuclei are not formed, because at any point the formation of a nucleus is a matter of chance, but within the limits of time within which laboratory experiments can be made it is safe to say that there is a region on *both* sides of the equilibrium point within which new nuclei will not be formed. By working only within this region, the phenomena of surface growth may be observed separated from the disturbing effect of nucleus formation. This is an essential distinction between a polymorphic transition and melting. With only a very few possible exceptions a solid cannot be carried any distance into the domain of a liquid without the formation of liquid nuclei, so that it is impossible to observe pure surface growth of a liquid into a solid.

The experimental conditions were such that the measurements of this paper are almost entirely concerned with the surface growth and not with formation of nuclei. It will be recalled that the method was to increase the pressure on one phase until a nucleus of the new phase appeared; and then to observe the rate of fall of pressure. All the internal evidence makes it probable that in most cases only one nucleus is formed, and that what is observed is the rate of advance of a single surface of separation. In support of this statement it is in the first place evident that as soon as a single nucleus is formed the progress of the reaction makes the formation of others much more difficult, and since nucleus formation is a matter of chance it is very unlikely that two nuclei are formed simultaneously. In the second place, the regularity of the points, always lying on smooth curves, makes it very probable that the conditions are always comparable; if sometimes we were measuring the rate of growth of surfaces starting from three nuclei and at other times from only one, we could not expect smooth curves. This is a matter of considerable importance for our interpretation of the results. We have noticed that the transition acceleration with decreasing pressure is in practically all cases greater than with rising pressure. Now this might be either because the surface of separation at a constant distance from equilibrium moves more rapidly with decreasing pressure, or because more nuclei of the high pressure phase are formed than of the low pressure phase, so that with decreasing pressure we are observing the rate of advance of more surfaces than with increasing. This particular point may be settled experimentally. If during a reaction with rising pressure we artificially raise the pressure into the region of the reverse transition, but not into the region of the formation of nuclei, the ensuing fall of pressure is due to a reversal of direction of transition at exactly the same surfaces which we had been previously observing. This was tried on several occasions, and no difference found between experiments under these and the usual conditions; the transition velocity with falling pressure was in all cases greater. It is therefore, very probable that in most cases only one nucleus was formed. But one nucleus would give rise to two surfaces, travelling in opposite directions along the cylindrical container. The limiting accelerations listed above were usually observed when between 50 and 90% of the substance had been transformed. If the nucleus were formed at random throughout the mass, this limiting acceleration would sometimes be observed with one and sometimes with two surfaces. But if the original nucleus were always formed near one end of the tube, one of the surfaces of separation would have run to the end of the tube and disappeared, so that the final observations would always be on one surface only. The regularity of the results makes it almost certain that this is what happened. It is, in any event, to be expected that nuclei will be formed near the ends in preference to the central portions, because the first effect of change of pressure would be felt near the open ends.

It was not possible to detect any connection whatever between transition acceleration and the total quantity of the substance that had been transformed. The data were of course at hand from which the total fraction of transformed material corresponding to each limiting acceleration could be determined. These fractions were written down against every point of all the curves plotted, and showed an utterly random distribution. This of course is what would be expected if the measurements are actually of the rate of advance of a single surface; the rate of this surface will not involve its position in the cylindrical tube.

Although the data of this paper are almost entirely concerned with the surface growth, some comment is called for on phenomena connected with nucleus formation. In some cases the observed pressure change did not initially assume a high velocity and decrease regularly, but at first increased, passed through a maximum, and then decreased. This is shown in Figure 2 for $AgNO_3$. It is without doubt due to the fact that the surface of the freshly formed nucleus has not yet attained the maximum extent allowed by the container; the total rate of transformation of one phase into the other is less because the small extent of the surface at which growth occurs more than compensates for the rapid linear advance of the surface. This phenomenon is probably exhibited by all substances, but for most substances the rate of advance of the surface in the region of nucleus formation is so rapid that it was not possible to make any measurements. The existence of the effect was established for $AgNO_3$, $KClO_3$, Acetamide, and Carbamide. The effect was in most cases shown only on the rising branch, on which the velocity was less, thus making it easier to observe. Carbamide, however, showed two examples on the falling branch, and acetamide one.

The question of the extent of the region in which no nuclei are formed is of interest. As has been said, no fixed limits can be assigned to this region, but the formation of a nucleus is a matter of chance. It is to be expected, therefore, that occasionally nuclei will be formed outside of the usual region. The effect of such a freshly formed nucleus will be to add more surfaces of separation, and so to increase the measured rate. Such effects were found for two substances, $AgNO_3$, and $KClO_3$. Several of the rate curves for $AgNO_3$ with rising pressure showed a secondary maximum or a region of arrest, to be explained in this way. The effect was so pronounced with $KClO_3$ that these secondary maxima appeared persistently throughout the entire region of observation with rising pressure at 0°. It is evident, therefore, that different substances differ greatly in the sharpness of the boundaries of the region of nucleus formation. For most substances the boundary is so sharp that no chance nuclei were ever formed in the region of observation. Furthermore with $AgNO_3$ and $KClO_3$ the boundary of the region at pressures above equilibrium was very much sharper than at pressures below; no secondary maxima or regions of arrest were observed with falling pressure. In the case of $AgNO_3$ the boundary was sufficiently sharp so that the extra velocity due to additional nuclei did not affect the limiting acceleration; the extra surfaces due to the extra nuclei having merged into the other surface before the final reading. This was shown by the regularity of the limiting accelerations, which lie on a fairly smooth curve, irrespective of whether there had been a secondary maximum or not. The disturbing effect with $KClO_3$, however, extended throughout the entire region of observation at 0°, so that it was not possible to establish any limiting acceleration. At higher temperatures the boundary for $KClO_3$ became much sharper and the curves became normal in every respect, as shown by Figure 4.

The question of the sharpness of the boundary of the region of nucleus formation is an interesting one for investigation, both experimental and theoretical. It has received little attention, but would seem capable of giving information about the mobility of the molecules in a crystal, or the number that must fall together in the right position in order to start a nucleus. It is probable that the formation of even a nucleus is a rather complicated matter. If nothing more were demanded than that two or three molecules come together in the right position, it is difficult to see why the region of nucleus formation should have any boundaries. The process may be something like this; two or three molecules fall together to form the beginning of the nucleus. These molecules by their orientation tend to attach other molecules to them in the same orientation; this tendency is greater at points farther removed from equilibrium. This process of agglutination until a full fledged nucleus is formed, is somewhat different from the ordinary surface growth. In the early stages there are disintegrating forces due to the comparatively small number of elements involved and the effects of surface tension, which vanish under the conditions of surface growth proper.

One of the most important results of these measurements is the establishment by an extrapolation that there is a region of no appreciable velocity. It has been stated that there may be actually some velocity in this region. The question arises, therefore, as to how justified the extrapolation is which entirely neglects this very small effect. The answer is that practically, except in one case, there was never the slightest doubt. Theoretically the possibility must be recognized that the velocity curves may turn gradually at the bottom and so vitiate the extrapolation, but practically, on diagrams of about ten times the scale of the published figures, there was no such effect. The only exception was $KClO_3$ at low temperature, and it has already been stated that for this substance the formation of nuclei was abnormally persistent over a wide range, and this effect is entirely competent to explain the character of the curves. So even in the case of this one exception, we have no reason to believe that the small reaction within the region of indifference would vitiate the extrapolation. One is the more strengthened in the belief that the small residual effect is due to a distinct mechanism when one considers that there must be corners and edges as well as surfaces of separation of the two phases, and that at corners and edges just this sort of an effect would be expected because of the lack of perfect homogeneity. Furthermore, observations of this small residual rate were made only after pressure had been carried artificially into the indifferent region. Under these conditions corners and edges must have been more numerous than if the reaction had been allowed by its own progress to enter this region, and therefore the observed transition velocity is very possibly greater than if the reaction had been allowed to run its natural course. In the following, this small residual effect will be entirely disregarded, if indeed it exists in most cases, as being an extraneous effect.

The general behavior of the acceleration curves next calls for comment. One feature common to nearly all the curves is that the acceleration with falling pressure is greater than with rising. This is what one would expect; that reaction runs more rapidly which is pushed by the external pressure. There may be a few exceptions, however, as in the case of $NH_4NO_3$.

With regard to the variation of the acceleration along the transition curves we may have either increasing or decreasing acceleration at the higher temperatures. It is to be recognized that on a normal transition curve there are two opposing tendencies; increase of pressure would be expected to decrease the transition acceleration, because of increase of viscosity, while increasing temperature would increase it. Phosphorus is a well marked example of decreasing acceleration, and camphor of increasing. The curves for $AgNO_3$ are remarkable; the acceleration passes through a maximum, much more pronounced for falling than for rising pressure, in the region of the rapid change of direction of the transition curve. It has already been stated that this is a region of anomalous behavior of other physical properties also. The accelerations with rising and falling pressure are usually affected in the same way as we move along a transition line, but the magnitude of the effects may be different, and in exceptional cases possibly the signs may be unlike.

The behavior of the region of indifference next concerns us. It is first to be noted that the apparent equilibrium within the region of indifference is somewhat different in its nature from the so-called "false equilibria" with which we are familiar in such cases as diamond and graphite or hardened steel. The usual explanation of the failure to react in such cases is that one of the phases has been cooled so far below the temperature of transition that the velocity of transition has become inappreciable because of the enormously increased viscosity. A similar example is afforded by those liquids which may be subcooled so far as to become glassy. In the cases described in this paper, however, the reaction does not run when two phases are in contact at only a slight distance from the equilibrium temperature. Such immobility cannot be a viscosity effect, because at lower temperatures, where the viscosity is greater, the reaction may run with high velocity.

Phenomena consistent with these have been observed before. In the case of $CBr_4$, Wahl[11] has observed optically that there is a region within which the reaction could not be observed to progress. This region was narrow, could be rather sharply observed, and was of approximately the same width as I have found above. The phenomenon has also been observed by Tammann[12] for the two varieties of phenol at low temperatures. I am not aware, however, that anyone has ever before actually measured the velocity at points outside the region and shown by an extrapolation that there is really a region of no appreciable velocity. I have been able to make such neasurements only because of an apparatus absolutely without leak.

The width of the region of indifference varies at different points on the equilibrium curve. We should expect rising temperature to decrease the width of the region, and rising pressure to increase it. With $NH_4SCN$ the region increases in width with falling temperature and rising pressure; the width is about twice as great at 0° as at 67°, although the width is in either case comparatively slight. $CBr_4$ shows a decreasing width with rising pressure and temperature; the effect of rising temperature here overbalances the effect of rising pressure. The band of indifference of Urethan decreases strikingly with rising temperature on all three transition curves; this has been described more fully in a preceding paper.[13] Phosphorus shows on the whole an increase of width with rising pressure and temperature, but has a flat minimum which is of interest. This points to a preponderating effect of temperature at low pressures, but at the higher pressures the retarding effect due to increased pressure overbalances the acceleration due to rising temperature. In a previous paper it was stated that the width of the band for phosphorus was inappreciable,[14] but this is now seen not to be true. The previous statement was deduced from a diagram like that of Figure 1; the advantage of the diagrams used in this paper is obvious. Phosphorus shows that there is no necessary connection between the width of the band and the acceleration of transition. The band becomes wider with increasing pressure and temperature, but the transition acceleration increases also. $CsNO_3$ shows an effect similar to phosphorus; at 177° and 190° the region of indifference has no appreciable width, but at 202° the width is 100 kgm. This is doubtless the effect of increasing pressure, as opposed to the effect of increasing temperature. $AgNO_3$ is interesting in this connection, as it is in many others; the width of the region, shown in Figure 13, passes through a minimum. On the equilibrium curve, temperature rises as pressure falls, as for $NH_4SCN$, so that one would expect a steady increase of width with rising pressure. The minimum seems to have some connection with the locality of the sharp bend of the equilibrium line.

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[11] W. Wahl, Trans. Roy. Soc. (A) 212, 137 (1912).
[12] G. Tammann, ZS. phys. Chem. **75**, 75–80 (1910–11).
[13] E, p. 122.
[14] B, p. 185.

The fact that there is an indifferent region introduces a possible error into determinations of the equilibrium values of pressure and temperature. Of course the fact that there is a region within which the reaction does not run does not prevent our attaching a definite meaning to the equilibrium coördinates. These are to be defined thermodynamically as the points at which the thermodynamic potentials of the two phases are equal. To apply this definition demands that at least at one point the reaction run without sticking. The error in the equilibrium coördinates due to the region of indifference is also of course operative at atmospheric pressure. In most cases it is only possible to shut the equilibrium temperature between an upper and a lower limit. The limits vary greatly for different substances; there are many solid transitions which are apparently as sharp as a melting point.

The actual point of equilibrium, defined thermodynamically as above, may be situated, as far as we can tell, at any point within the region of indifference. In all the preceding work, the equilibrium point has been assumed to be in the middle of the indifferent band. In most cases this can lead to only very small error, because the width of the band is small compared with the total pressure. But in a case like that of $AgNO_3$ at 0° the error from this cause may become appreciable. An attempt to correct the equilibrium point by displacing it from the center in proportion to the transition velocities from above and below might lead to better results, but we could not be sure of them, because we have seen that there is no necessary connection between the width of the band and transition velocity. It is conceivable that the true equilibrium point might lie on that side of the center of the band toward the smaller velocity.

Before proceeding to the final part of the discussion, which will be occupied with an attempt to find the implications as to mechanism of these new facts, it will pay to very briefly recapitulate the nature of the facts. There are two cardinal facts; in the first place there is a region of indifference surrounding the equilibrium point within which the transition does not run even when the phases are in contact, and secondly the transition velocity at points equally distant from the equilibrium point is in nearly every case greater with falling than with rising pressure. There is no uniformity among different substances as to the direction of variation of either the width of the band of indifference or of the transition velocity with rising temperature along the transition line.

The first conclusion to be drawn is that the mechanism of a polymorphic transition cannot be the same in nature as that of a vaporization. Equilibrium between a liquid and its vapor is a dynamic equilibrium in which two independent processes are involved; molecules of the liquid leave the surface of the liquid and enter the vapor phase, and molecules of vapor impinge on the liquid surface and enter the liquid phase. At those pressures and temperatures for which these two streams of molecules in opposite directions each carry the same quantity of matter per unit time, there is equilibrium between liquid and vapor. The velocity of these two streams of molecules may be represented as a function of pressure at constant temperature as in Figure 20. The equilibrium pressure is located at the point of intersection of the two curves. It is immediately evident from the diagram that at points near the equilibrium point the velocity of vaporization or of condensation is equally great at points equidistant from the equilibrium point on either side.

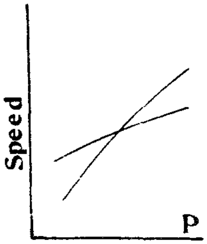

FIGURE 20. Curves showing at constant temperature the probable velocity of vaporization of a liquid or condensation of its vapor as a function of pressure. The equilibrium pressure is that at which the two curves cross.

Each of the two cardinal facts mentioned above shows that this cannot be the mechanism of polymorphic transition. If such a mechanism were consistent with the existence of a region of indifference, the curves for velocity of transition in opposite directions would have to be of the form in Figure 21. Further, if the mechanism were consistent with unsymmetrical transition velocities on opposite sides of the transition point, when there is no appreciable region of indifference, one at least of the transition velocity curves must have a bend at the equilibrium point, as shown in Figure 22. But the curves of both Figures 21 and 22 involve non-analytical singularities of a kind that we are not willing to admit in our physical phenomena, at least not on this scale of magnitude.

The fact that polymorphic equilibrium is not a dynamic equilibrium is significant with respect to the random distribution of velocity of temperature agitation among the molecules. In a vapor there is almost certainly, and in a liquid quite probably, a very close approach to Maxwell's distribution of velocities. In such a distribution there are always some molecules with a velocity so much above the average that they can pierce the transition layer and enter the other phase. The fact of the existence of the region of indifference shows that with polymorphs there are no molecules at the equilibrium point with velocities high enough above the average to pierce the transition layer (or its equivalent) and enter the other phase. That is, in a crystal the random temperature agitation is random only within a restricted range; the velocity never rises above a definite limit. This is certainly consistent with the known definiteness of crystal structure as shown by X-ray photographs. The region within which the velocities are confined in all probability does not have a sharp boundary, but is more or less nebulous at the edges. We recognize that the abruptness of this boundary within which the velocities are confined may vary greatly from substance to substance, and may also vary with the same substance at different pressures and temperatures.

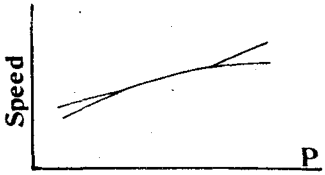

Fig. 21.

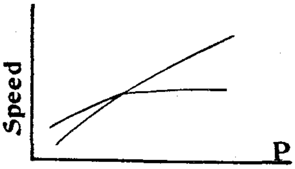

Fig. 22.

FIGURE 21. Hypothetical curves showing what must be the nature of the velocity between two polymorphic modifications in those cases where there is a band of indifference.

FIGURE 22. Hypothetical curves showing what must be the nature of the transition velocity between two polymorphic modifications in order to explain asymmetry of velocity in those cases where there is no band of indifference.

We have ruled out the reciprocal passage of molecules between polymorphic phases; we must therefore find some other mechanism which will carry the molecules from one phase to the other. As I have mentioned in a previous paper,[15] the fact that the conditions of equilibrium involve the constants of both phases shows that a transition from one phase to another does not take place because the first phase, at the equilibrium pressure and temperature, suddenly becomes inherently unstable, and falls apart into its elements, which then build themselves up into some other arrangement which under the conditions does happen to be stable. The instability of one phase is only a relative instability, into which the properties of the other phase enter. The driving force from one phase to the other is doubtless to be found in a definite orienting force exerted by the one phase on the molecules of the other. The same orienting force comes into play when a crystal separates from solution; there is a field of force like a skin over the crystal which compels the molecules being freshly deposited to orient themselves definitely with respect to the regular assemblage of molecules already laid down. In the same way, when two polymorphs are in contact, each phase reaches into the other and strives to orient the molecules of the other into its own position. Above the equilibrium point the orienting forces of one phase prevail, and below it those of the other. This struggle for mastery between the orienting forces of the two phases is a static rather than a dynamic struggle, like a tug of war rather than a game of tennis.

It is possible to represent graphically some of the counter-play of forces on the molecules. We will go to the extreme of simplification and suppose that at any constant temperature and pressure all possible configurations, whether stable or unstable, of the molecules of a crystal may be defined by a single position coördinate. Corresponding to each arrangement there is a definite potential energy. We plot potential energy against position coördinate. If the configuration is a stable one, the potential energy is a minimum. If the substance has two arrangements of possible stability (polymorphism) there will be two minima, and the lowest one will correspond to the absolutely stable form. At an equilibrium point the two phases are equally stable and the two minima are at the same level. At pressures above equilibrium pressure (at constant temperature) the minimum of one phase becomes the absolute minimum, and vice versa. Such a state of affairs is indicated in Figure 23. Let us now consider the curve corresponding to equilibrium. If a molecule is to pass from the phase

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[15] D, p. 108.

I to the phase II, it must pass over the intervening maximum of energy. This passage may take place if the range of irregular thermal agitation is so great that some molecules occasionally possess an amount of energy greater than the average equal to the height of the hill, but otherwise the transition will not take place, even when the phases are in contact. In this way we get a region of indifference.

If there is a region of indifference, its width is fixed by the position of the first curve beyond the equilibrium curve on which the height of the hill can be surmounted by the haphazard temperature agitation, aided by the orienting forces of the more stable phase, which away from the equilibrium point prevail over those of the unstable phase. According to the specific effect of pressure and temperature on the shape of the potential energy curves and on the random distribution of temperature agitation, it is easy to see that the band may become broader or narrower at higher temperatures on the transition line. It is conceivable, although unlikely, that the height of the hill to be surmounted should so increase on both sides of the equilibrium point that the reaction will never run, no matter how far one goes from the equilibrium point. Or the other extreme is more probable, that at points sufficiently far from equilibrium, the hill, together with one of the minima, should totally disappear, resulting in the absolute instability of the corresponding phase. This is Ostwald's labile state, or the unstable portion of James Thomson's isothermal.

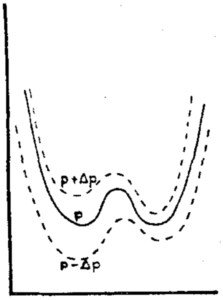

FIGURE 23. Shows at constant temperature the potential energy of position against position coördinate in the neighborhood of a transition point for a substance which has two polymorphic modifications.

The pressure acceleration of the transition velocity is evidently intimately connected with the sharpness of the region of random temperature agitation. If all the molecules once in every few oscillations receive enough energy to surmount the barrier, the transition is cataclysmic; but on the other hand, is very slow if only once in a while a molecule receives enough energy to slip over. A transition increasing in acceleration with increasing temperature means that the border of the velocity domain becomes sharper at the higher temperature, and a retarded acceleration at higher pressures means a more nebulous boundary. In general the results above do show a more nebulous boundary at higher pressures. One would expect, therefore, that the velocity domain of the phase of smaller volume would have the more nebulous boundary. This is precisely what the greater acceleration with falling as compared with rising pressure means. During the falling pressure transition, the low pressure modification, with the sharper velocity boundary is tumbling over the hill into the lower minimum, and is running with the greater speed.

It is suggested by the figures that in extreme cases, when the intervening hill is low, and the domain of temperature distribution is wide, that not only should there be no region of indifference, but that the equilibrium should be of the nature of a liquid-vapor equilibrium, and should involve equality of streams of molecules in two directions. Such transitions must be of great velocity, and would not be expected to be within the range of these observations. This may perhaps be the nature of the equilibrium in the neighborhood of a triple point between two solids and a liquid, on the upper end of the ice I–III curve, for example.

This analysis does not pretend to be an explanation; it makes no attempt to explain why the various factors vary with pressure and temperature in the way in which we have supposed they may. It is merely an attempt to state the nature of the elements that may enter the problem.

## SUMMARY.

The rate at which one polymorphic modification is transformed into another may be measured by the time rate of change of pressure at constant temperature during the transition. Data for the velocity of a number of such transitions are given in this paper. It is probable that all the measurements of this paper have to do essentially with the rate of advance of a single surface separating the two phases. The rate of advance increases rapidly as pressure is displaced from the pressure of equilibrium between the phases. As a rough comparative

measure of the speed of the transition we may use the pressure acceleration of the speed at zero velocity. The acceleration of nearly every transition is not symmetrical with respect to the direction of the transition, but is greater in the direction accompanied by decreasing pressure. Furthermore, there exists for most substances a distinct region on both sides of the equilibrium point within which the transition will not run even when the two phases are in contact. This is the "region of indifference." The acceleration of the transition and the breadth of the band of indifference vary with pressure and temperature along the transition curve; the manner of variation is different for different substances.

It results from these facts that an equilibrium between polymorphs cannot be due to a balance between two transitions running in opposite directions, as is the case for vaporization. This implies that in a crystalline solid the velocities of temperature agitation of the molecules cannot be distributed over a wide range of velocities, as is demanded by Maxwell's distribution law for gases or liquids, but must be confined within a restricted range. The boundaries of this range may be more or less sharp. The sharpness of the boundary is an important factor in determining the acceleration of the transition velocity.

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# POLYMORPHISM AT HIGH PRESSURES.

By P. W. Bridgman.

Investigations on Light and Heat made and published with aid from the Rumford Fund.

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## INTRODUCTION.

In a number of preceding papers [1] the phase diagrams and the other thermodynamic data have been given for a number of polymorphic transitions under high pressures. In this paper complete data are given for ten new transitions, and incomplete data for five or six others. This brings the total number of polymorphic substances investigated to date to 37, including 34 new phases whose existence was not known before. In addition to substances which do show polymorphism, 94 other substances have been examined to 12000 kgm. at 20° and 200° without result. These substances are enumerated here.

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[1] P. W. Bridgman, (A) Proc. Amer. Acad., **47**, 439–558 (1912).
(B) Jour. Amer. Chem. Soc., **36**, 1344–1366 (1914).
(C) Phys. Rev., **3**, 126–141 and 153–203 (1914).
(D). Phys. Rev., **6**, 1–33 and 94–112 (1915).
(E) Proc. Amer. Acad., **51**, 53–124 (1915).
(F) Proc. Nat. Acad., **1**, 513–516 (1915).
(G) Proc. Amer. Acad., **49**, 4–114 (1913).
(H) Proc. Amer. Acad., **51**, 579–625 (1916).
(I) Proc. Amer. Acad., 52, 57–88 (1916).

Throughout the rest of this paper reference to these papers will be made by letter.

This paper brings to a close, at least for the present, this series of investigations. The reason for this is not at all that all polymorphic substances have been studied, or that all interesting cases have been exhausted. The field is merely opened by these results. The reason is a practical one. Most of the substances that now suggest themselves for examination are not common chemicals that can be supplied by the large chemical houses, but require special preparation, for which I have not the facilities.

The number of substances investigated is nevertheless probably large enough to justify our pausing a little for examination of the entire field. The purpose of this paper, beyond the presentation of new data, is to coördinate somewhat this mass of results. To this end all the substances examined have been collected into various groups, chemical or not, according to the various clues which suggested an examination of them, and part of the discussion is concerned with these possible clues. The discussion also takes up a number of other general considerations connected with polymorphism, and tries to picture what may be the mechanism of a polymorphic change. It must be emphasized, however, that we cannot hope to get a complete explanation or description of polymorphism from data such as these. Data of many other kinds, particularly crystallographic data, are needed before the picture is complete.

## New Data for Individual Substances.

Acetic Acid.— Several attempts were made before this substance could be obtained commercially in sufficient purity to show a good freezing point. The sample finally used was from the J. T. Baker Chemical Co., and by analysis was better than 99.9% pure. It is necessary to keep the acid as much as possible from contact with the air, because it rapidly absorbs moisture; to this end the glass stoppered bottle was kept sealed with paraffine.

The acetic acid was placed in the nickel steel cup, inverted under mercury. After prolonged contact with the steel at high temperatures and pressures there seems to be some slight chemical action; the corners of the melting curve show somewhat more rounding, which in any event is slight, and on taking the apparatus apart a few bubbles of gas are set free. To avoid error from this effect measurements were begun about twice as far above the melting pressure as usual, and the cylinder was kept in cold water over night between runs. At the highest point attempted, 12000 kgm. and 165°, the reaction becomes so much more rapid that it did not seem worth while to make measurements. Runs were made with three fillings of the apparatus, two at high pressures and one near atmospheric pressure.

The equilibrium values of pressure and temperature are shown in Figure 1, the observed changes of volume in Figure 2, the latent heat and change of internal energy in Figure 3, and the numerical results are collected in Table I. There is a second solid phase; the existence of this phase was first discovered by Tammann. No peculiarities are presented by the phase diagrams. It is to be noticed that the direction of curvature of the $\Delta v$ curves on both melting lines is the normal one for solid-liquid.

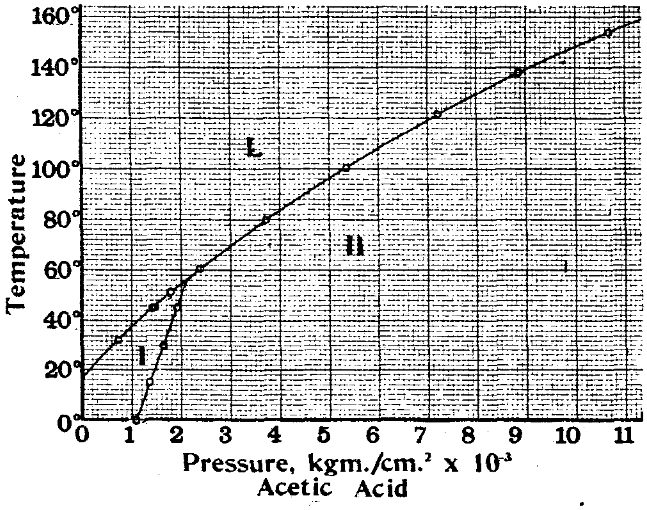

FIGURE 1. Acetic Acid. The observed equilibrium pressures and temperatures.

There are a number of measurements of the thermodynamic data at atmospheric pressure. For the melting temperature there is 17.5°

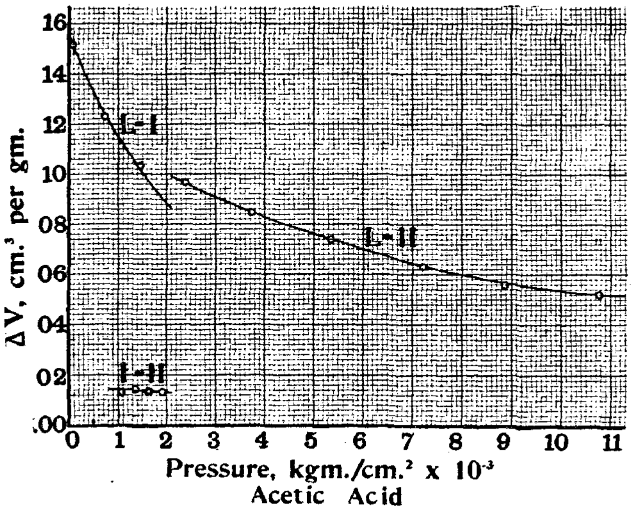

Figure 2. Acetic Acid. The observed changes of volume.

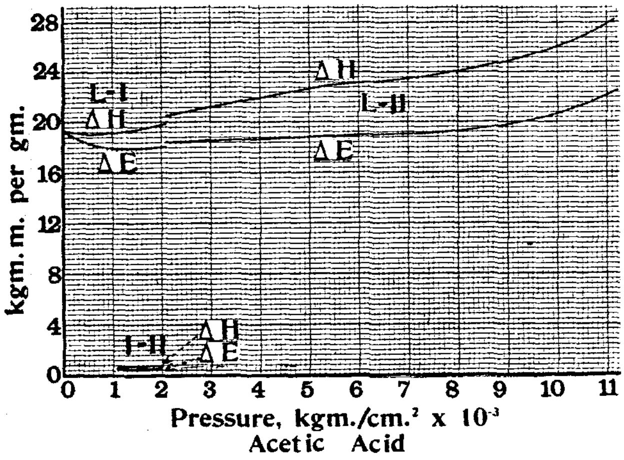

Figure 3. Acetic Acid. The calculated latent heats and the changes of internal energy.

## TABLE I.

### ACETIC ACID.

| Pressure | Temperature | ΔV cm³/gm. | $\frac{d\tau}{dp}$ | Latent Heat kgm.m./gm. | Change of Energy kgm.m./gm. |
|---|---|---|---|---|---|
| LIQUID — I. | | | | | |
| 1 | 16°.68 | .1560 | .02351 | 19.23 | 19.23 |
| 500 | 27 .8 | .1330 | 2101 | 19.01 | 18.34 |
| 1000 | 37 .7 | .1148 | 1869 | 19.09 | 17.94 |
| 1500 | 46 .5 | .1006 | 1658 | 19.38 | 17.87 |
| 2000 | 54 .3 | .0887 | 1470 | 19.75 | 17.98 |
| LIQUID — II. | | | | | |
| 2000 | 54°.1 | .1003 | .01600 | 20.51 | 18.50 |
| 3000 | 69 .4 | .0908 | 1463 | 21.27 | 18.55 |
| 4000 | 83 .4 | 828 | 1343 | 21.98 | 18.67 |
| 5000 | 96 .3 | 761 | 1240 | 22.68 | 18.88 |
| 6000 | 108 .2 | 700 | 1151 | 23.18 | 18.98 |
| 7000 | 119 .3 | 644 | 1072 | 23.57 | 19.06 |
| 8000 | 129 .6 | 598 | 1000 | 24.08 | 19.30 |
| 9000 | 139 .3 | 561 | 931 | 24.85 | 19.80 |
| 10000 | 148 .3 | 534 | 864 | 26.04 | 20.70 |
| 11000 | 156 .6 | 520 | 800 | 27.94 | 22.22 |
| I–II | | | | | |
| 1074 | 0 | .0140 | .0544 | .70 | .55 |
| 1534 | 25 | 136 | | .74 | .54 |
| 1994 | 50 | 131 | | .78 | .52 |
| TRIPLE POINT, I–II — LIQUID. | | | | | |
| 2100 | 55°.7 | L–II<br>.0992 | .01585 | 20.58 | 18.50 |
| | | L–I<br>.0862 | .01432 | 19.79 | 17.98 |
| | | I–II<br>.0130 | .0544 | .79 | .52 |

by Sonstadt,[2] 16.55° by Pettersson,[3] 16.75° by Rudorff,[4] 16.75 to 17.00° by Raoult,[5] 16.59° by de Visser,[6] 15.62° by Abegg,[7] 16.54° by Meyer,[8] and 16.68° by Faucon.[9] In Table I, I have adopted the value given without reference in the recent edition of Stelzner, which is the same as the value of Faucon. For the latent heat there is 43.66 cal. between 2.9° and 5.6° by Pettersson,[3] 46.4 at the melting point by Marignac,[10] 46.3 by de Visser,[6] and 45.8 by Meyer.[8] The value of Table I, computed from $\Delta v$ and $\frac{d\tau}{dp}$ is 45.1 cal. For the change of volume on melting Pettersson [3] gives 0.1252 cm.$^3$ per gm., de Visser [6] 0.1595, Meyer [8] 0.1578, and Block [11] 0.1278. I find by extrapolation from 78 kgm. 0.1560. The specific heat of the solid is given by Guillot [12] as 0.627 at 4.5°, and that of the liquid at 16.5° as 0.473 by Massol and Guillot.[13] Schiff [14] gives for the liquid the formula $0.444 + 0.0_3709\ (t + t_1)$. There can be no doubt but that the value for the solid is erroneous.

The effect of pressure on the melting point has been measured by de Visser [6] up to 30 kgm.; he gives for the initial slope 0.02355 degrees per kgm. Meyer [8] calculates from his data 0.0233 for the initial slope, and I find above 0.02351. Tammann [15] finds a considerably lower value, 0.0220. The only measurements to higher pressures are by Tammann, who also gives the only previous data on the transition line between the two solids. His coördinates of the transition line differ widely from mine. At 0° his equilibrium pressure is 1145 kgm. against 1074 of mine, and at 50° it is 2170 against 1994. The difference cannot be explained by the width of the band of indifference. Tammann's melting curves both lie appreciably below mine; his triple point differs by 230 kgm. and 1.8° from mine. Tammann does not give the changes of volume.

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[2] Sonstadt, Jahrber. Fort. Chem. 1878, **34**, and Chem. News, **37**, 199 (1878).
[3] O. Pettersson, Jour. prak. Chem. (2), **24**, 296.
[4] Rudorff, Ber. D. Chem. Ges., **3**, 390.
[5] Raoult, Ann. Chim. Phys. (6) **2**, 66, 71 and 75.
[6] de Visser, Rec. Trav. Pays Bas, **12**, 101 (1893).
[7] Abegg, ZS. phys. Chem., **15**, 213 (1894).
[8] J. Meyer, ZS. phys. Chem., **72**, 225 (1910).
[9] Faucon, quoted in the Tables of the French Physical Society under date of 1910.
[10] Marignac, quoted by J. Meyer, reference (8) above.
[11] H. Block, ZS. phys. Chem., **78**, 385 (1911–12).
[12] Guillot, Paris, J. B. Baillière et fils (1895). Thesis (?).
[13] Massol and Guillot, C. R., **121**, 108 (1895).
[14] Schiff, Lieb. Ann., **234**, p. 322.
[15] G. Tammann, Kristallisieren und Schmelzen, p. 275.

The difference of compressibility, thermal expansion, and specific heats between the several phases may be found in the usual way.[16] Direct experimental values were found for the difference of thermal expansion of the liquid and I at 78 kgm., and for the difference of compressibility of II and I. For the difference of expansion ($\Delta\beta$) I found 0.00050, which is in rather good agreement with the value of Block,[11] 0.00048. It is somewhat of a puzzle that Block seems to be able to get good values for $\Delta\beta$, whereas his values for $\Delta v$ are often widely in error. Using the experimental value of $\Delta\beta$ and the values for $\frac{d\Delta v}{dp}$ and $\frac{d\Delta H}{dp}$ that may be computed from the table, we find $\Delta\alpha = 0.0_464$ and $\Delta C_p = -0.016$ cal. The negative value for $\Delta C_p$ is improbable in spite of the experimental statement of this by Guillot.[12] If we assume that $\Delta C_p = 0$, the approximation usually made, the value of $\Delta\alpha$ becomes $0.0_465$ and the value of $\Delta\beta$ $0.0_356$. This shows again the insensitiveness of $\Delta\alpha$ to the value of $\Delta C_p$. The fact that $\Delta\beta$ calculated with this assumption differs no more from the experimental value makes it probable that $\Delta C_p$ is really very small at atmospheric pressure.

The values of $\Delta\alpha$ and $\Delta\beta$ calculated at other points on the L–I and the L–II curves are shown in Table II, making the usual assumption that $\Delta C_p = 0$. This is probably a bad assumption on the L–I curve at 2000 kgm., because the value for $\Delta\beta$ becomes too small, and also because $\Delta\alpha$ does not check with the values for L–II and I–II. Also error is probably introduced by a too rapid increase of $\Delta H$ on the upper end of the L–II curve. If the calculations are carried through at 10000, $\Delta\beta$ is negative. On the I–II curve the experimental data showed that the difference of compressibility of I and II is not more than $0.0_51$. This gives us a maximum value of $\Delta\beta$ $0.0_52$. As mentioned above, the values given for $\Delta\alpha$ (L–I and L–II) do not check at 2000 with $\Delta\alpha$ (I–II); probably $\Delta\alpha$ (L–I) is too large.

In an unsuccessful search for other modifications pressure was carried to 12500 kgm. at 30° and 170°. It should be remembered that Tammann found some evidence for another form below 0°; I did not investigate this.

ACETAMIDE.—This substance has a second modification of the solid, not known before, the transition point being at 6000 kgm. at room temperature, and the transition line running nearly vertically. The material was obtained from Eimer and Amend. Two series of runs were made, separated by an interval of seven months. The first series of runs gave four points on the transition curve. The acetamide was used without purification for this run; it was placed in the inverted nickel steel cup under mercury. There was a rather large amount of impurity as shown by the moist appearance of the original material, and the very large amount of rounding of the corners of the melting curve. This rounding was so large that it was not possible to make accurate measurements on the upper end of the transition line, near the triple point, and much less to make even rough determinations of the melting data. This did not at that time seem a very serious matter, however, since the primary interest of this work was not in measuring more melting curves, and since the transition itself was sharp, showing that the impurity did not form mixed crystals. On working up the data it appeared, however, that there were some very slight indications that there were really three modifications instead of two. If there were three modifications, the volume of one of them must have been so close to one of the others that it was desirable to use as pure material as possible. A second series of runs was accordingly made with material which had been twice crystallized in the thermostat at constant temperature. Large perfectly trans

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[16] D, p. 100.

TABLE II.

Acetic Acid.

| Pressure | $\Delta\alpha$ | $\Delta\beta$ |
|---|---|---|
| | L–I | |
| 1 | $.0_464$ | $.0_350$ |
| 1000 | 37 | 26 |
| 2000 | 20 | 15 |
| | L–II | |
| 2000 | $.0_411$ | $.0_46$ |
| 4000 | $.0_577$ | $.0_44$ |
| 6000 | 59 | $.0_59$ |
| 8000 | 42 | $.0_52$ |

parent crystals were obtained, $\frac{1}{4}$ to $\frac{3}{8}$ of an inch in diameter, showing very well the hexagonal primitive form and the characteristic faces on the ends. The purity was still not perfect as shown by a slight remaining rounding of the melting curves, but it was nevertheless possible to make satisfactory measurements of the melting.

The observed equilibrium pressures and temperatures are shown in Figure 4, the observed differences of volume in Figure 5, the calculated latent heats and changes of internal energy in Figure 6, and the numerical values are collected in Table III. I did not think it worth while to try for a point on the melting curve nearer 12000 kgm., as rather inconvenient manipulation would have been necessary to obtain it, and even then satisfactory measurements of $\Delta v$ would not have been possible without taking the apparatus further beyond 12000 kgm. than I cared to. Furthermore, such a point was not necessary to establish the non-existence of other solid forms, because at 173° I found no new solid form up to beyond 12000. The inconvenience of manipulation referred to is caused by the very great subcooling that the liquid will support; after determining the point at 157.5° and 8800 kgm. it was necessary to lower the temperature 20° and raise the pressure to 12700 kgm. before the liquid could be forced to freeze. This means a superpressure of 6200 kgm. or a subcooling of 45°.

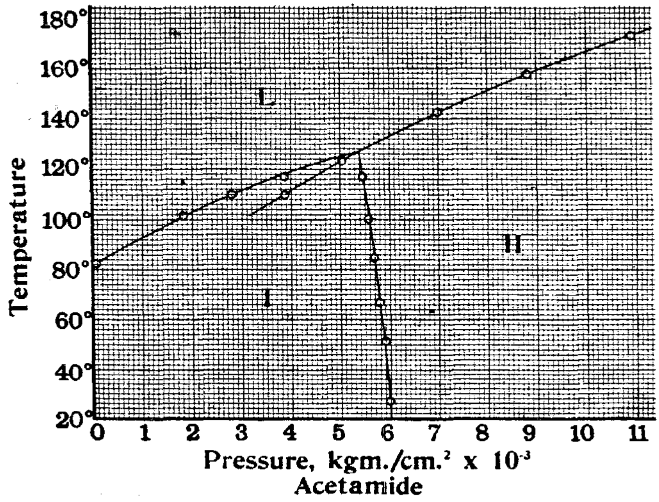

FIGURE 4. Acetamide. The observed equilibrium pressures and temperatures.

With the use of the purer sample the minute irregularities in the transition, which had given rise to the suspicion that there might be another modification, disappeared, but still other very minute consistent irregularities remained which leave me uneasily suspicious that

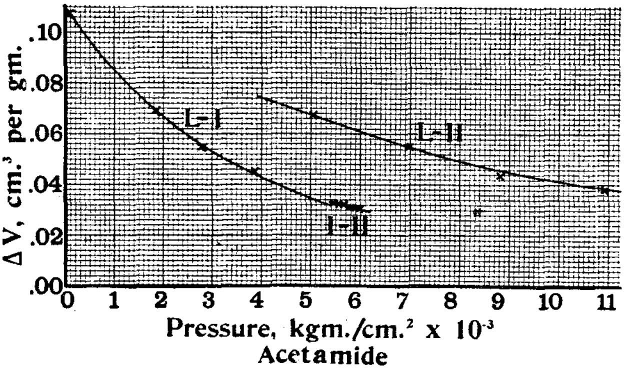

FIGURE 5. Acetamide. The observed changes of volume.

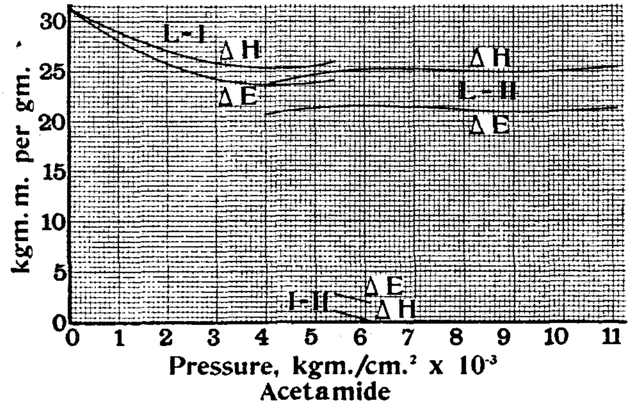

FIGURE 6. Acetamide. The calculated latent heats and the changes of internal energy.

somewhere below 75° the transition line may perhaps split into two inclined at a very slight angle. It would be very difficult to verify this suspicion, however, because of the increasing sluggishness of the

## TABLE III.

### Acetamide.

| Pressure | Temperature | ΔV cm³./gm. | $\frac{d\tau}{dp}$ | Latent Heat kgm.m./gm. | Change of Energy kgm.m./gm. |
|---|---|---|---|---|---|
| Liquid — I | | | | | |
| 1 | 81°.5 | .1098 | .01245 | 31.3 | 31.3 |
| 1000 | 93 .1 | .0852 | 1076 | 29.0 | 28.1 |
| 2000 | 103 .1 | .0668 | 0929 | 27.1 | 25.7 |
| 3000 | 111 .7 | .0533 | 0792 | 25.9 | 24.3 |
| 4000 | 119 .0 | .0429 | 0662 | 25.4 | 23.7 |
| 5000 | 125 .0 | .0346 | 0538 | 25.6 | 23.9 |
| Liquid — II | | | | | |
| 4000 | 111°.5 | .0746 | .01209 | 23.7 | 20.8 |
| 5000 | 122 .9 | 676 | 1083 | 24.7 | 21.3 |
| 6000 | 133 .1 | 609 | 0979 | 25.3 | 21.6 |
| 7000 | 142 .5 | 546 | 0897 | 25.3 | 21.5 |
| 8000 | 151 .1 | 492 | 0831 | 25.1 | 21.2 |
| 9000 | 159 .1 | 448 | 0774 | 25.0 | 21.0 |
| 10000 | 166 .55 | 412 | 0720 | 25.2 | 21.0 |
| 11000 | 173 .5 | 383 | 0672 | 25.5 | 21.2 |
| I–II | | | | | |
| 6000 | 20°.0 | .0302 | —.357 | .25 | 2.06 |
| 5930 | 40 .0 | 305 | .241 | .40 | 2.21 |
| 5835 | 60 .0 | 309 | .190 | .54 | 2.35 |
| 5720 | 80 .0 | 315 | .161 | .69 | 2.49 |
| 5590 | 100 .0 | 321 | .144 | .83 | 2.63 |
| 5440 | 120 .0 | 328 | .132 | .98 | 2.76 |
| Triple Point, Liquid — I–II | | | | | |
| 5390 | 127°.0 | L–I .0319 | .00491 | 26.0 | 24.3 |
| | | L–II .0649 | .01040 | 25.0 | 21.5 |
| | | I–II .0330 | —.128 | 1.03 | 2.81 |

transition at low temperatures. I made an attempt at 0°, where the separation of the lines would be greater than at higher temperatures, but the transition is so very sluggish that II would not change into I even when carried 3000 kgm. into the region of I. Under the conditions it was useless to try to establish the existence of a transition with an abnormally small volume change and equilibrium pressure very close to that already measured, and I abandoned the attempt.

Acetamide is a substance of unusual interest because at atmospheric pressure it forms, in addition to the ordinary stable form of the solid, an unstable modification of such persistence that Körber [17] has been able to measure its melting curve up to 3000 kgm. Körber, in fact, says that the unstable form was always the one that crystallized out of the subcooled melt, and that inconvenient manipulation was necessary to force the stable form to appear. I was very anxious to measure the melting curve of the unstable form, because the general relations between melting curves of stable and unstable forms is still undetermined, in spite of work of Wahl [18] and Körber. Not once, however, did this unstable from appear in my apparatus, although several times the liquid was forced to freeze by raising pressure across the melting line at constant temperature. Once I thought that I had the unstable form, and measured the melting data for it, but it turned out afterward that this was merely a point on the melting curve of II prolonged into the region of I. That the new modification II is identical with the previously known unstable form is made impossible by the direction of its melting curve, and the volume relations. II is more dense than I, while the unstable form is less dense, and the melting point of the unstable form is about 10° below that of I, while the melting point of II, if it could be realized at atmospheric pressure, would be at least 30° below that of I. It may be that the failure of the unstable form to appear is connected in some way with the material of the container. For this experiment the acetamide was placed in a nickel steel container, whereas Körber used glass. It will be recalled that an analogous effect of the container was found in the case of ice [19]; the unstable ice VI crystallized out of the melt in the usual metal container, and the stable V could be forced to appear only by bringing glass somewhere into contact with the liquid.

The fact that the line L–II can be carried so far into the region of I is of interest. Ordinarily, when measuring points on the transition curve I–II, working from lower to higher temperatures, II can be carried only an inappreciable distance into the region of I at temperatures above 100°. That is, if I has been recently present in the apparatus, so that II carries in its crystalline structure a nuclear memory of the structure of I, then II will change to I as soon as pressure is manipulated so that I is the absolutely stable form. But if I has been melted, and the liquid forced to crystallize to II, the sojourn in the liquid state now effaces all memory of I, and II will show very little tendency to change to I on being carried into its region of stability, but will melt to the liquid instead, if the temperature is not too far below the triple point. If, however, temperature is too far below the triple point (at 101.5° in an actual case), II will spontaneously change to I before the melting curve can be reached.

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[17] F. Körber, ZS. phys. Chem., **82**, 45–55 (1913).
[18] W. Wahl, Trans. Roy. Soc., **212**, 117 (1912).
[19] A, p. 502.

It was fortunate that I could obtain a very satisfactory measurement of the change of volume when II changed to I at 101.5° at a point 1800 kgm. distant from the transition line. This evidently gives, by a method independent of that usually used, the difference of compressibility between I and II. It turns out that II, the phase with the smaller volume, is on the average $0.0_512$ cm.$^3$/gm. per kgm. more compressible than I between 3800 and 5600 kgm. The difference of compressibility between I and II along the transition line was also determined in the usual way from the difference of slopes of the isothermals above and below the transition point. This method shows that II is more compressible than I over the entire length of the transition line, the difference being pronouncedly greater at the lower temperatures. The observed values for $\Delta\alpha$, which show a gratifying regularity, are reproduced in Figure 7. At 101.5° the agreement by the two independent methods, $0.0_512$ against $0.0_514$, is unexpectedly good. The small difference may quite possibly be due to the difference of pressure range.

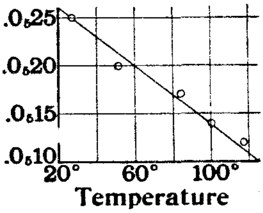

FIGURE 7. Acetamide. The observed differences of compressibility between the two solid phases.

From the difference of compressibility between I and II the difference of thermal expansions and the difference of specific heats may be calculated in the usual way. The results are collected in Table IV. I is throughout more expansible than II, and has a smaller specific heat.

The difference of compressibility between liquid and solid may be approximately calculated in the usual way, on the hypothesis that $\Delta C_p = 0$. The results are shown in Table V. The behavior of $\Delta \alpha$ for L–II is remarkable, at first increasing with rising pressure and then falling. The effect probably really exists, and is not to be explained by experimental error, because the figures indicate that at the triple point I is less compressible than II, a result already reached by independent method. This is not the first time that we have found that a new phase in the neighborhood of the triple point is abnormally compressible.

## TABLE IV.

### ACETAMIDE.

Difference of thermal expansion and specific heat between I and II.

| Temperature | $\Delta\beta$ | $\Delta C_p$ gm.cal./gm. |
|---|---|---|
| 20° | $+.0_420$ | $-.013$ |
| 40 | 29 | 15 |
| 60 | 34 | 15 |
| 80 | 39 | 15 |
| 100 | 42 | 13 |
| 120 | 43 | 12 |

The directly measured difference of expansion between L and I at approximately atmospheric pressure was $0.0_354$. The value found by Block[11] was $0.0_342$. The assumption that $\Delta C_p = 0$ probably does not give values of $\Delta \beta$ at higher pressures worth recording, because on this assumption we find that at atmospheric pressure $\Delta\beta$ has a value much too large, 0.001 against 0.0005. This indicates that at least the initial values of $\Delta\alpha$ are a little too high.

A number of measurements have been made on acetamide at atmospheric pressure by other observers. For the melting point of the stable variety there is 82° to 83° by Hofmann,[20] 81.5° by Block,[11] and 80.1° by Körber.[17] The value for the specimen used above was 81.5°. The change of volume has been found to be 0.1507 cm.$^3$/gm. at atmospheric pressure by Block,[11] which is in inexplicable disagreement with the value found above. The effect of pressure on the melting point has been measured by Körber,[17] who used the comparatively inaccurate method of varying temperature at constant volume. His melting curve shows considerably more curvature than mine, starting at a temperature lower by 1°.4, it is 3°.8 higher at 1600 kgm., and 1°.1 higher at 3000. This must have been a result of his method; the lower initial melting point and much premature rounding of his curves shows that his specimen must have been the impurer of the two. It

TABLE V.

ACETAMIDE.

Difference of Compressibility between the Liquid and the Two Solids.

| Pressure | $\Delta\alpha$ | |
|---|---|---|
| | L–I | L–II |
| 1 | $0_4 41$ | |
| 1000 | 30 | |
| 2000 | 21 | |
| 3000 | 15 | |
| 4000 | 10 | $0_5 54$ |
| 5000 | $0_5 83$ | 67 |
| 6000 | | 74 |
| 7000 | | 72 |
| 8000 | | 60 |
| 9000 | | 47 |
| 10000 | | 39 |
| 11000 | | 33 |

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[20] Hofmann, Ber. D. Chem. Ges., **14**, 2729.

is this evident inaccuracy in the method which leaves me unconvinced as to the cogency of his proof that the melting curves of the stable and unstable forms are approximately parallel.

CARBAMIDE.— This was obtained from Eimer and Amend and was used directly without further purification. The purity was fairly high, as was shown by the melting point. At atmospheric pressure I found for my specimen 131.7°, which agrees within the limits of error of the previous work with the only published value that I have found, 132° by Liubarvin.[21] At least six different runs were made with this substance. For all except the least important it was hammered dry and cold into the requisite forms. It may be easily melted into the forms, but not without some slight decomposition, so that it seemed best to avoid this possibility. Under pressure decomposition takes place with increasing rapidity above 150° so that accurate measurements could not be made above this. The carbamide was placed in different sorts of containers for the different runs. At first it was hammered dry into the nickel steel shell and used with a mercury seal. But this resulted in rupture of the shell. The dissolving action of kerosene is slight, so that it was possible to make one run with the pressure transmitted directly to the carbamide by kerosene. This gave good values for the equilibrium values of pressure and temperature, but not consistent values for $\Delta v$, presumably partly because of a very slight dissolving action. In the final arrangement the carbamide was hammered into compact cylindrical forms and placed loosely in detached hunks beneath the surface of mercury, and prevented from rising to the surface by a clip of obvious design. Carbamide is quite unusual with regard to its distortion during a transition. The growth of one modification at the expense of another takes place in such a direction that a cylindrical block increases in diameter. This increase of thickness may develop considerable pressure, as shown by the bursting of the nickel steel shell. Even the blocks which were placed loose beneath the surface of mercury had, after several transitions, so swelled laterally as to tightly fill the shell, just as if they had been melted into position. This lateral growth, besides being very inconvenient because of the rupture that it may produce, may also produce very appreciable error in the measurements of $\Delta v$, because of internal strains. Because of this effect the measurements of $\Delta v$ of the first runs were irregular and had to be discarded, and with the final arrangement, loose hunks under mercury, the apparatus had to be set up again after so many transitions had run as to jamb the carbamide tightly against the walls of the shell. At higher temperatures there is another source of error in measurements of $\Delta v$ because of incipient decomposition.

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[21] Liubarvin, Ber. D. Chem. Ges., **3**, 305.

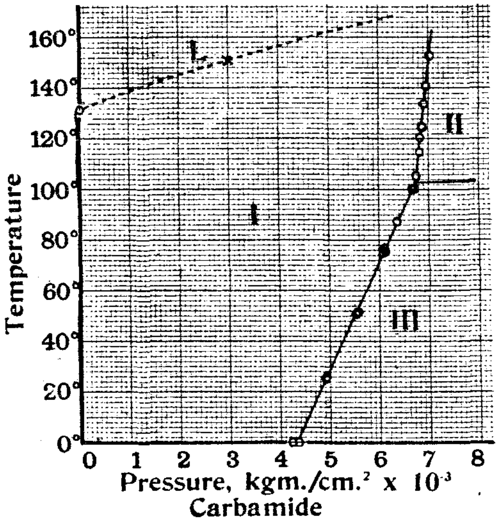

FIGURE 8. Carbamide. The observed equilibrium pressures and temperatures.

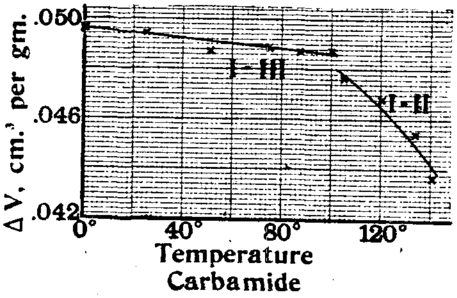

FIGURE 9. Carbamide. The observed differences of volume.

The equilibrium curves, with the observed pressure and temperature points, are shown in Figure 8, the values of $\Delta v$ determined after the proper procedure had been discovered are shown in Figure 9, the calculated values of latent heat and the change of internal energy in Figure 10, and the numerical values are collected in Table VI.

There are three modifications of the solid, but it was possible to make observations on only two of the transition lines because of the unusual volume relations. At lower temperatures there is only one transition line, I–III, but at 102.3° there is a triple point, and the line splits in such a way that almost all the change of volume remains with one of the lines, I–II, and almost all the heat of transition with the other. The change of volume between II and III is so small that it was not possible to measure points on this line, the motion of the piston accompanying this transition being only 0.001 of an inch. Apart from the evidence afforded by the sharp change of direction of I–III, the existence of the line II–III was verified by direct observation, by varying temperature at constant volume and plotting the resulting change of pressure. A discontinuity in this curve of the appropriate order of magnitude was found, but of course this method does not give accurately the coördinates of the transition line.

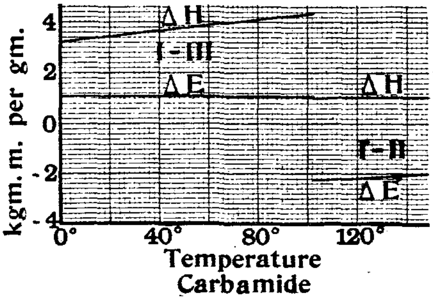

FIGURE 10. Carbamide. The calculated latent heats and the changes of internal energy.

Along with an abnormally small change of volume and a fairly high latent heat goes, of course, a very small slope for the line II–III. This slope was found by calculation from the data for the other lines at the triple point to be only 0.7° per 1000 kgm., by far the smallest slope yet found. It should be pointed out, however, that this calculated value was found from the difference of two very nearly equal quantities, so that it may be in error by a rather high percentage. The relation of the curves for the change of volume makes it perfectly certain, however, that the slope is positive.

There is the usual variation of the velocity of transition with temperature. At temperatures above 130° the transition is so rapid that no measurements of the rate were possible, whereas at 0° it was a matter of an hour or two for the pressure to approach within values differing by 700 kgm. from above and below. It was possible at 0°, however, by artificially changing the pressure and watching the subsequent reaction, to shut the pressure within limits only 60 kgm. apart.

TABLE VI.

CARBAMIDE.

| Pressure | Temperature | ΔV cm³/gm. | $\frac{d\tau}{dp}$ | Latent Heat kgm.m./gm. | Change of Energy kgm.m./gm. |
|---|---|---|---|---|---|
| | | | I–III | | |
| 4320 | 0°.0 | .04962 | .04210 | 3.22 | 1.08 |
| 4795 | 20 .0 | 4942 | " | 3.44 | 1.07 |
| 5270 | 40 .0 | 4921 | " | 3.66 | 1.07 |
| 5745 | 60 .0 | 4901 | " | 3.88 | 1.06 |
| 6220 | 80 .0 | 4880 | " | 4.09 | 1.06 |
| 6695 | 100 .0 | 4860 | " | 4.31 | 1.05 |
| | | | I–II | | |
| 6737 | 100°.0 | .0482 | .1803 | .98 | −2.25 |
| 6848 | 120 .0 | 465 | " | 1.01 | −2.17 |
| 6959 | 140 .0 | 442 | " | 1.01 | −2.06 |
| 7070 | 160 .0 | 413 | " | .99 | −1.93 |
| | | | TRIPLE POINT | | |
| 6750 | 102°.3 | I–III<br>.04858 | .04210 | 4.33 | 1.05 |
| | | I–II<br>.0480 | .1803 | 1.00 | −2.24 |
| | | II–III<br>.0006 | .00065 | 3.33 | 3.29 |

The superpressure required to start the reaction also varies greatly; on one occasion 1750 kgm. beyond the equilibrium point was necessary at 0°.

Carbamide is an unusual organic compound with respect to its melting as well as with respect to its solid transitions. At 150°, in the search for other solid modifications, the melting data were approximately determined. The melting pressure is shown by a cross in Figure 8 and the approximate location of the melting curve is shown by the dotted line. The change of volume at 150° is very low for a melting, about 0.01 $cm.^3/gm.$ This, together with the unusual flatness of the melting curve, a flatness that one associates with the melting of a metal rather than of an organic compound, aroused the suspicion that this point might really belong to a solid transition instead of to a melting. To test this, another run was made at 124°, slightly below the melting point at atmospheric pressure, and no transition of any kind found. It was for this run that the carbamide was melted into the form, as mentioned above. That the melting curve has approximately the location shown is also indicated by the fact that on one occasion, on trying to extend the I–II line to 170°, the transition entirely disappeared, doubtless because of melting. It has already been explained that the decomposition makes it useless to try for accurate coördinates of the melting curve.

The direct experimental measurement of the difference of the compressibility of the several phases did not give results accurate enough to justify an attempt to calculate $\Delta\beta$ and $\Delta C_p$. It seems established, however, that I is less compressible than II, and that the difference is in the vicinity of the order of $0.0_54$ $cm.^3/gm.$ per kgm., which is fairly high. It is also probable that I is less compressible than III, but the difference between I and III is considerably less than the difference between I and II. The difference of compressibility between I and II, large as it is, is not nearly large enough to account for the rapid drop of $\Delta v$ between I and II with rising temperature. It is very probable, therefore, that II is more expansible than I, and that the difference is of the order of 0.0001 $cm.^3/gm.$ per degree.

CAMPHOR.— This material was obtained from Eimer and Amend, "gum camphor, refined, powdered," and was used without further purification. It was hammered cold into the inverted cup, and pressure transmitted to it by mercury. The reason for trying this substance was that it is known to have an abnormally steep melting curve, the rise of temperature being about 130° for 1000 kgm.[22] This suggested that it might be abnormal in other respects, and might possibly show other modifications. It does as a matter of fact have at least six modifications, the complication of its phase diagram being equalled or surpassed only by water and ammonium nitrate. It is unfortunate that it was not possible to make measurements with as great accuracy as was possible for many other substances. In fact the uncertainty in many of the measurements of the change of volume is so great that I have not attempted to estimate from them the most probable value of the change of volume or to calculate the latent heats. There can be no question, however, that the existence of the transition curves in approximately the situations indicated has been established. It was a matter of some difficulty, involving considerable time, to be sure of as much as this. More than a month was spent in getting the data given below; to obtain accurate values of the changes of volume and the latent heats in addition would have taken many times longer, if indeed it would have been possible at all. In the present state of our knowledge of polymorphism it did not seem that all this effort on a single substance would have been worth while.

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[22] G. A. Hulett, ZS. phys. Chem., **28**, 629 (1899).

The phase diagram is shown in Figure 11, the curves for $\Delta v$ in Figure 12, the latent heat and the change of internal energy in Figure 13, and the numerical values in Table VII. The melting curve indicated in Figure 11 is taken from the data of Hulett.[22]

It will give an idea of the various difficulties and assist in forming an estimate of the accuracy of the work to describe the curves somewhat in detail. On the III–IV curve the reaction is very slow and the limits of indifference wide. This does not affect the accuracy of the change of volume, for which fairly satisfactory values were obtained, but may affect the equilibrium values, and so the slope and the latent heat. The III–IV curve is, however, determined with considerably greater accuracy than any of the others. On the II–V and the IV–V curves it is difficult to measure either the equilibrium or the $\Delta v$ values because of the very small change of volume and the slight separation of the curves. Let us suppose that we are trying to get a point on the IV–V curve, lowering pressure from above. The pressure steps have to be made very small, because of the small change of volume, and so the measurements consume much time. Furthermore, after the reaction has once started, one cannot release the pressure far enough to ensure the completion of the reaction without overstepping the II–V line, and so getting the reaction II–V mixed up in the effect; no amount of time will give a good value for $\Delta v$. It does no good to attempt to ensure the completion of the reaction in the narrow region

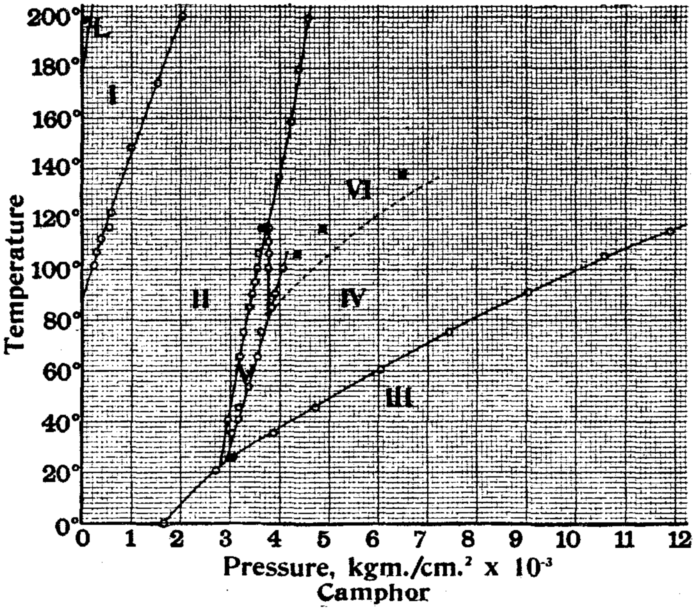

FIGURE 11. Camphor. The observed equilibrium pressures and temperatures. On the line IV–VI measurements could not be made on the reversible transition. The crosses show the points where metastable IV changes spontaneously to VI.

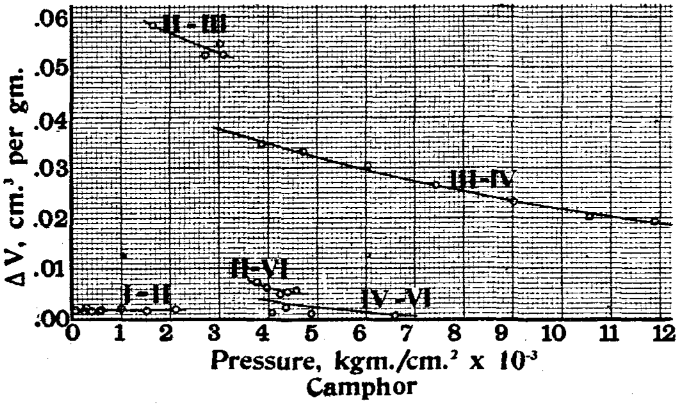

FIGURE 12. Camphor. The observed differences of volume.

between IV and II by raising or lowering the temperature, as is possible in the case of some other substances, because the boundaries of the region of existence of V are so nearly vertical that a change of temperature does not carry the pressure far enough away from the equilibrium value to ensure completion of the transition. The total change of volume from IV to II can be determined with some accuracy, however, by taking pressure steps long enough to step over the region of V.

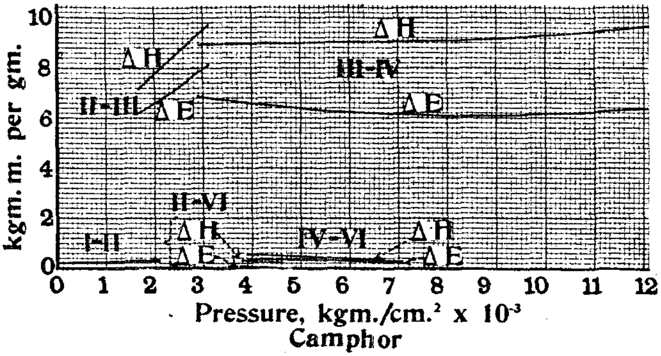

Figure 13. Camphor. The calculated latent heats and the changes of internal energy.

TABLE VII.

CAMPHOR.

| Pressure | Temperature | ΔV cm³/gm. | $\frac{d\tau}{dp}$ | Latent Heat kgm.m./gm. | Change of Energy kgm.m./gm. |
|---|---|---|---|---|---|
| | | | I–II | | |
| 0 | 87°.1 | .00187 | .0635 | .106 | .106 |
| 500 | 117 .6 | " | .0589 | .124 | .115 |
| 1000 | 146 .2 | " | .0555 | .141 | .122 |
| 1500 | 173 .2 | " | .0528 | .158 | .130 |
| 2000 | 199 .1 | " | .0510 | .173 | .136 |

*Table VII, Continued.*

| Pressure | Temperature | $\Delta V$ $cm^3./gm.$ | $\frac{d\tau}{dp}$ | Latent Heat kgm.m./gm. | Change of Energy kgm.m./gm. |
|---|---|---|---|---|---|
| II-III | | | | | |
| 1660 | 0° | .0585 | .0223 | 7.2 | 6.2 |
| 2130 | 10 | 563 | .0200 | 8.0 | 6.8 |
| 2660 | 20 | 540 | .0178 | 8.9 | 7.5 |
| III-IV | | | | | |
| 3000 | 25°.3 | .0377 | .01260 | 8.93 | 7.83 |
| 4000 | 37 .6 | 348 | 1204 | 8.96 | 7.57 |
| 5000 | 49 .4 | 320 | 1148 | 8.99 | 7.39 |
| 6000 | 60 .6 | 296 | 1092 | 9.03 | 7.26 |
| 7000 | 71 .2 | 273 | 1037 | 9.07 | 7.15 |
| 8000 | 81 .3 | 252 | 0982 | 9.12 | 7.10 |
| 9000 | 90 .8 | 234 | 0926 | 9.21 | 7.10 |
| 10000 | 99 .8 | 218 | 0870 | 9.32 | 7.14 |
| 11000 | 108 .2 | 202 | 0815 | 9.46 | 7.24 |
| 12000 | 116 .1 | 188 | 0760 | 9.62 | 7.36 |
| IV-V | | | | | |
| 2880 | 20° | | | | |
| 3460 | 60 | | .0687 | | |
| 3900 | 90 | | | | |
| II-V | | | | | |
| 2800 | 20° | | .133 | | |
| 2959 | 40 | | .119 | | |
| 3137 | 60 | | .107 | | |
| 3333 | 80 | | .098 | | |
| 3548 | 100 | | .089 | | |
| 3783 | 120 | | .082 | | |

*Table VII, Concluded.*

| Pressure | Temperature | $\Delta V$ cm³./gm. | $\frac{d\tau}{dp}$ | Latent Heat kgm.m./gm. | Change of Energy kgm.m./gm. |
|---|---|---|---|---|---|
| II–VI | | | | | |
| 3782 | 120° | .0069 | .084 | .321 | .060 |
| 4009 | 140 | 61 | .093 | .272 | .028 |
| 4217 | 160 | 57 | .101 | .244 | −.003 |
| 4407 | 180 | 54 | .110 | .222 | −.016 |
| 4582 | 200 | 52 | .119 | .206 | −.032 |
| VI–IV | | | | | |
| 4000 | 87°.5 | .0037 | .025 | .53 | .39 |
| 5000 | 106 .0 | 24 | .019 | .47 | .35 |
| 6000 | 121 .5 | 14 | .014 | .39 | .31 |
| 7000 | 134 .0 | 06 | .008 | .29 | .25 |
| V–VI | | | | | |
| 3780 | 90° | | ∞ | 0 | |
| 3780 | 110 | | | | |

In determining points on the V–VI line there is the same difficulty as on the IV–V line; the difficulty increases toward the triple point at the upper end.

On the VI–IV line the points could not be determined with any approach to accuracy because this is a reaction that will run in only one direction. If pressure is released, starting with IV, the reaction to VI will run with a discontinuous change of volume, but on increasing the pressure again within reasonable limits the transition cannot be made to run backwards. This does not mean at all that IV is an unstable form. The behavior here is much like that on the rising branch of the curve for mercuric iodide. The amount by which the phase VI may be carried into the region of IV without the transition running is very large. At 110°, the transition from VI to IV did not run when pressure was increased to 9000 kgm. The points on this curve had to be obtained by lowering the temperature after every reading far enough to reach the region where the reaction to III runs. On increasing temperature again beyond the III–IV line, the reaction from III to IV was certain to run, so that one could be sure in this way of starting with IV. One consequence of the great lag of the reaction from IV to VI is that the changes of volume determined from these measurements are certain to be in error, unless the compressibility of the two phases should by accident happen to be the same, because these are not the differences of volume at the equilibrium point, but at some other point. The measured difference is presumably too large. The difficulty of measurement is further increased by the fact that the difference of volume is excessively small. The discontinuity in the piston displacement at the last point measured on this curve, at 137°, was only 0.0018 of an inch.

The transition from V to IV shows the same lag as that from VI to IV, although the lag is not so obstinate. If temperature is raised on the phase II to somewhere between 50° and 80°, the pressure being at about 2000 kgm., and then pressure increased at this temperature, the reaction from V to IV will not run, even if the pressure is increased several thousand kilograms beyond the transition value. It is impossible in this way to obtain points on the IV–V line, but only the II–V line will be found. This curious disappearance of the IV–V line was the cause of much mystification before the explanation was found. To be sure of getting points on the IV–V line, the phase IV should be produced by first increasing pressure at low temperature, so as to force the appearance of III, and then raising temperature across the III–IV line. If the phase IV has once been formed, and pressure is decreased so as to produce V, the transition may be made to run in the reverse direction by increasing pressure again. That is, the phase IV will be produced from V if IV has existed in the apparatus only a short time previous. Under such circumstances there seem to be some nuclei left about which the formation of IV can begin. A precisely similar behavior has been found for some of the varieties of ice.

The points on the II–VI line could be obtained with fair definiteness, and the values of $\Delta v$ were also self consistent enough to warrant including them in Figure 12. The concave side of the equilibrium line II–VI is toward the temperature axis, which is the reverse of the case for II–V. There are indications that the curve II–VI may split again at its upper end, there being a modification VII; an appearance at 200° like that of two equilibrium points close together suggested this. The points for $\Delta v$ (II–VI) at 180° and 200° lie higher than the others would demand, and suggest the same possibility. Furthermore, the direction of curvature would be explained by the existence of a new modification. I was not able to settle the point, however, and have drawn only one curve in the diagram. This is the curve which best fits the equilibrium points. It is very probable, however, that at the triple point II–V–VI the curve II–VI should be steeper than drawn, and the curve II–V less steep. This is indicated by the latent heat relations.

On the II–III line only two points were determined, but the line has nevertheless been indicated in Table VII as having a curvature in the normal direction. The existence of curvature in this direction is demanded by the latent heat relations of II–III–IV–V at the approximately quadruple point, and the actual amount of curvature can be very approximately calculated from the data for the other curves. But to detect this curvature experimentally would have been difficult, because the breadth of the band of indifference is sufficient to conceal the effect on a curve so short.

On the I–II line a considerable element of uncertainty is introduced because the change of volume is so small that equilibrium points cannot be obtained, but the transition runs entirely to completion when it has once started. The transition is fairly rapid, however, and runs without much trespassing of the equilibrium pressure. The points shown in the diagram are the means of the pressures at which the transition ran spontaneously from above and below; these two pressures differed by from 50 to 100 kgm. The values of $\Delta v$ cannot be determined with much percentage accuracy because of their smallness; it was not possible to tell from the data whether $\Delta v$ increases or decreases with rising temperature, and in the computations $\Delta v$ has been assumed to be constant. The points on the I–II line were not determined at the same time as the other points, but nearly one year later. The same specimen of camphor was used.

In view of the uncertainty of the $\Delta v$ values on the II–V, IV–V, and V–VI curves, I have not attempted in Table VII to give even the most probable values for the change of volume or latent heat. The individual determinations of these changes of volume are chaotic; except for being of the same order of magnitude they offer no justification for the choice of any one set of values. I have, therefore, not included the experimental points in Figure 12. One is in a position to say this much, however; on the lower ends of the curves II–V and IV–V the two changes of volume should add up to the total change II–IV which is indicated in Figure 12, and which was determined with fair consistency experimentally; with a little less confidence, but still with fair accuracy, one may say that near 120° the changes of volume II–V and V–VI must add to the experimental values given in the figure for II–VI; and with much less accuracy one may demand that at the triple point IV–V–VI the changes IV–V and V–VI check with the value given in the figure for IV–VI. It must be remembered that the change IV–VI could not be determined on the equilibrium line, and so in all probability is not accurate.

With regard to computing the latent heat, the difficulty of determining $\frac{d\tau}{dp}$ accurately is greatly exaggerated because some of the transition curves run so nearly vertical; a very small change in the angular direction of a curve produces an enormous percentage change in the value of $d\tau/dp$, and so in the latent heat. Under the circumstances the only means of getting $\Delta v$ and $\Delta H$ is one of trial and error, demanding that the additive relations at the triple points shall hold. It did not seem to me that the accuracy of the rest of the work would justify such an attempt. In the figures and the table I have, therefore, given the values of $\Delta v$ and $\Delta H$ for only five of the curves. In order of certainty these are: III–IV, II–III, II–VI, I–II, and VI–IV, the last being of a higher order of uncertainty than the others.

On the III–IV line satisfactory measurements were made of the velocity of transition and the width of the band of indifference. These results have been described in a previous paper.

On the III–IV curve also rough values of the difference of compressibility could be determined. III is more compressible than IV over the entire curve; at about 7000 kgm. the difference is of the order of $0.0_525$ cm.$^3$ per gm. per kgm., rising to perhaps twice this value at the lower end, and falling to not less than two thirds of it at the upper end. The uncertainty in these values, and also those of $\Delta H$ and $\Delta v$ is so great that it did not seem worth while to try to get $\Delta\beta$ or $\Delta C_p$. Because of incompleteness of the transition etc. it was not possible to get good values of $\Delta\alpha$ or any of the other curves.

In a recent paper Wallerant[23] has described camphor as having three modifications at atmospheric pressure. It crystallizes from the melt in the cubic system. On cooling, the cubic modification changes at 97° to a feebly doubly refracting rhombohedric form, which at −28° is transformed again to a strongly doubly refracting rhombohedric form. These transition points of Wallerant are both about 10°

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[23] F. Wallerant, C. R., **158**, 597 (1914).

higher than mine. A linear extrapolation of my points on the II–III line would indicate −36° as the transition temperature at atmospheric pressure, and the probable curvature of this line would bring it down to somewhat below −40°. The discrepancy may possibly be due to still another modification, with a triple point on the line II–III below 0°. My point corresponding to Wallerant's 97° is at 87°. The discrepancy cannot be due to impurity of my specimen, because there was no preliminary rounding whatever of the corners of the isotherms. There is, however, a possible uncertainty of two or three degrees in my value 87°, as may be judged from the irregularity of the points at the lower end. I do not consider that this uncertainty can be possibly large enough to account for the discrepancy.

Wallerant's paper did not become known to me until all the other transition lines of camphor except I–II had been investigated. If it had not been for his paper, this transition would have entirely escaped me, because as has been previously explained, only in rare cases have I made especial search at atmospheric pressure for new modifications, but have accepted absence of mention of such transitions as presumptive evidence that there are none. This example brings out that in exceptional cases, where the transition line is very steep, or the change of volume very small, my method of exploration may possibly leave new transitions undiscovered. Camphor is a particularly unfavorable case, because both the transition line is very steep, and the change of volume very small.

Potassium Acid Sulfate.— This was obtained from Eimer and Amend, of the "tested purity" grade. Two sets of runs were made; the one gave all the high pressure points, and the other, with the low pressure apparatus, gave two points at nearly atmospheric pressure. Immediately before use it was heated to 100° in vacuum for several hours to remove the moisture. For the high pressure runs it was hammered cold into an open steel shell, and pressure was transmitted directly to it by kerosene. For the low pressure run it was melted into a glass tube, the glass removed, and the specimen placed loose in the pressure chamber, in direct contact with the kerosene.

$KHSO_4$ has four modifications. The existence of forms other than the ordinary low temperature form does not seem to have been known before, although there are two transition points and three modifications at atmospheric pressure. I was fortunate not to miss altogether the other modifications. At room temperature no new form was found out to 12000 kgm., although a transition line was crossed at 7000 or 8000. The reason for this is that at 20° the reaction is very sluggish, so that a superpressure of 5000 kgm. will not start it. At 200° a small transition was found at about 2000 kgm., so small that I had to repeat the work before I could be sure of it. If the transition had been at 1000 instead of 2000, it would have been so near the end of the stroke that it would have been missed altogether. This experience shows that by making runs out to 12000 at 20° and 200° one cannot be sure that there are no transitions in the region unless it is certain that there are no transitions at atmospheric pressure between 20° and 200°; it is not sufficient warrant for the absence of a transition at atmospheric pressure, even for a common chemical, that no one has noticed it and tabulated it. It is possible, therefore, that some of the substances which I have examined for transitions without result may really have transitions, since I have always assumed that if a substance has not been tabulated as polymorphic it has no transition at atmospheric pressure. I have found one or two other examples of new phases at atmospheric pressure not known before.

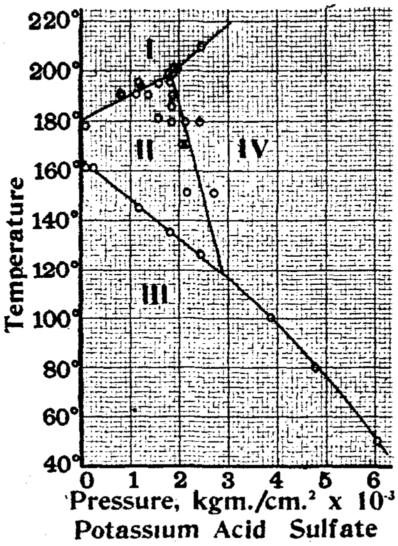

FIGURE 14. Potassium Acid Sulfate. The observed equilibrium pressures and temperatures.

The equilibrium values of pressure and temperature are shown in Figure 14, the values of $\Delta v$ in Figure 15, the computed values of latent heat and change of internal energy in Figure 16, and the collected numerical results in Table VIII. As is evident from the irregularity of the points, there are difficulties in the way of accurate measurement that need comment. On the II–IV line the change of volume is so small that it was not possible to obtain equilibrium values of the pressure even by the method of artificially varying pressure after the

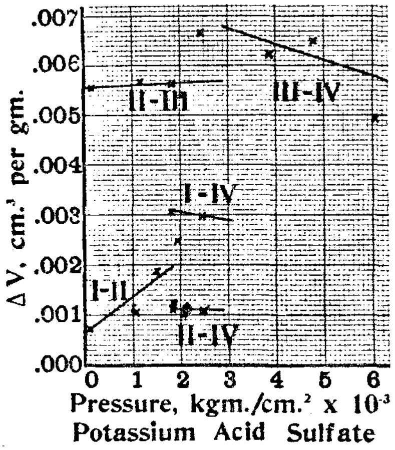

FIGURE 15. Potassium Acid Sulfate. The observed differences of volume

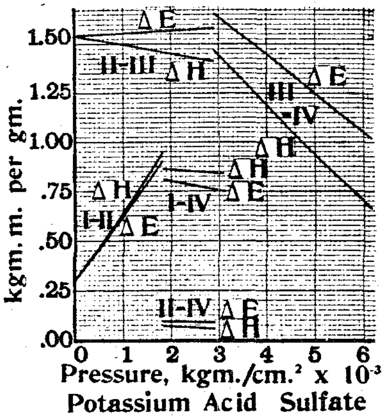

FIGURE 16. Potassium Acid Sulfate. The calculated latent heats and the changes of internal energy.

transition was partly completed. When the transition once started it ran rapidly to completion in less than fifteen minutes. The points shown in Figure 14 are the pressures of spontaneous starting of the transition, from IV to II with decreasing pressure, and from II to IV with increasing pressure. The true equilibrium line lies somewhere in the region limited by these points; there is no reason why it should lie in the center of the region. The actual position of the curve as shown was determined from the conditions of compatibility at the triple points. The curve I–II also has a small change of volume, but it was large enough so that the equilibrium values of pressure and temperature could be found when both phases were present simultaneously. On the line III–IV, which has the largest $\Delta v$ of all, equilibrium points could be found in the regular way. The difficulty with this line is in the rapidly increasing slowness of the reaction toward the lower temperatures. At 50°, the equilibrium value could not be found to better than 750 kgm., even when both phases were present together, and it required more than one hour for the transition to run to apparent completion more than 2500 kgm. below the equilibrium pressure. This slowness doubtless has something to do with the bad $\Delta v$ value at 50° on III–IV. It was quite out of the question to try for lower points on this line; it has been mentioned that at 20° the transition will not start at 12000, and at 50° it required a superpressure of 4000 kgm. to start the transition. The points on the II–III line presented little difficulty.

TABLE VIII.

POTASSIUM ACID SULFATE.

| Pressure | Temperature | $\Delta V$ cm³./gm. | $\frac{d\tau}{dp}$ | Latent Heat kgm.m./gm. | Change of Energy kgm.m./gm. |
|---|---|---|---|---|---|
| | | I–II | | | |
| 1 | 180°.5 | .00066 | .0099 | .302 | .302 |
| 1000 | 190 .4 | .00137 | " | .642 | .628 |
| 2000 | 200 .3 | .00209 | " | 1.000 | .958 |
| | | I–IV | | | |
| 2000 | 201°.5 | .00307 | .0169 | .862 | .801 |
| 3000 | 218 .4 | .00290 | " | .843 | .756 |

*Table VIII, Continued.*

| Pressure | Temperature | ΔV cm³./gm. | $\frac{d\tau}{dp}$ | Latent Heat kgm.m./gm. | Change of Energy kgm.m./gm. |
|---|---|---|---|---|---|
| II–IV | | | | | |
| 1810 | 200°.0 | .00113 | −.075 | .071 | .093 |
| 2075 | 180 .0 | .00112 | " | 68 | 91 |
| 2340 | 160 .0 | .00111 | " | 64 | 90 |
| 2610 | 140 .0 | .00111 | " | 61 | 90 |
| 2875 | 120 .0 | .00110 | " | 59 | 90 |
| II–III | | | | | |
| 1 | 164°.2 | .00556 | −.0158 | 1.54 | 1.54 |
| 1000 | 148 .4 | 561 | " | 1.50 | 1.55 |
| 2000 | 132 .6 | 566 | " | 1.45 | 1.57 |
| 3000 | 116 .6 | 571 | " | 1.41 | 1.58 |
| III–IV | | | | | |
| 3000 | 116°.4 | .0068 | −.0182 | 1.46 | 1.66 |
| 4000 | 97 .5 | .0064 | −.0199 | 1.19 | 1.45 |
| 5000 | 76 .1 | .0061 | −.0229 | .93 | 1.24 |
| 6000 | 51 .4 | .0058 | −.0268 | .70 | 1.05 |
| TRIPLE POINT, I–II–IV | | | | | |
| 1830 | 198°.6 | I–II<br>.00197 | .0099 | .939 | .903 |
| | | II–IV<br>.00113 | −.075 | .711 | .092 |
| | | I–IV<br>.00310 | .0169 | .865 | .809 |
| TRIPLE POINT, II–III–IV | | | | | |
| 2900 | 118°.2 | II–III<br>.00570 | −.0158 | 1.41 | 1.58 |
| | | II–IV<br>.00110 | −.075 | .057 | .089 |
| | | III–IV<br>.00680 | −.0181 | 1.47 | 1.66 |

Because of the smallness of the change and the width of the band of indifference on the lines I–II and II–IV, several of the early measurements of the change of volume were not good and had to be discarded. These comprised three points on II–IV, one on I–II, and one on I–IV. In virtue of the relations at the triple points one can be fairly certain of the values of $\Delta v$ for these lines.

By a curious grouping of the discrepancies of the early measurements of I–II and II–IV, it looked as if possibly there were still another modification with excessively small volume change. Subsequent careful exploration failed to substantiate this surmise.

The order of determination of the points was as follows: first one on I–IV, then several on I–II and II–IV, then the points on III–IV, then II–III, II–IV, I–II, and I–IV again, and finally two points with the low pressure apparatus. The complete run at high pressures with the apparatus set up with one filling and pressure never entirely released extended over seven days; there was no decomposition or change in the $KHSO_4$ in this time as shown by the fact that the last determined I–IV points fall exactly in line with the first.

The point with the low pressure apparatus on II–III does not require especial comment. To obtain the $\Delta v$ value of I–II special procedure was necessary because of the sluggishness of the transition. On passing over the line from II to I, the temperature was raised to 190°, and then lowered, in order to ensure completion of the transition. The sluggishness was so great that it was not possible to obtain an equilibrium point on I–II at low pressures. At 180° the transition from I to II could not be started by a pressure of 1000 kgm., the limit of the apparatus.

There are no previous data for the polymorphic transitions for comparison, the existence of the modifications not having been known before. The melting point of $KHSO_4$ at atmospheric pressure is given as 200° by Mitscherlich[24] and 210° by Schultz-Sellack.[25] I found it to be at any rate higher than 200°.

The accuracy of the measurements does not justify an attempt to compute the difference of compressibility of the four forms. It can be stated, however, that what difference of compressibility there is will be found to be very small. In this connection it should be noted that along the line I–II $\Delta v$ increases with rising pressure, and that along III–IV it falls. Both these effects are unusual. The change along I–II points to a thermal expansion of I abnormally greater than that of II. The direct measurements with the low pressure apparatus showed that the expansion of I is of the order of 0.0005 cm.$^3$ per gm. greater than that of II, and that of II 0.00004 greater than III. On the III–IV line, the indications are that III is considerably more expansible than IV.

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[24] Mitscherlich, Pogg. Ann., **18**, 152 (1830).
[25] Schultz-Sellack, Jahresber., 1871, 217.

It is of interest to notice that the triple point II–III–IV is the first case of its kind that has been found. All three transition lines meeting at this point are abnormal in that the form at the higher temperatures has the smaller volume.

$KHSO_4$ was suggested as a subject for investigation by a remark on page 95 of Groth's Chemical Crystallography. $KHSO_4$ forms with $NH_4HSO_4$ three series of mixed crystals, of different systems, only one of these being known as a possible form for the pure $KHSO_4$. It suggested itself that $KHSO_4$ might be really trimorphic. We have here, at higher pressures, other modifications as suspected, but they are four in number instead of only three. It would be most interesting to find whether either of the new modifications stable at high temperatures at atmospheric pressure is really of the same crystalline system as the mixed crystals of $KHSO_4$ and $NH_4HSO_4$.

AMMONIUM ACID SULFATE.— The reason for investigating this substance was the same as for $KHSO_4$, namely that the existence of other forms is suggested by the fact that under ordinary conditions these two substances form a series of mixed crystals belonging to three different systems. And as a matter of fact, just as in the case of $KHSO_4$, it is found that there are new forms at high pressures. However, there are, instead of the three forms suggested by the mixed crystal relations, demonstrably four forms, and I am morally certain that there is still another. Furthermore, the phase diagrams of $KHSO_4$ and $NH_4HSO_4$ show no obvious relation to each other. The relationship of isopolymorphism between the two substances must evidently be more complicated than is indicated by the mixed-crystals system at atmospheric pressure.

The substance used for this investigation was obtained from Eimer and Amend, and was of the "tested purity" grade. Two separate lots were used, which were practically identical in behavior. The analysis on the bottle showed only a minute trace of impurity. There was, however, considerable absorbed moisture. I removed this as far as possible immediately before use by keeping it in vacuum for six hours in the melted condition (at 160°). In spite, however, of the excellence of the analysis and the precautions to remove moisture, there was still a large amount of impurity left, judging by the lack of sharpness of the freezing point. In fact, this is the most impure substance, judged by this criterion, for which I have ventured to publish data. At 150° it is entirely melted; at 139° it is mushy, perhaps $\frac{1}{10}$ melted, and even as low as 110°, it is perceptibly moist with the remnants of the melt. It may be of course that even the absolutely pure substance does not behave at the melting point like a simple substance; it might be, for example, that there is a reversible dissociation into something like $(NH_4)_2SO_4$ and $H_2SO_4$, although this precise dissociation is not likely.

Partly as a consequence of the width of the domain of melting, and partly as a result of the smallness of the change of volume, the data obtained for this substance are unsatisfactory in many respects. The changes of volume are probably in error by large amounts. These changes as measured fall far short of satisfying the additive conditions at the triple point I–II–III. Furthermore, the additive conditions for ΔH are far from being satisfied at this point. The way in which the observed data should be adjusted so as to satisfy this condition does not readily suggest itself. I have preferred, therefore, not to try at all to deduce the latent heats and the changes of energy from the data, and in Table IX have tabulated only the equilibrium pressures

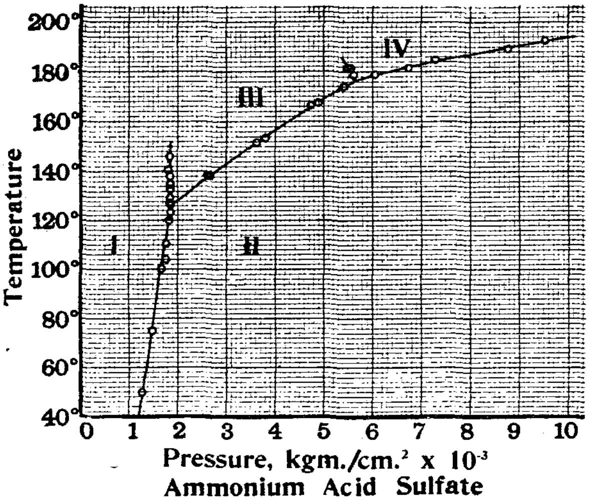

FIGURE 17. Ammonium Acid Sulfate. The observed equilibrium pressures and temperatures.

and temperatures and the sometimes very uncertain values for the changes of volume.

The phase diagram is shown in Figure 17, the measured changes of volume in Figure 18, and the numerical results in Table IX. The phase diagram itself may probably be accepted as substantially correct. The results could be repeated, and there was no rounding of the corners of the transition curves, making it unlikely that the impurity or chemical dissociation affects the transition pressures and temperatures. The approximate location of the melting curve may be estimated from the fact that points at higher temperatures than those shown on the lines I–III and III–IV were attempted, but were not possible because of the closeness of the melting.

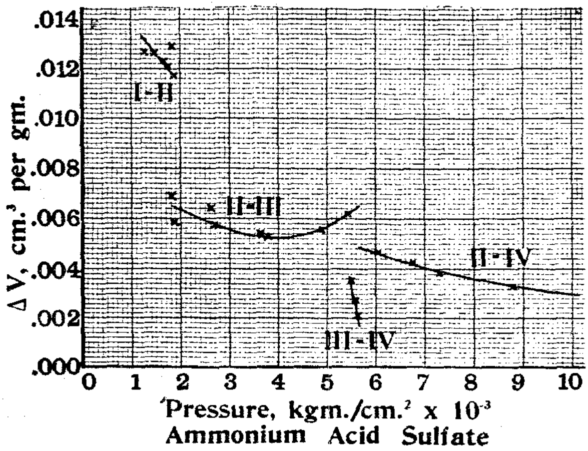

FIGURE 18. Ammonium Acid Sulfate. The observed differences of volume.

In addition to the phases shown, it is almost certain that the II–IV line should split at higher pressures with the appearance of a fifth phase. It was practically impossible to determine the exact location of the transition, because of the very small change of volume, but the existence of the transition was made practically certain in the following way. At room temperature pressure was raised to about 11000 and then temperature was raised to 198.5°, the pressure rising to something over 12000. Then the pressure was lowered at constant temperature, but I could find no point in the expected place (11200) on the II–IV

## TABLE IX.

### AMMONIUM ACID SULFATE.

| Pressure | Temperature | ΔV cm³./gm. |
|---|---|---|
| | I–II | |
| 1220 | 40° | .01330 |
| 1370 | 60 | 1295 |
| 1520 | 80 | 1259 |
| 1670 | 100 | 1224 |
| 1810 | 120 | 1188 |
| | I–III | |
| 1860 | 130° | .00529 |
| 1860 | 150 | |
| | II–III | |
| 2000 | 128°.4 | .00635 |
| 3000 | 143 .4 | 555 |
| 4000 | 156 .9 | 524 |
| 5000 | 169 .3 | 569 |
| | III–IV | |
| 5650 | 177°.0 | .00168 |
| 5530 | 181 .0 | 265 |
| | IV–V | |
| 6000 | 178°.3 | .00466 |
| 7000 | 183 .1 | 406 |
| 8000 | 187 .1 | 360 |
| 9000 | 190 .6 | 325 |
| 10000 | 193 .8 | 300 |
| | TRIPLE POINT, I–II–III | |
| 1860 | 126°.2 | |
| | TRIPLE POINT, II–III–IV | |
| 5660 | 176°.9 | |

line. Thinking that by some careless oversight I might have run across the II–IV line while raising temperature, I lowered the temperature at 8000 to 140°, which is far enough to compel the transition from IV to II. This procedure was made necessary because on the transition line II–IV the superpressure required to force IV to change to II is so great that it cannot be reached in the limits of this work. The reverse transition, II to IV, however, runs with little transgression of the transition line. After lowering temperature to 140°, I then raised the pressure to 12000 and the temperature to 198.5°, the pressure rising to 12500. The rise of pressure accompanying this last change of temperature was larger than usual. Furthermore, as judged by the rise of pressure, the process of attaining temperature equilibrium extended over an unusually long interval of time, and then suddenly the supposed process of temperature equalization ceased. The result of all this manipulation was that, provided the II–IV line continues as it starts, the phase II must certainly have been in the apparatus at 12500 kgm. and 198.5°. Now on lowering pressure the transition to IV should have been found at about 11200. No such transition was found, however, down to 7500. The explanation, of course, that suggests itself is that the line II–IV does not continue as it starts, but encounters a triple point and splits into two. The apparent sluggishness in reaching temperature equilibrium was really due to the new transition. The point 12500 and 198.5° is therefore in the domain of the new phase, V, and the point at 7500 and 198.5° is doubtless in the domain of IV, the transition from IV to V not being noticed because of the smallness of the change of volume. This was verified by lowering temperature at 7500 from 198.5° to 181.6°, and then lowering pressure further. The transition IV–III was found in the location expected.

Figure 18, for $\Delta v$, requires some comment. The points on II–IV lie smoothly and normally, and the results are probably near the truth. The shape of the $\Delta v$ curve for II–III is unusual in the marked rise on approaching the triple point II–III–IV. This rise, however, is consistent with the marked drop of $\Delta v$ for III–IV with falling temperature. The rise and the drop both mean the same thing, that near the triple point the thermal expansion of III becomes unusually large. Furthermore, the curves for II–III, III–IV, and II–IV satisfy the additive condition at the triple point without forcing. It is probable that affairs are really as measured, and that III does increase markedly in expansion near the triple point. At the triple point I–II–III, however, the state of affairs was not nearly so satisfactory. The change of volume on I–II was not so consistent as it should be, considering the rapidity and sharpness of the reaction. On the line I–III it was not possible to obtain any consistent measurements at all. Four measurements were made, which varied irregularly with temperature from 0.0032 to 0.0078. These points are not shown in the figure, and in the table the listed value was obtained from the conditions at the triple point. This value is evidently uncertain because of the lack of agreement of the two points on the lower end of II–III. Besides these four points, on the I–III line, the only other points not shown in the figure are two of the higher pressure points obtained with the first set-up with a patched cylinder, which later developed a perceptible leak.

The reaction velocity phenomena were as follows. On the line I–II the transition is rapid and easy to measure near the triple point, but becomes so rapidly slower with falling temperature that it did not pay to try for points below 50°. On the I–III line the transition is rapid, in both directions, but is strikingly more rapid with falling pressure. The transition II–III is slow over the entire length of the curve, but III–IV is rapid. It has already been mentioned that on the II–IV line the transition in the direction from II to IV runs easily, although very slowly, with falling pressure, but that the reverse transition from IV to II cannot be forced within the limits of the apparatus at the higher temperatures. About 5000 kgm. superpressure is necessary.

The unusually large error in the results makes it of no use to try for $\Delta\alpha$, $\Delta\beta$, or $\Delta C_p$.

Since these other forms were not known before, there are no other values for comparison. Even the melting point is not listed; it is without doubt greatly affected by small quantities of moisture.

Figure 19 suggests the simplest conceivable mixed crystal diagram for $KHSO_4$ and $NH_4HSO_4$ at atmospheric pressure that will explain the three known series of crystals at room temperature. Whether this surmise is really correct or not cannot be verified until the crystalline forms of $KHSO_4$ at atmospheric pressure have been determined. But in any event, the possibility of so simple a diagram shows that there is no necessary connection between the new high pressure modifications of $NH_4HSO_4$ and those of $KHSO_4$. In fact, the entire dissimilarity of the phase diagrams shows that such a connection is not likely.

Cuprous Iodide.— Two lots of this material were used; the first was 55 gm. from Eimer and Amend, with which the existence of a transition was discovered, and the second lot was of 100 gm. from Hoffmann and Kropff, with which the final measurements were made.

It was hammered cold into an open steel shell, and pressure transmitted directly to it by kerosene.

There are two modifications; the transition is of the ice type, and within the temperature range of this work the second phase exists only at the higher pressures. The experimental values of pressure and temperature and change of volume are shown in Figure 20, the computed values of latent heat and change of internal energy in Figure 21, and the collected numerical results in Table X. The diagram seems normal in every way. This is one of the few examples found in this

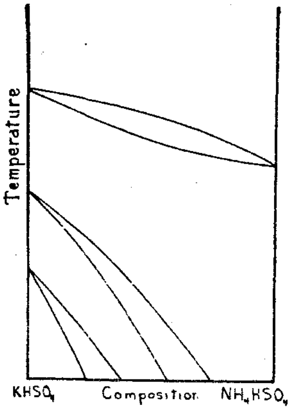

Fig. 19.

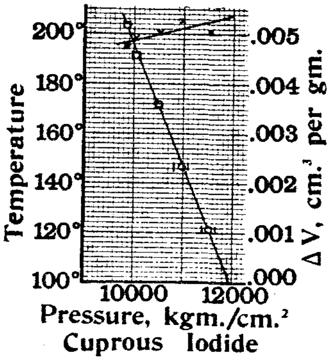

Fig. 20.

FIGURE 19. Possible mixed crystal diagram between Ammonium Acid Sulfate and Potassium Acid Sulfate.

FIGURE 20. Cuprous Iodide. The observed equilibrium pressures and temperatures (circles) and the observed differences of volume (crosses).

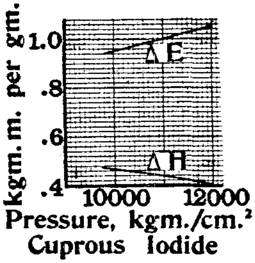

FIGURE 21. Cuprous Iodide. The calculated latent heat and the changes of internal energy.

investigation of a substance showing a new phase at 200° and not at 20°.

The transition is singular in that there is a region 150 to 250 kgm. wide within which the transition velocity is very slow, whereas outside of this band the velocity increases with unusual rapidity. At any given temperature the equilibrium point was found by artificially changing pressure until a point was found at which the pressure did not change by as much as 0.5 kgm. in 5 minutes. These stationary points are the equilibrium points shown in the diagram. Points were also found on each side of the equilibrium points at which the transition was perceptibly running in opposite directions. These points are also indicated in the diagram.

TABLE X.

CUPROUS IODIDE.

| Pressure | Temperature | $\Delta V$ cm³./gm. | $\frac{d\tau}{dp}$ | Latent Heat kgm.m./gm. | Change of Energy kgm.m./gm. |
|---|---|---|---|---|---|
| 11950 | 100°.0 | .00535 | −0.04926 | .405 | 1.045 |
| 11544 | 120 .0 | 525 | " | .419 | 1.025 |
| 11138 | 140 .0 | 515 | " | .432 | 1.006 |
| 10732 | 160 .0 | 505 | " | .444 | .985 |
| 10326 | 180 .0 | 495 | " | .455 | .966 |
| 9920 | 200 .0 | 485 | " | .466 | .947 |

Within the limits of error the transition line is straight. It would extrapolate to a transition point at atmospheric pressure at 690°. The melting point of this substance is listed at 638°. Gossner[26] makes the statement that, on heating, the ordinary cubic form of $Cu_2I_2$ changes to a doubly refracting form. He does not state the temperature of transition, except to say that it is very high, but it must probably be considerably lower than 638° if the optical observation is to be possible at all. It is therefore improbable that the phase found above is the same as that mentioned by Gossner. There must be at least three modifications. From the meagre data at hand a phase diagram in its general features like that of AgI does not seem improbable.

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[26] B. Gossner, ZS. Kryst., **38**, 110 (1903).

The difference of compressibility can be determined in the usual way from the difference of slope of the isothermals above and below the transition. All five determinations gave fairly consistent results. The low temperature form is less compressible, and a fair average for the difference is $0.0_64$ cm.$^3$ per gm. per kgm. The measurements were not accurate enough to give the variation of this along the transition line. Using this value for $\Delta\alpha$, we may find as average values over the entire range for $\Delta\beta$, 0.00006, and for $\Delta C_p$, 0.34 kgm. cm. (0.0014 gm. cal.). The high temperature form is the more expansible and has the higher specific heat. The relations of the two phases, so far as the signs of $\Delta\alpha$, $\Delta\beta$, and $\Delta C_p$ go, is exactly that of water and ordinary ice.

Ammonium Iodide.— This substance has a new modification, which is striking because of the large change of volume, about 14%. The transition line runs steeply, with pronounced concavity toward the temperature axis, and crosses the line of atmospheric pressure at about −17.6°. It should, therefore, be comparatively easy to study this form under atmospheric conditions.

In all, three different sets of runs were made. The first set was for purposes of exploration. No transition at 20° was found between 12300 and 3500 kgm., and at 200° the only transition found up to 12300 was at approximately 2000. The second set included the measurements with the high pressure apparatus, and included all points between 20° and 200°. The material was from Hoffmann and Kropff, dried in vacuum before use, and hammered cold into the steel shell. Pressure was transmitted directly by kerosene. The third run with the low pressure apparatus, by the method of varying pressure at constant temperature, gave the point at 0°. The substance for this run was obtained from Eimer and Amend, and was used directly without any preparation. It contained slight traces of moisture, and judging from its light yellow color, it could not have been quite so pure as the previous sample.

The experimental values of pressure and temperature and of the change of volume are shown in Figure 22, the calculated values of the latent heat and the change of internal energy in Figure 23, and the numerical values are collected in Table XI.

At 200° the velocity of transition is very high, but it becomes less at lower temperatures until at 50° the velocity is so low that a value of $\Delta v$ too low was obtained, the reaction appearing to have stopped before it really had. At 25° an attempt was made to avoid this source of error by raising the temperature to 100° in the middle of the run to ensure completion of the transition, and then bringing it back again. The attempt seems to have been not entirely successful. At 0° the pressure was lowered to atmospheric pressure for an hour to ensure completion, with apparently satisfactory results.

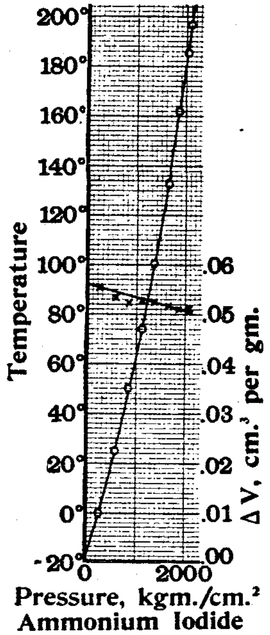

FIGURE 22. Ammonium Iodide. The observed equilibrium pressures and temperatures (circles) and the observed differences of volume (crosses).

At the two upper points of the curve there was a very slight rounding of the corners noticeable at the last reading, about 180 kgm. before the transition. The behavior was not like that usually found for an impure substance, and there is therefore a slight possibility that there may be a third phase, the transition line splitting in the neighborhood of 160° into two lines diverging at a very slight angle. If such is the case, the coördinates above are for the left hand of the two branches.

The amount of transgression of the transition line possible before the transition starts is a matter of 100 or 200 kgm. at the lower temperatures. A greater degree of subpressure than of superpressure may be supported.

The direct measurement of the difference of compressibility of the two phases did not give results so accurate as are sometimes obtained, but the conclusion is fairly safe that the difference is not greater than $\pm 0.0_51$ cm.³ per gm. per kgm. Using these limits for $\Delta\alpha$, we find fairly close limits for the quantities $\Delta\beta$ and $\Delta C_p$ when calculated in the usual way. Over the entire length of the curve both $\Delta\beta$ and $\Delta C_p$ are negative on either assumption for $\Delta\alpha$; that is, the high temperature phase is less expansible and has the lower specific heat. At 100 kgm. (−8°) $\Delta\beta$ lies within the limits $-0.0_426$ to $-0.0_454$ cm.³ per gm. per degree Centigrade, and $\Delta C_p$ between −0.023 and −0.026 cal. per gm. At 2053 kgm. (190°) the limits of $\Delta\beta$ are $-0.0_58$ to $-0.0_421$, and for

$\Delta C_p$ −0.013 to −0.014. The closeness of these limits for $\Delta C_p$ is unusual, and we may accept the value with some confidence.

It is interesting that the existence of a second modification has been suspected by a number of observers because of the behavior of the mixed crystals with $NH_4Br$ and $NH_4Cl$. In fact, Wallace [27] made a search down to −16° without result; if he had gone only two degrees farther he would have found what he was looking for.

Ammonium Bromide.— This was obtained from Eimer and Amend, U. S. P. Before use it was dried in vacuum at 100°. Whatever impurity there may be present does not form mixed crystals and so affect the transition point, because the transition was always very sharp and rapid, with no rounding of the corners whatever. Very little transgression of the transition line is possible before the transition starts; the maximum observed was a superpressure of 30 kgm.

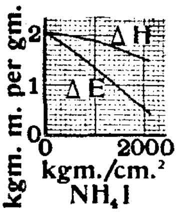

Figure 23. Ammonium Iodide. The calculated latent heat and the change of internal energy.

The equilibrium pressure was shut within limits which were usually 3 kgm. apart, and in the extreme case only 6 kgm. apart. The transition line is very steep; it runs from about 138° at atmospheric pressure to 680 kgm. at 200°. All the measurements had, therefore, to be made with the low pressure apparatus. Two sets of readings were made with the same sample. The repetition was necessary because the first set of $\Delta v$ points was irregular. There seems to be some sort of internal strain set up by the transition which it is necessary to remove by seasoning. For the second run the transition was carried backward and forward a number of times before any measurements were made. The agreement of the pressure-temperature points of the two runs was very close. The substance was placed in the open steel shell, and pressure transmitted to it by kerosene.

The observed equilibrium points and the changes of volume are shown in Figure 24, the computed values for the latent heat and the change of internal energy in Figure 25, and the numerical values are collected in Table XII. Both sets of pressure-temperature points are shown, but only the $\Delta v$ values of the second set of determinations. The direction of curvature of the transition line, convex toward the pressure axis, is noteworthy. It is in the same direction as for $NH_4I$.

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[27] R. C. Wallace, C. Bl. f. Min. 1910, p. 33.

It was known previously that $NH_4Br$ has a second modification. Its existence was first suspected from the behavior of the mixed crystals with $NH_4I$, and was afterward established by Wallace[27] by microscopic examination. No satisfactory measurements of the physical constants are extant, however. Wallace was able to state only that the high temperature modification has the larger volume. Not only is this true, but the change of volume is one of the highest known for solids. That Wallace did not comment on this shows the difficulty of making quantitative estimates of the change of volume from any alteration of appearance. I can verify this from my own experience. The transition temperature found by Wallace was 159° against 137.8° above. There would seem to be no question but that

TABLE XI.

AMMONIUM IODIDE.

| Pressure | Temperature | $\Delta V$ $cm^3./gm$ | $\frac{d\tau}{dp}$ | Latent Heat kgm.m./gm. | Change of Energy kgm.m./gm. |
|---|---|---|---|---|---|
| 1 | −17°.6* | .0561* | .0699* | 2.05 | 2.05 |
| 245 | 0 .0 | 554 | 748 | 2.02 | 1.89 |
| 503 | 20 .0 | 547 | 807 | 1.99 | 1.71 |
| 742 | 40 .0 | 540 | 870 | 1.94 | 1.54 |
| 964 | 60 .0 | 534 | 938 | 1.89 | 1.38 |
| 1171 | 80 .0 | 528 | .1012 | 1.84 | 1.22 |
| 1362 | 100 .0 | 523 | .1096 | 1.78 | 1.07 |
| 1537 | 120 .0 | 518 | .1189 | 1.71 | .92 |
| 1698 | 140 .0 | 514 | .1287 | 1.65 | .78 |
| 1848 | 160 .0 | 510 | .1388 | 1.59 | .65 |
| 1988 | 180 .0 | 507 | .1490 | 1.54 | .54 |
| 2118 | 200 .0 | 504 | .1593 | 1.50 | .43 |

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* Extrapolated.

the lower value is the more accurate, when one considers the contrast in the methods. The lower value was obtained with both phases in contact, and is the average of results differing only 3 kgm. for opposite directions of the transition. Wallace's value was obtained with

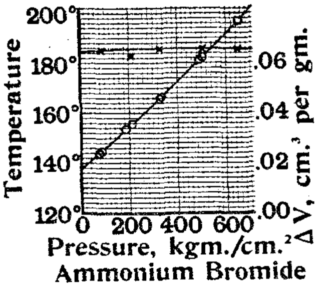

Fig. 24.

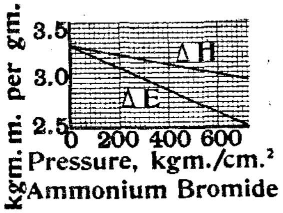

Fig. 25.

FIGURE 24. Ammonium Bromide. The observed equilibrium pressures and temperatures (circles) and the observed differences of volume (crosses).

FIGURE 25. Ammonium Bromide. The calculated latent heat and the change of internal energy.

TABLE XII.

AMMONIUM BROMIDE.

| Pressure | Temperature | ΔV cm³./gm. | $\frac{d\tau}{dp}$ | Latent Heat kgm.m./gm. | Change of Energy kgm.m./gm |
|---|---|---|---|---|---|
| 0 | 137°.8 | .0647 | .0800 | 3.32 | 3.32 |
| 100 | 146 .0 | 652 | 834 | 3.28 | 3.21 |
| 200 | 154 .5 | 656 | 869 | 3.23 | 3.10 |
| 300 | 163 .4 | 659 | 903 | 3.18 | 2.99 |
| 400 | 172 .4 | 660 | 937 | 3.14 | 2.87 |
| 500 | 181 .9 | 659 | 972 | 3.09 | 2.76 |
| 600 | 191 .8 | 658 | .1006 | 3.04 | 2.65 |
| 700 | 202 .0 | 655 | .1040 | 2.99 | 2.53 |

the microscope, in which temperature control is not so easy, and was not an equilibrium measurement between two phases simultaneously present. I have already stated that the character of the transition rules out the possibility of there having been any effect exerted by impurities on the data given here.

Ammonium Chloride.— This was Kahlbaum's purest, and was dried in vacuum at 100° before using. Its behavior is very similar to that of $NH_4Br$. The transition line is also steep, and the transition temperature at atmospheric pressure is higher than that of $NH_4Br$. All the measurements were made in the low pressure apparatus, and only three points were determined, up to 203° and 275 kgm. The reaction is just as sharp as for $NH_4Br$, and the pressure limits no wider; in fact it would be difficult to imagine two substances more alike in the character of the transition.

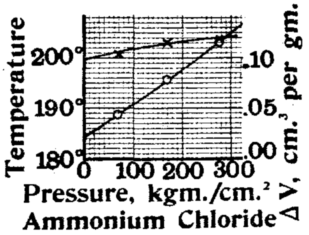

Fig. 26.

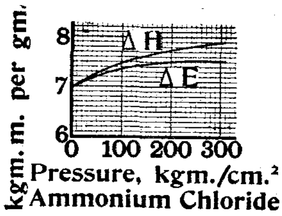

Fig. 27.

Figure 26. Ammonium Chloride. The observed equilibrium pressures and temperatures (circles) and the observed differences of volume (crosses).

Figure 27. Ammonium Chloride. The calculated latent heat and the change of internal energy.

The observed equilibrium pressures and temperatures and the changes of volume are shown in Figure 26, the computed values of the latent heat and the change of internal energy are shown in Figure 27, and the numerical values are collected in Table XIII. The transition curve is distinctly convex downward, as it is also for $NH_4I$ and $NH_4Br$. The $\Delta v$ curve is concave downward. Such is also the case for $NH_4Br$, but in the case of $NH_4Cl$ the $\Delta v$ curve has not yet reached the maximum in the range shown, whereas the corresponding curve for $NH_4Br$ does pass through a maximum in the range of the experiment.

It has been known previously that $NH_4Cl$ also has a second form at high temperature which has a greater volume than the low temperature form. Wallace [27] gives 159 ° for the transition point, against 184.3° above. There is no previous measurement of $\Delta V$.

Further comment will be made on the behavior of these three halogen compounds under the discussion by groups. This is the best example of a thorough going isopolymorphism that I have found.

THE ANTIMONY SULFIDES.— Antimony may form either the trivalent or the pentavalent sulfide, $Sb_2S_3$ or $Sb_2S_5$. $Sb_2S_3$ exists in two distinct modifications; a black form occurring in nature, and a red form, said to be amorphous, which is formed chemically by precipitation. The red modification may be changed to the black at atmospheric pressure by heating to about 200°. $Sb_2S_5$ has also apparently two forms; the commercial form is a bright orange powder, but when made in the laboratory by precipitation with $H_2S$ from solution it is red in color, very much like the red $Sb_2S_3$. The materials used in this investigation were obtained from two sources; the black $Sb_2S_3$ and the orange $Sb_2S_5$ were from Eimer and Amend, and the red $Sb_2S_3$ and red $Sb_2S_5$ were made especially for me at the Chemical Laboratory of Harvard University.

TABLE XIII.

AMMONIUM CHLORIDE.

| Pressure | Temperature | ΔV cm³./gm. | $\frac{d\tau}{dp}$ | Latent Heat kgm.m./gm. | Change of Energy kgm.m./gm. |
|---|---|---|---|---|---|
| 0 | 184°.3 | .0985 | .0645 | 6.98 | 6.98 |
| 100 | 190 .9 | .1087 | 675 | 7.47 | 7.36 |
| 200 | 197 .8 | .1160 | 705 | 7.73 | 7.50 |
| 300 | 205 .0 | .1212 | 735 | 7.88 | 7.52 |

The behavior of $Sb_2S_3$ will be first described. The black modification shows no peculiarity; there is no new form to 12000 at 20° or 100° or 200°. The red $Sb_2S_3$ changes to the black $Sb_2S_3$ on heating in a peculiar way. Up to temperatures of 150° and pressures up to 12000 kgm. no change in the red is produced; this was verified by taking the apparatus apart after exposure to these values. If red $Sb_2S_3$ is heated at 12000 kgm. however, some sort of change takes place between 150° and 200° accompanied by a decrease of volume. If after exposure to 200° at 12000 kgm. the apparatus is cooled, still under 12000 kgm., and then opened, it will be found that the change from red to black $Sb_2S_3$ has not taken place completely; the mass is mostly transformed, but there is still an appreciable quantity of the red left throughout the mass, and particularly on the surfaces. The complete transition from red to black $Sb_2S_3$ (which of course is irreversible) takes place in two stages; the first stage is as just described at a pressure of 12000 kgm. somewhere between 150° and 200°, and the second stage takes place on releasing pressure from 12000 kgm. at the constant temperature of 200° at a pressure of about 10000 kgm. This second stage in the transition is also accompanied by a drop of pressure, that is, by a decrease of volume. I have verified by trial that this second stage in the decomposition is irreversible, as indeed it must be. This second drop is remarkable; if the second stage in the transition were purely a pressure effect, thermodynamics shows that it must be accompanied by an increase of volume, instead of a decrease. The observed effect must be due to friction; the highest pressures so increase the frictional resistance to the second stage of the transition that it cannot run, but as pressure decreases the frictional resistance falls rapidly until it is low enough for the transition to proceed.

In addition to the irreversible change to the black $Sb_2S_3$ at high temperatures, red $Sb_2S_3$ shows a reversible transition at lower temperatures. Several measurements of this reversible transition were made. The transition line runs from 7800 kgm. at 0° to 12000 kgm. at 32°, and the change of volume is approximately 0.010 cm.$^3$ per gm. It was not possible, however, to make any very accurate measurements of this transition. The reaction is sluggish, and there is some impurity present, as shown by the rounding of both corners of the $\Delta v$ curves, an unusual feature. It seems that it is not possible to obtain $Sb_2S_3$ entirely pure, but it always contains some free S and $Sb_2 S_5$ and other sulfides. The result above shows that this impurity is soluble in both phases. It is not likely, therefore, that any chemical refinement would give a much more satisfactory material, and I did not try for further more accurate measurements on the transition curve.

The importance of the existence of this second modification of red $Sb_2S_3$ is to be insisted on, however, because it shows that ordinary red $Sb_2S_3$ cannot be amorphous, but must be crystalline. No substance is known with a transition of an amorphous phase. The universal opinion that $Sb_2S_3$ is amorphous is doubtless due to its appearance, which is as a precipitate, too fine to show any crystalline structure. Tammann, however, states that amorphous red $Sb_2S_3$ may be obtained by rapidly cooling the melt of black $Sb_2S_3$. We see now, however, that the subcooled red $Sb_2S_3$ cannot be a glass, but is another crystalline modification formed from the subcooled melt.

The system of the two modifications of $Sb_2S_3$ now becomes strikingly like that of red and white phosphorus. Red $Sb_2S_3$ corresponds to white phosphorus, and black $Sb_2S_3$ to red phosphorus. Red phosphorus and black $Sb_2S_3$ are both the absolutely stable forms, but there is frictional resistance to change so that the unstable forms will not change to the stable forms, even when inoculated, at low temperatures. The unstable white phosphorus and the red $Sb_2S_3$ furthermore each have a second modification, and the transition curves to these new modifications are very similar in location for the two substances. A careful study of the system $Sb_2 S_3$ would be interesting, both for its own sake, and for the light which it might throw on the vexed question of the relation of the forms of phosphorus.

The results obtained with the two forms of $Sb_2S_5$ are curious and interesting, but have little real value because it is not possible to produce an $Sb_2S_5$ which is even approximately pure; the pure substance probably does not exist, but spontaneously decomposes into other sulfides and free sulfur. I have found the statement that under usual chemical methods of manufacture 40% of $Sb_2S_5$ is a high proportion. The behavior of the two specimens of $Sb_2S_5$ becomes intelligible from this point of view. The red modification is one in which there is a large proportion of red $Sb_2S_3$. At low temperatures it shows the same transition as the red $Sb_2S_3$, but with corners considerably more rounded, and the transition is at a somewhat higher pressure. This is as one would expect as the result of an increased proportion of dissolved impurity. On heating to 200° under a pressure of 12000 kgm. and then releasing pressure at 200°, decomposition takes place of what $Sb_2S_5$ there is to black $Sb_2S_3$ and free S. At approximately 4000 kgm. and 200° the melting point of the free S is reached. This gave the appearance of some sort of a new transition until the true nature of the effect was discovered. The presence of free sulfur may be easily proved by digesting with $CS_2$, or in some cases acicular crystals of sulfur several mm. long were formed. The melting point of the sulfur is much rounded, showing that the $Sb_2S_3$ is soluble in sulfur, as one might expect. The appearance of the $Sb_2S_3$ remaining after the decomposition is black, exactly like that of red $Sb_2S_3$ subjected to the same treatment. It is noteworthy that the decomposition in two stages, particularly the decrease of volume at 10000 kgm. at 200° could not be detected.

The orange $Sb_2S_5$ behaves as if it contained a larger proportion of

$Sb_2S_5$. At room temperature it shows no transition at all to 12500 kgm.; probably the impurity of $Sb_2S_3$ is so effectively dissolved in the other components that its transition is suppressed. On raising temperature at 12000 on the orange $Sb_2S_5$ a partial decomposition takes place with decrease of volume, and on releasing pressure at 200° the decomposition is completed near 5000 with further decrease of volume. This second point at 5000 was verified by repetition with another sample. On further release of pressure to 4000 the melting point of the free sulfur is reached. The end product is free sulfur and black $Sb_2S_3$, just as in the previous case. It is curious that the manner of decomposition of the orange $Sb_2S_5$ should be more like that of the red $Sb_2S_3$ than that of the red $Sb_2S_5$, although the latter doubtless contains a larger proportion of red $Sb_2S_3$ than the orange $Sb_2S_5$. The most obvious difference between the decomposition of red $Sb_2S_3$ and orange $Sb_2S_5$ is that with the latter the decrease of volume at 200° occurs at a considerably lower pressure, 5000 against 10000. Probably the explanation is partly to be found in complicated mixed crystal relations.

No further attempt was made to straighten out the relation of these various sulfides; a complete investigation would be a matter of great difficulty, demanding first of all accurate chemical analysis of the various substances. As such, the subject is beyond the scope of this investigation.

DOUBLE SULFATE OF ZINC AND POTASSIUM. [$K_2Zn(SO_4)_2$].—This was obtained from Eimer and Amend. Analysis showed Fe, .0004%; Na, none; chloride, none. The salt crystallizes from solution with six molecules of water. In this form it shows no new modification to 12000 kgm. at 20° or 100°. The substance was also examined in the anhydrous condition; the water was removed by heating in vacuum to 140° for 1½ hours. The anhydrous salt also has nothing new to 12000 at room temperature, but does have a transition at higher temperatures. The transition is well marked. Figure 28 shows the relation between pressure and temperature. The general order of magnitude of the change of volume is 0.015 cm.$^3$ per gm., but there were various irregularities. Once or twice an appearance was found as of a second small transition like a satellite of the main transition. The curves of volume against pressure at constant temperature present an appearance which one might at first take for the usual rounding in the presence of an impurity, but more careful reading shows not a rounding but a sharp break in the direction of the isothermal above the main transition. This is shown in Figure 29.

In view of the known chemical behavior of such double salts, there seems little question as to the proper interpretation. We have here not a polymorphic transition in the proper sense of the word, but a reversible decomposition of the double salt into two simple salts. The decomposition is further complicated by the formation of mixed crystals between the various products. The correctness of this surmise would be at once verified if the extrapolated transition at atmospheric pressure agreed with the known temperature of decomposition. I have not been able to find, however, that this point has ever been determined; the decomposition point of the hydrous salt is well known, but no work seems to have been done on the anhydrous salt.

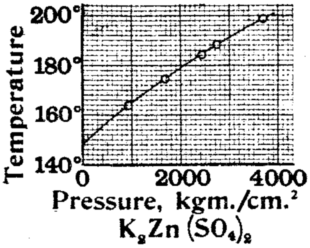

Fig. 28.

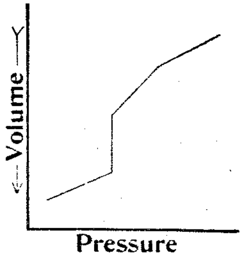

Fig. 29.

FIGURE 28. Zinc Potassium Double Sulfate. The observed equilibrium pressures and temperatures. This transition is probably a decomposition of the double salt into two simple salts.

FIGURE 29. Zinc Potassium Double Sulfate. Shows the general nature of the change of volume.

If this explanation is correct, it removes this substance at once from the range of this work, and I did not try for further data. It is obvious, however, that there is here an immense field, as yet untouched. For instance, it would be of great interest to examine the other double sulfates of this series. The careful work of Tutton shows that the hydrous salts of the series present remarkable similarities. I am not aware that any measurements have ever been made on the decomposition of double salts, or on mixed crystals, under pressure. The work above showed that accurate measurements on mixed crystal equilibria under pressure is going to involve very tedious manipulation, because diffusion takes place so slowly.

THALLIUM.— This substance was procured for the purpose of making the nitrate from it; the measurements on the pure metal were undertaken only because it happened to be on hand, and because it is known to be polymorphic. Only a small quantity was available; I procured all there was to be obtained in this country — 3 ounces from Merck and 2 ounces from Eimer and Amend. This was only a quarter of the amount required to fill the apparatus.

The transition point at atmospheric pressure to the second modification is known to be at about 225°, and the transition is of the ice type. The effect of pressure has been investigated by Werner[28] up to 3000 kgm. He found that the slope of the transition line is only $-2°$ for 1000 kgm., and that the change of volume is excessively small, much too small to allow accurate measurements in the present apparatus with the small quantity of material available. It should, however, be easy to detect the presence of the transition. The particular interest of this present examination is in finding whether there is not a third modification, giving a diagram like that of AgI. The change of volume to this third modification would be larger than that between the two known forms and should be readily detectible if present. The principal result obtained by my work is that there is no third modification up to 12000 at 200°. At higher temperatures the presence of the transition line investigated by Werner could be detected, and his conclusions verified within my wide limits of error. My data are not inconsistent with his value for the slope of the curve, and my value for the change of volume agrees with his at 2500 kgm. My data do not indicate, however, the unusually large increase of $\Delta v$ found by him toward the high pressure end of the line, but would indicate approximate constancy of $\Delta v$. The pressure limits of indifference to the transition are wide, and become rapidly wider at the higher pressures. Werner's data indicate the same thing, although his measurements were made with increasing temperature at nearly constant pressure, whereas mine were made at constant temperature.

Ammonium Potassium Phosphate [$(NH_4)_2KPO_4$].—This substance was obtained from Eimer and Amend, "tested purity," and showed Cl 0.0005%, Fe 0.0002%, with traces of $SO_3$ and CaO. It crystallizes with four molecules of water. For this experiment the water was removed by heating to 100° in vacuum for two hours. The dried salt was then hammered cold into an open steel shell, and pressure transmitted directly to it by kerosene.

There is a transition to a new form at high pressures and at temperatures in the neighborhood of 100°, but in this vicinity the substance is apparently unstable, so that it was not possible to make any accurate measurements. About all that I could do was to determine the general character and location of the transition. Two runs were made. The first was an exploring run; it showed the existence of a new modification with a transition near 11000 kgm. at 100°. No transition was found at room temperature to beyond 12000. Presumably, therefore, the transition is of the ice type. The second run, for purposes of measurement, verified the existence of the transition, and gave for the more accurate coördinates at 100°, 11480 kgm., with a change of volume of about 0.0035 cm.$^3$ per gm. Temperature was then lowered to 86°, and the transition pressure found to be at least as high as 11800, verifying that the transition is of the ice type. The apparatus was now left over night. The next morning at 128°, instead of one transition point, two were found, at 9600 and 8900 kgm. The transition line has split, and there are three modifications. But a disquieting fact was that the total change of these two new transitions was less than half that found for the single transition the day before. These two transitions were verified by repetition. Temperature was now raised to 141°, and again two transitions found at still lower pressures, at very roughly 8000 and 8700. The apparatus was again allowed to rest, this time over Sunday. On Monday morning temperature was raised to 90° and pressure to 12500, preparatory to a run. It took an unusually long time to reach pressure equilibrium, the pressure continuing to drop. After approximate equilibrium had been reached, no trace of the transition was to be found within 1800 kgm. of the previous location, and again on raising temperature to 100° the transition had entirely disappeared.

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[28] M. Werner, ZS. anorg. Chem., **83**, 275 (1913).

The evident explanation of these effects is that $(NH_4)_2KPO_4$ has three modifications, with transition coördinates approximately as given. But in the region in which the transition runs the substance is itself chemically unstable, and gradually changes to some other substance with decrease of volume. The fact that the change of volume is a decrease shows that the instability has been brought about by the high pressure and not by the high temperature. The change is not accompanied by change of color, nor setting free of $NH_4$, to judge by the lack of odor.

It is unfortunate that the decomposition does not allow closer study of these transitions, because they are of a rather rare type. At the triple point it seems that three lines, all of the ice type, come together. The only previous example of this is $KHSO_4$.

It is quite possible that the decomposition is a splitting up into two simple salts, as suspected for $K_2Zn(SO_4)_2$. The apparent irreversibility may be simply an effect of diffusion, which must be very slow at high pressures.

POTASSIUM BINOXALATE. — This was obtained from Eimer and Amend. It was hammered cold into the open steel shell, and pressure transmitted directly to it by kerosene. It crystallizes with one molecule of water. This, like $(NH_4)_2KPO_4$, is a substance with a new modification in a region where it is chemically unstable, so that it was not possible to more than establish the existence of the transition. The instability is greater than that of $(NH_4)_2KPO_4$. At room temperature no transition was found to 12000. At 100°, there is a transition at about 8000 kgm., with a change of volume of about 0.0009 $cm.^3$ per gm. This transition was verified by repetition. Presumably it is of the ice type. On setting up the apparatus again, no transition could be found at 100°, nor yet at 200°. On cooling and opening the apparatus there was an almost explosive evolution of gas, and the substance was found completely decomposed. For some reason the decomposition must have taken place at a temperature lower than 100° on the second run.

## GENERAL SURVEY OF ALL SUBSTANCES EXAMINED.

The purpose of this section is two-fold. It is, first, to give a list of all those substances among which I have made unsuccessful search for other modifications. Information of this sort is doubtless of value, but too much weight must not be attached to it because failure to find a new form does not prove that none is capable of existence — the frictional resistance to the transition may be too great to allow it to start. In the second place, the purpose is to give some idea of the nature of the clues that I have followed in the attempt to coördinate the various data and to find some method of predicting whether a given substance is likely to have new polymorphic forms. This purpose will be served by grouping the various substances according to the clue that suggested their investigation. This purpose will also be furthered by including in the groups the substances which do have new forms and the data for which I have already published. The following contains, therefore, a collection by groups of all the substances that I have investigated either for melting or polymorphic forms. Of course for those substances whose melting curves have been measured, the range over which search for polymorphic forms has been made is of necessity more restricted than for the other substances. Since the same substance has sometimes been suggested by more than one clue, cases will be found of the inclusion of the same substance in more than one group.

The search for new modifications has been made, unless stated to the contrary, up to 12000 or 13000 kgm. at 20° and 200°. For substances melting within the range, search was made out to 12000 at room temperature and the highest temperature of the melting curve. I have not especially mentioned these cases. If no transition is found on either of these isothermals, and if it is certain that there is no transition at atmospheric pressure between 20° and 200°, it is very unlikely that there will be found to be a transition at any pressure less than 12000 at any temperature between 20° and 200°. Benzol shows the character of the transition curves of possible exceptions. I have not, except in a few cases, made any examination for polymorphism at atmospheric pressure between 20° and 200°, but have accepted the absence of any mention in the literature of polymorphism as probable evidence that there are no other forms. In several cases, however, I have found forms at atmospheric pressure not previously listed. But it is very probable that if there is a transition at atmospheric pressure the transition line will run across the isothermal at either 20° or 200°, and so be discovered by the run to high pressures. It is not probable that any large transitions have been overlooked. Another restriction to which this investigation is subject is that the change of volume of the transition must be large enough to detect with this apparatus. It is not likely that transitions with a change of volume of much less than 1/100% would have been detected, although if the transition were known to exist, measurements could be made on still smaller transitions. This means that transitions as small as many described by Cohen[29] in his papers on the allotropy of the metals are beyond the reach of this apparatus.

It will pay to give a preliminary discussion of the nature of the various clues. The clue of chemical similarity is perhaps the most obvious of all. If one substance shows polymorphism, one may expect others built up on the same chemical scheme also to have polymorphism. The expectation is especially strong if the substances differ only by the replacement of one element by another which usually stands to it in the relation of isomorphism. In particular the substitution of one atom for another would be expected to have less disturbing effect on polymorphic relations when it takes place in a complex than in a simple molecule. Evidently crystalline form is an important factor in the matter of chemical replacement; the chances are much greater for polymorphic similarity in those cases where the known forms belong to the same systems. In the following list of substances the crystalline system is given in all those cases where it is known.

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[29] Ernst Cohen, Proc. Amst. Acad., numerous papers in 1914 and 1915.

There are some cases where the existence of mixed crystals suggests the existence of other forms. If one salt, A, crystallizes isomorphously with another, B, in a form which pure A does not show, then one may expect that possibly this strange form may correspond to a modification of A stable at higher pressures.

Salts with water of crystallization might possibly be expected to show other forms under pressure, since the water molecules are a much less tenacious part of the compound and might be compelled to assume new bonds by the high pressure. Similar reasoning suggests that possibly double salts will assume other forms under pressure.

Substances which crystallize in unstable forms from the melt, like a number of organic compounds, would seem to be promising material, because the molecules are known to have the capability of being built up into several arrangements. In the same way, one might hope for polymorphism among those inorganic compounds which are known to exist in several forms as minerals, but whose relationship of monotropy or enantiotropy is not known. And with some plausibility one might expect that in some cases the transition lines of substances with transitions at higher temperatures than 200° at atmospheric pressure might be brought down by pressure to the region of this investigation.

Finally a suggestion may be mentioned growing out of Tammann's [30] theory of polymorphism. He has suggested that according to his theory there is a particularly good chance of polymorphism among those substances whose melts are associated. It is known that most organic acids are associated in the liquid, and accordingly one would expect frequent polymorphism here. A number of this class of substances were investigated.

In the following, the substances investigated under each group will be first of all simply enumerated, those showing polymorphic forms being marked with an asterisk. In the text comments are made on the various substances as they are called for.

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[30] G. Tammann, Gött. Nach., 1912, p. 1.

## *Chemically Related Groups.*

(1) NITRATES.—$KNO_3$*, I trigonal, II rhombic; $NH_4NO_3$*, I cubic, II tetragonal, III monoclinic, IV rhombic, V tetragonal; $CsNO_3$*, I cubic, II trigonal; $RbNO_3$*, I trigonal, II cubic, III trigonal; $TlNO_3$*, I cubic, II trigonal, III rhombic; $AgNO_3$*, I trigonal, II rhombic; $NaNO_3$, trigonal; $LiNO_3$, trigonal, rhombic, cubic; $HgNO_3$; $Hg(NO_3)_2$; $Pb(NO_3)_2$ cubic, monoclinic; $Al(NO_3)_3$. This group has already been sufficiently commented on in a previous paper. The results are given again here for the sake of completeness and convenience of reference.

HALOGEN COMPOUNDS.—$CCl_4$*, $CBr_4$*, I cubic, II monoclinic; $SiCl_4$; $C_2Cl_6$*, I cubic, II triclinic, III rhombic; $HgI_2$*, I rhombic, II tetragonal; $Hg_2I_2$; AgCl, cubic; AgBr, cubic; AgI*, I cubic, II cubic quasi-hexagonal; $NH_4Cl$*, I cubic, II cubic; $NH_4Br$*, I cubic, II cubic; $NH_4I$*, I cubic, II cubic; TlI*, I cubic, II rhombic; KI, cubic; NaI, cubic; LiI; $Cu_2I_2$*, cubic; $SbI_3$, trigonal. It seems to be a general rule in the case of halogen compounds that compounds formed by the substitution of one halogen for another crystallize isomorphously, although there are exceptions. The effect of the substitution, in case the substance is polymorphic, is to displace the temperature of transition.

$CCl_4$ and $CBr_4$ each have three forms and might at first glance seem to be a very good example of isopolymorphism. But this relation is probably illusory, because the general character of the phase diagrams is so different, one being the inverse of the other. The crystalline form of ordinary $CCl_4$ seems not to be known, so that we cannot say whether the ordinary modifications of $CCl_4$ and $CBr_4$ correspond or not.

Possibly $SiCl_4$ may show other forms at lower temperatures and higher pressures than those reached here, in analogy with $CCl_4$, because in many compounds an atom of silicon is capable of replacing an atom of carbon. The expectation need not be very strong, however, because the physical properties of corresponding Si and C compounds are often very different, $SiO_2$ and $CO_2$ for example.

Other compounds analogous to $C_2Cl_6$ are known to be polymorphic, and it would be most interesting to examine them under pressure. These are $C_2Cl_4Br_2$, $C_2Cl_2Br_4$, $C_2Br_6$. Preparation of them was beyond my chemical resources.

$HgI_2$ is unique among all substances so far examined in that its transition line has a maximum temperature. $HgBr_2$ under ordinary conditions has a form isomorphous with the high temperature form of $HgI_2$, while the form of $HgCl_2$ is not isomorphous with either the iodide or the bromide. It would be interesting to try the effect of pressure on the bromide and the chloride.

$Hg_2I_2$ is a substance listed in the Tables of the French Physical Society as dimorphous enantiotropic, with a transition point at 70°, on the authority of Yvon, in 1873. I have been unable to find the reference, but there is later work by Varet.[31] He recognizes three forms in all, but probably two of these forms are unstable and there is no enantiotropic transformation point. One variety is yellow-green; it is precipitated under some conditions, but is very unstable. The other new variety is red; it is unstable under ordinary conditions. The reaction from the ordinary yellow to this begins at about 70°, but is not complete until 245° is reached. It does not seem to be a transition of the ordinary type. In the present work, no new form was found up to 12000 kgm. at room temperature, or to 13000 at 125°. In a further search for the second modification, $Hg_2I_2$ was heated in a dilatometer at atmospheric pressure. Between 57° and 83° there can be no discontinuity of volume of so much as one part in 3000. The only evidence that I have found for the transition is a fairly rapid deepening of the yellow color to a brown, on heating through 70°. This evidence cannot be regarded as sufficient, however.

AgCl and AgBr are cubic and crystallize isomorphously with each other and the high temperature modification of AgI. We would expect therefore to find at low temperatures at atmospheric pressure a transition of each of these substances to another form isomorphous with the low temperature modification of AgI. No such transition point seems to be known, and I do not know whether the search has been made. Analogously one would expect at higher pressures at room temperature to find another modification of AgCl and AgBr corresponding to AgI (III). No such form was found, however, nor was there anything at 200°. In my previous paper I referred to the low temperature modification of AgI as hexagonal. There are many authorities for this, but the most recent work[32] seems to establish that this is cubic, although imitating very closely the hexagonal form by its peculiar method of twinning.

The series of three salts $NH_4Cl$, $NH_4Br$, and $NH_4I$, is the most noteworthy example of any that I have found of complete isopolymorphism. Not only are the three substances so similar that they will form mixed crystals with each other, but the three phase diagrams of the pure substances are very similar, except for the absolute value of the temperature of transition at atmospheric pressure. These cases of polymorphism are also unique in another particular; the high and low temperature forms of each substance belong not only to the same crystalline group, the cubic, but also to the same sub-group, the pentagonikositetrahedric.

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[31] R. Varet, Ann. Chim. Phys., **8**, 79 (1896).
[32] F. Wallerant, Cristallographie, p. 275.

In large features the phase diagrams are much similar; the transition lines are unusually steep, the changes of volume are unusually large, the curvature of all three lines is unusual in that they are convex toward the pressure axis, and the transitions are all sharp and rapid with a very narrow band of indifference. In finer detail, however, the several diagrams do not show regular variations. The atomic weights of Cl, Br, I are approximately 35, 80, and 127. Br is very nearly half way between Cl and I and we should expect the properties of the $NH_4Br$ transition to be midway between those of $NH_4Cl$ and $NH_4I$. This regularity does not exist. The transition temperatures at atmospheric pressure are 184.3°, 137.8°, and −17.6°, in order of increasing atomic weight. The Br salt is much nearer the Cl end. It is noteworthy that the order of these transition points is the exact reverse of the usual order of the boiling or melting points of homologous halogen compounds — the chlorine compound having the lower boiling point and behaving as if it had less internal cohesion. The slopes of the transition lines at atmospheric pressure do not even follow the order of atomic weight, being 0.0645, 0.0800, and 0.0699. At the same temperature, however, the order is normal, being at 185°, for example, 0.065, 0.098, and 0.153. Again Br is nearer the Cl end. The changes of volume at atmospheric pressure are 0.0985, 0.0647, and 0.0561 cm.$^3$ per gm. In this respect Br falls nearer the I end. The changes of volume per gm. molecule, which afford perhaps a juster comparison, are 5.26, 6.34, and 8.14. Br is now much closer to Cl. It is rather surprising that the molecular changes of volume of all three salts are not more nearly equal; the easy isomorphism would lead one to expect it. Finally, the latent heats of transition are 6.98, 3.32, and 2.05 respectively in kg. m. per gm., or 374, 327, and 247 kg. m. per gm. mol. According to the first method of comparison Br is closer to I, but by the second, which is more significant, is closer to Cl.

TlI is marked with an asterisk because it is known to have two modifications. The transition takes place on heating to about 168°, and is well marked by the striking change of color, from yellow to red. I was not able, however, to make any measurements under pressure, or even to detect the transition, although I made careful search at 20°, 130°, and 200°. The explanation is to be found in the great sluggishness of the transition, which is much worse in this respect than $HgI_2$. Gernez [33] has found at atmospheric pressure that the yellow modification may be heated to 200° without initiating the transition, and that the red may be cooled to liquid air without starting the reverse transition. He has kept the unstable red modification for two years at room temperature without the transition starting. Furthermore he finds that the transition from one phase to the other is always slow, even when inoculated. The material is evidently unsuitable for experiment under high pressures. The only possibility is in finding some catalytic agent that will hasten the transition.

KI is cubic and isomorphous with the ordinary modification of $NH_4I$ and the high temperature modifications of $NH_4Br$ and $NH_4Cl$; it is strange that it showed no new form. KBr and KCl are also isomorphous with KI; judging by analogy with the ammonium salts one would might expect that these two would be more likely to show a new transition than the KI. They should be tried. KI is a substance for which Spring [34] announced a large permanent change of volume after exposure to 10000 kgm. at room temperature. I could find no evidence for it.

NaI is not isomorphous with KI, but it is with AgI. One would expect new forms.

LiI crystallizes with three molecules of water, the "melting point" at atmospheric pressure being at about 72°, at which it gives up two molecules of water. At 200° this transition point was found displaced to about 10000 kgm. with a change of volume of approximately 0.015 $cm.^3$ per gm. It is a matter of considerable difficulty to produce anhydrous LiI, and I did not make the attempt.

$SbI_3$ has been erroneously described as having enantiotropic transition points; it has already been discussed under melting.

Among other compounds of this class which could be profitably examined is $PbI_2$: this is known to have a transition at atmospheric pressure, but at a temperature so high as to be beyond the range of this work.

Summarizing the behavior of the halogen compounds, polymorphism seems to be of frequent occurrence in this group, but except for the three ammonium salts, the phase diagrams of the various salts have no relation whatever to each other. The frequency of polymorphism cannot be ascribed to any detailed resemblance in the structure of the crystals of the several salts, but must be due to some rather general property of the halogen elements. It may be, for example, that the atoms of the halogens are surrounded by rather complicated fields of force which offer the possibility of being grouped in a variety of stable configurations.

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[33] D. Gernez, C. R., **138**, 1695, 1904, and **139**, 278 (1904).

[34] W. Spring, Rap. au Cong. international Phys., **1**, 402 (1900).

SULFIDES.— $K_2S$*; FeS trigonal; $Sb_2S_3$* rhombic; $Sb_2S_5$; BaS cubic; PbS cubic; SrS cubic; CdS trigonal; CaS cubic; CuS hexagonal; $Ag_2S$ cubic*; $HgS_1$ cubic, trigonal.

$K_2S$ was found to have a form not previously known. By analogy $(NH_4)_2S$ would be expected to have a new form, but it is not chemically stable enough to allow the trial. An attempt to remove the moisture from $Na_2S$ sufficiently to allow a trial did not succeed.

$Ag_2S$ has been known for a long time to have a second modification above 170°. The transition is accompanied by the evolution of a considerable quantity of heat, as may be shown by taking the cooling curve. There is also a discontinuous change in the electrical resistance on passing through the transition point. No attempt seems to have been made to detect the change of volume, but it has been assumed that in analogy with $Cu_2S$ the change of volume would probably be very small. I made particularly careful search for any evidence of a transition of $Ag_2S$ under pressure. Search was made by the usual methods out to 12000 kgm. at four temperatures, 20°, 120°, 150°, and 200°, and in addition the method of varying temperature from 130° to 200° at approximately 1000 kgm. was used. The change of volume of the transition must certainly be less than 0.0002 cm.$^3$ per gm. and it is probably safe to put the limit at a half or a third of this. This experiment answers one question about which there has been some conjecture; it has been thought that at higher pressures the difference of volume of the phases might become appreciable, the transition point at atmospheric pressure corresponding to a maximum or minimum of the transition curve. This experiment shows that the smallness of the difference of volume persists at high pressures; there is no reason to suppose that the transition line will show more than the normal amount of curvature.

$Cu_2S$ is a substance with a transition point at 103°, the character of the transition being very much like that of $Ag_2S$. The change of volume has in this case, however, been demonstrated to be so small as to be imperceptible. In view of this known fact it did not seem worth while to make an examination under pressure. The result would in all probability be negative as for $Ag_2S$.

HgS has two modifications; a very unstable black modification, cubic, and a stable red one. It is known that under pressure the black form changes to the red. I made trial on the red modification over the usual range without result; a negative result is not surprising.

The antimony sulfides have already been extensively commented on, and the other sulfides seem to require no special mention. In the only two cases where the phase diagrams of sulfides have been measured, they have been found to be simple and quite normal in type.

SULFOCYANIDES.—KSCN, II rhombic; $NH_4SCN$, I rhombic, II monoclinic.

Although the phase diagrams of these two substances differ in appearance, it has been shown in the discussion of individual data that the substances probably are isopolymorphic when modifications in the same crystalline system are put into correspondence. Each substance probably has three modifications instead of two, and the appearance of difference in the phase diagrams is to be explained by the different temperatures of transition.

It is known that RbSCN, CsSCN, and TlSCN also have other modifications. It would be interesting to try them.

HALOGENATES.—$KClO_3$* monoclinic; $KBrO_3$ trigonal; $KIO_3$ monoclinic; $NaClO_3$ cubic, trigonal, rhombic; $KClO_4$, rhombic.

The relation between the first three potassium salts is not simple. The ordinary form of $KClO_3$ is monoclinic, and that of $KBrO_3$ is trigonal pseudo-cubic. $KBrO_3$ is, however, isodimorphous with $KClO_3$, forming a series of mixed crystals with a gap. It is possible that the ordinary form of $KBrO_3$ is the high pressure form of $KClO_3$. If the usual form of $KClO_3$ corresponds to the second unstable form of $KBrO_3$, the latter might be expected to have three modifications under pressure, but no new one was found. $KIO_3$ is monoclinic, but will not form mixed crystals with either $KClO_3$ or $KBrO_3$. $NaClO_3$ is cubic, and shows no isomorphism with $KClO_3$. It is known, however, to have two unstable forms, one trigonal and one rhombic, and might on this account be expected to have other forms under pressure.

It would be interesting to try $RbClO_3$ and $CsClO_3$, since these are probably isomorphous with $KClO_3$.

The first one of this group of substances which I investigated was $KClO_3$, my reason was simply that its well known chemical instability seemed to suggest the possibility of readily taking new groupings.

$KClO_4$ is chemically more unstable, however, but shows no new modification under pressure, so the apparent connection between chemical instability and polymorphism turns out after all to have been merely accidental.

In experimenting with members of this group, the danger of explosion has to be particularly guarded against. One very severe explosion resulted from the combination of $NaClO_3$ with the kerosene with which pressure was transmitted. To avoid this possibility, the salts were placed in a nickel steel shell, and pressure transmitted by mercury. There is always the possibility, however, that the shell may split, and the kerosene find its way to the chlorate, as it does for many other substances. In view of this possibility I felt justified in confining my exploration with $NaClO_3$ and $KClO_4$ to only 20°. It remains to this day unexplained why the first three potassium salts did not explode when exposed to direct action of kerosene up to 200° and 12000 kgm.

## *Organic Compounds.*

HALOGEN COMPOUNDS WITH ONE CARBON ATOM.— $CCl_4$*, $CBr_4$*, $CHCl_3$, $CHBr_3$, $CHI_3$.

The first two of this series have already been discussed. It is unfortunate that $CI_4$ is too unstable to investigate. The other members of the series are formed by replacing a halogen by an atom of hydrogen. Since this substitution is known to produce important changes in the chemical form, it is perhaps not surprising that polymorphism disappears under the substitution. It is known, however, that lower members of the series, such as $CH_2Cl_2$ show polymorphism again; the effect of pressure on these substances should be tried.

BENZOL AND MONO-SUBSTITUTION PRODUCTS.— $C_6H_6$*; $C_6H_5OH$*; $C_6H_5Cl$; $C_6H_5Br$; $C_6H_5NO_2$; $C_6H_5NH_2$; $C_6H_5CO_2H$ monoclinic; $C_6H_5NHC_2H_3O$ (acetanilid); acetophenone. It is strange that so few of the crystalline forms are known in this group.

In the comparatively complicated molecules of series like this, one would not expect a substitution for a single atom to have so large an effect on polymorphism as in series of simpler molecules, and it is therefore perhaps surprising that polymorphism is not of more frequent occurrence. But it is to be remembered that the second modification of benzol is stable only at the extreme pressures of the range. It is possibly worthy of remark that the only other member of this series showing polymorphism is the one nearest it in molecular weight.

Because of an accident the exploration with acetanilide at room temperature was made only between 6000 and 12000. At 200°, the approximate coördinates of the melting curve were found to be 6400 kgm. with a change of volume of 0.070 $cm.^3/gm.$ The melting point at atmospheric pressure is in the neighborhood of 113°.

BENZOL DERIVATIVES WITH SUBSTITUTIONS FOR TWO ATOMS.— Orthokresol *; para nitro-phenol, monoclinic; para toluidine; anethol.

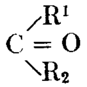
A unique feature of the phase diagram of o-kresol is that the melting curve of I can be realized at *higher* temperatures in the region of II. It is perhaps surprising that p-toluidine does not show another form like o-kresol, since the substituting groups differ so little in molecular weight.

CAMPHOR GROUP.— Camphor *, trigonal; monobrom-camphor; dibromcamphor; phenylated camphor.

Ordinary camphor has one of the most complicated phase diagrams; it was a surprise that at least monobrom camphor did not have other forms. Monobrom camphor decomposes at about 4000 kgm. at 200° when pressure is released from 12000. Apparently there is no free bromine among the decomposition products. Dibrom camphor at 200° decomposed when pressure had been released from 12000 to 8000 with so large an increase of volume that pressure rose again to 12000. The products of the decomposition were a suffocating gas and much free bromine. There were many indications that neither of these brom camphors began to decompose at 200° until reaching the melting curve. The high pressure as well as the high temperature must play some part in the decomposition, because both of them melt quietly on heating to 200° at atmospheric pressure. Pressure was transmitted to these substances by mercury, so that the kerosene could have had no part in the decomposition.

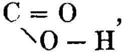
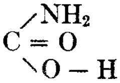
Runs on phenylated camphor were made at 0°, 20° and 180°. The melting point at atmospheric pressure is −23°; at 180°, 12000 kgm. was not high enough to freeze it.

### *Compounds of the Type*



$$C \begin{cases} R^1 \\ = O \\ R_2 \end{cases}$$

Carbamide *, tetragonal; Acetic Acid *; Acetamide *, trigonal; Acetone; Formamide; Formic Acid; Oxamide, monoclinic; Propionic Acid.

The first three of these substances form a sub-group very closely related chemically. They are built up by using the radicals $NH_2$, $CH_3$, or OH in various combinations. It is possible, by combining any two of these radicals with $\overset{\diagup}{\underset{\diagdown}{C}}=O$ to build up six compounds. Three of these have polymorphic forms, the fourth, acetone, is liquid throughout most of the range so that new forms would not be expected, the fifth is carbonic acid,  $C\begin{matrix}\diagup O-H\\ =O\\ \diagdown O-H\end{matrix}$, which does not exist in a free state, and the sixth,  $C\begin{matrix}\diagup NH_2\\ =O\\ \diagdown O-H\end{matrix}$, does not exist. We therefore have polymorphism in all possible cases in this group. The phase diagrams, near the triple point, are similar, in that the transition lines are steep, and the latent heat of transition is small. In closer detail, however, the transition lines are different; those of carbamide and acetic acid are both straight with a positive slope, whereas that of acetamide is distinctly curved and has a negative slope. The diagrams further differ in that carbamide has three modifications and acetic acid and acetamide have only two. We have seen, however, that it is not impossible that acetamide has a third form at lower temperatures, and Tammann[35] says that acetic acid probably has a third form considerably below zero. It may be that the apparent relationship of the phase diagrams is only accidental; what we know about the crystalline forms would suggest this. The system of acetic acid is not known, carbamide is tetragonal, and acetamide is hexagonal. It has turned out of late years, however, that nearly every so-called hexagonal crystal has been found on careful examination to belong to some other system, the apparent hexagonal form being due to the particular manner of twinning. The evidence from the crystalline forms is not conclusive, therefore.

It is worthy of remark that the three radicals forming the compounds just discussed are all of nearly the same molecular weight, $OH = 17$, $NH_2 = 16$, and $CH_3 = 15$. The four other members of the group listed above are formed by using radicals of quite different molecular weight, and in no case were other forms found. Particularly careful search was made in the case of formamide and propionic acid. Two specimens of propionic acid were used, from different sources. Neither was very pure, but the second was somewhat purer than the first.

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[35] G. Tammann, Kristallisieren und Schmelzen, p. 277.

The first specimen showed a small discontinuity like a transition in the region of rounding just before melting, which did not appear with the second specimen. Runs were made with this at 0°, 10°, and 21°, pressure being transmitted by mercury. Only at 20° could the pressure be raised to 12000, since at lower temperature the mercury freezes. The approximate coördinates of the melting curve are: 7200 kgm. at 0°, 8100 at 10°, and 9100 at 21°. Two specimens of formamide were also used, both impure. The maximum temperature of examination was 26°. The impurity of the best specimen was considerable, because its melting point was below 0°, whereas that of the pure formamide is listed as +3°. The formic acid used was also very impure, but it showed nothing except a very rounded melting curve at 26° and 50°. If purer materials were easily available, it might pay to try these substances again. My specimens were made especially to order, and are apparently of as great purity as can be produced in an ordinary commercial laboratory.

The oxamide was tried unsuccessfully at 20° and 200°. It decomposed slightly at 200°, as shown by a slight evolution of gas on taking the apparatus apart. Pressure was transmitted directly to the oxamide by kerosene; all the other substances of this group were submerged beneath mercury.

COMPOUNDS WITH TWO BENZOL RINGS.— Diphenylamine, monoclinic; Benzophenone, rhombic, monoclinic.

Neither of these had new forms, although benzophenone has several unstable forms.

SUBSTANCES WITH SUGGESTIVE MIXED CRYSTAL RELATIONS.— $KHSO_4$*, rhombic; $NH_4HSO_4$*, rhombic; $MgSO_47H_2O$, rhombic; $FeSO_47H_2O$, monoclinic.

The first two of these do not crystallize isomorphously, although belonging to the rhombic system. They form a series of mixed crystals with a gap, the third form being monoclinic. The relations of the several forms have been fully treated in the detailed discussion, and a possible mixed crystal diagram has been indicated. This diagram would demand that the first new form of $KHSO_4$ at high temperatures be monoclinic, and the second new high temperature form rhombic.

The trichromates of K and $NH_4$ have the same relations as the acid sulfates, and should be tried.

$MgSO_47HO_2$ and $FeSO_47H_2O$ do not crystallize in the same system, but form a series of mixed crystals with a fairly wide gap. If one extrapolates the density of the crystals with preponderating

$FeSO_47H_2O$ content to pure $MgSO_47H_2O$, a greater density is found than that of the natural form of $MgSO_47H_2O$. One is therefore led to expect a second denser form, and to look for it at higher pressures. This matter is discussed in greater detail in Groth's Chemical Crystallography, p. 92. It was a matter of great surprise to me to find that there is no other form at high pressures, for I had regarded this evidence as establishing a greater *à priori* probability for the existence of a new form than for any other substance within my knowledge. Two runs were made on $MgSO_47H_2O$. The first was with the pure salt at 20°. For the second run the salt was inoculated with $FeSO_47H_2O$. and runs made at 20° and 100°. Higher temperatures were not tried for fear of running into a decomposition of the hydrate. The expected new form should have been found at low temperatures rather than high. Runs were made with $FeSO_47H_2O$ at 20° and 100°. At 20° nothing was found out to 12000 kgm., but at 100° a decomposition point of the hydrate was found. This will be described in detail under the group of substances with water of crystallization.

SUBSTANCES WITH WATER OF CRYSTALLIZATION.— Every one of the following list of substances has been investigated both with and without its water of crystallization, except $LiNO_3$, which was tried only in the anhydrous state. For brevity, both the hydrous and the anhydrous salts are specified in the following list only when the crystalline forms are known; in many cases the crystalline form of the anhydrous salt is not known.

Oxalic Acid, $+2H_2O$ — monoclinic, anhydrous — rhombic; Potassium Oxalate, $+H_2O$ — monoclinic; Potassium Acid Oxalate, $+H_2O$*, anhydrous-monoclinic; Potassium Tetroxalate; Potassium Tartrate, $+1/2H_2O$ — monoclinic; Ammonium Potassium Phosphate, $(NH_4)_2KPO_4$*,$+4H_2O$; $FeSO_47H_2O$ — monoclinic; $MgSO_47H_2O$ — rhombic; $CuSO_45H_2O$ — triclinic; $Hg_2(NO_3)_22H_2O$*; $LiNO_33H_2O$.

All of these substances were tested at the two temperatures 20° and 100°, instead of at 20° and 200°, as for most other substances. All of them were hammered into an open steel shell and pressure transmitted directly by kerosene. With one exception, the water of crystallization was removed by heating to 100° in vacuum for several hours. Potassium Tartrate was dehydrated in vacuum at 160°, since 100° was not high enough.

It is a curious freak of chance that the first substance with water of crystallization that I tried, $Hg_2(NO_3)_2$, showed a transition in the hydrous condition, but not when anhydrous. I was prepared to expect a number of such cases, but the tetroxalate and $FeSO_47H_2O$ were the only others. $(NH_4)_2KPO_4$ has a transition when anhydrous, but not when hydrous. It is to be remembered, however, that only one of these substances belongs to the class whose hydrates have a transition point at atmospheric pressure. The investigation of other substances with known transition points of the hydrates offers a field of considerable interest, but the phenomena are more complicated than those of the simple polymorphic transitions considered here.

The phenomena with $FeSO_47H_2O$ are as follows. It showed two transition points. At 10,000 kgm. there is a transition with fairly large decrease of volume somewhere near 80°; that is, this transition is of the ice type. At 85° there is no transition between 10,000 and 2000, but on cooling at 2000 there is a much smaller reverse transition near 35°. This point is doubtless connected with the known dehydration points at atmospheric pressure, and the other point at 10,000 is probably of the same nature, rather than a true polymorphic transition. It is evident that the dehydration diagram of this substance must be rather complicated. I made no further attempt to study it; there are several serious difficulties in the way of a complete investigation.

DOUBLE SALTS.— $(NH_4)_2KPO_4$, anhydrous,* and with $4H_2O$; $KNH_4SO_42H_2O$; $ZnK_2(SO_4)_26H_2O$, anhydrous* and with $6H_2O$.

All of these runs were made at 20° and 100°, except $K_2Zn(SO_4)_2$. The first substance has been described under the last heading, and $K_2Zn(SO_4)_2$ has been described in a separate section. It is likely that both of these transitions are really decompositions to the simple salts, rather than a true polymorphic change.

An attempt to make $KNH_4SO_42H_2O$ anhydrous by heating for a number of hours at 150° in vacuum was without success.

SUBSTANCES WITH UNSTABLE FORMS.— Benzophenone, has four modifications including a monoclinic and a rhombic form; Para-nitro phenol; Acetamide*, trigonal; Menthol; Acetophenone; Monochloracetic Acid, has several unstable forms; Propionic Acid; $SbI_3$, trigonal; Sulfur*, has a stable rhombic and monoclinic form, and numerous unstable forms; Phosphorus,* cubic, trigonal; Antimony, trigonal; Selenium, two monoclinic and a trigonal form; Arsenic, cubic, trigonal; Iodine, rhombic, monoclinic.

The first seven of these substances have already been commented on, either in this or in earlier papers; the melting curves of several of them have been determined. Phosphorus[B] has been made the subject of a special paper. Antimony, besides being very near phosphorus in the periodic table, forms a number of stable and unstable modifications at low temperatures. It was somewhat of a surprise that there were no new forms under pressure.

Sulfur is known to have a number of modifications, both stable and unstable, several of them more or less obscure in character, but there are at least two well defined forms that stand to one another in the relation of enantiotropy. I found no other well defined forms to 12000 at 20°, 100°, or 200°. It is possible, however, that a small percentage of the form insoluble in $CS_2$ was formed by pressure, because the specimen which had been subjected to pressure was not completely soluble. One might possibly expect a new modification of sulfur by an irreversible transition, like that of black phosphorus, because of the proximity of the two elements in the periodic table. On one occasion a piece of sulfur was kept at 12500 kgm. and 200° for six hours, but with no permanent change.

The runs on selenium were started with the amorphous variety, which had been fused immediately before the experiment. It showed no transition to 12000 at room temperature. Pressure was maintained on the amorphous selenium at 7000 for 16 hours with no effect. At 7000 kgm. the selenium was then heated to 200°. There was a transition with decrease of volume somewhere between 20° and 200°, which was doubtless due to the formation of the crystalline phase. At 200°, the new phase showed no transition to 12000 kgm., and also showed none between atmospheric pressure and 12000 after cooling again to room temperature. On releasing pressure the selenium was found to have a density of 4.69. The density of the metallic modification of Se is given as 4.79, and that of the amorphous as 4.29 by Saunders.[36] (It may be mentioned incidentally that several wildly inaccurate values for the density have found their way into some Tables. The Chemiker Kakendar, for instance, gives the density of the amorphous variety as 5.68, and that of the crystalline form as 6.5). There seems little reason to doubt that the substance formed above was metallic selenium, the somewhat small density might easily be due to fissures or occluded kerosene. It is known that ordinary amorphous selenium will crystallize slowly at atmospheric pressure at 200°; the effect of high pressures seems, therefore, to be to lower somewhat the temperature of crystallization. This is as one would expect. Before trying the experiment it seemed plausible to me to expect a new form stable at atmospheric pressure analogous to black phosphorus, because of the similar position of these two elements in the periodic table.

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[36] A. P. Saunders, Jour. Phys. Chem., **4**, 491 (1900).

Arsenic, again, is near phosphorus in the periodic table, and I thought that there might be another form like black phosphorus, but none was found. The substance used was distilled arsenic; pressure was transmitted to it by mercury, so as to avoid the possibility of poisonous compounds with the kerosene. After the run, the surface of the arsenic was found wet with mercury, and the appearance was that of amalgamation. The mouth of the steel shell was also amalgamated, a thing which I have never observed before under similar conditions of pressure and temperature.

Iodine was tried at 30° and 200° to 12000 kgm. without result. At 200° a much rounded melting point was found near 5000. Because of the great chemical activity of Iodine it was placed in a steel shell beneath water, instead of being allowed to come in contact with kerosene or mercury, as usual. In this way chemical action was largely reduced, but at 200° the Iodine apparently goes slowly into solution.

SUBSTANCES EXISTING AS MINERALS IN TWO FORMS.— $Pb(NO_3)_2$; HgS, trigonal, cubic; $CaCO_3$, rhombic, trigonal.

Pb $(NO_3)_2$ is said by Morel[37] to have two mineral forms. HgS crystallizes in a black and a red form; the black is very unstable. Both of these substances have been commented on in previous sections.

The $CaCO_3$ was investigated in the form of calcite. In nature $CaCO_3$ occurs in two forms, as calcite and arragonite, the latter being the more dense. There has been considerable speculation as to the relation of these two forms, but it seems to have been finally settled that at ordinary temperatures calcite is the stable form, the reversible transition from calcite to arragonite running at fairly high temperatures. This means that the phase diagram must be of the ice type. There is, therefore, a possibility that a permanent change from calcite to arragonite might be brought about by increase of pressure, if the pressure could be carried far enough into the region of stability of the arragonite to force the reaction from calcite to run, and the pressure then released in a region where the reverse transition from arragonite to calcite does not run because of viscosity. The experiment was tried of maintaining pressure on calcite at 12500 kgm. for six hours at 200°, cooling it, and then releasing pressure, but there was no permanent change of density. Since the reaction from arragonite to calcite does not run at atmospheric pressure and room temperature, it would run still less at higher pressures at room temperature. Therefore, if arragonite had been formed at all in this experiment, the reverse transition would not have run, and we infer that 12500 kgm. at 200°, is not a high enough pressure to bring the calcite into a region where the reaction to arragonite spontaneously runs. The specimen used was not large enough to enable me to tell whether there was any reversible change in the calcite itself up to 12000 at 200°.

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[37] J. Morel, Bull. Soc. Min. France, **13**, 337 (1890).

SUBSTANCES WITH REVERSIBLE TRANSITIONS AT HIGHER TEMPERATURES.— $K_2CrO_4$, rhombic, triclinic; $K_2CrO_7$, two triclinic forms; $Cu_2I_2$*, cubic.

These all have transitions near a red heat; it is not known to what type they belong. If they were of the ice type, a high pressure might bring the transition down to the range of this work. Reason has been given for supposing that the transition found at 200° for $Cu_2I_2$ is not the same as the previously known high temperature transition.

ORGANIC ACIDS.— Carbolic*; Acetic*; Monochloracetic; Stearic; Tartaric, monoclinic; Benzoic, monoclinic; Oxalic, rhombic; Citric; Propionic; Formic.

Pressure was transmitted to all of these substances by mercury. I have already mentioned we would expect from Tammann's theory of polymorphism that enantiotropic transitions would be particularly common among the organic acids. Out of ten substances tried, only two examples were found, and these were not new examples, but were known before. If ten is a sufficient number of instances to give the law of chances a fair test, this is not favorable to the theory.

In the above list, monochloracetic acid, which has been previously mentioned, has unstable forms, but this sort of transition is not contemplated by Tammann's theory. Oxalic acid has also been mentioned; it was tried with and without the water of crystallization. Citric acid is listed as crystallizing with water, but as I could produce no change in its appearance by heating to 100° in vacuum for several hours, I assumed that it had been supplied in the anhydrous condition.

The tartaric acid was dried in vacuum at 100°. At 12000 kgm. it showed no change on heating from room temperature to 200°, but on releasing pressure at 200°, decomposition began at 5000 with an increase of volume large enough to bring the pressure back to at least 7100. The product of decomposition was a sticky putty-like mass, which swelled up and overflowed the mouth of the tube when the apparatus was opened after cooling. The Nickel steel shell was split. Professor Kohler was kind enough to examine the substance, and found that it was one of the anhydrides of tartaric acid.

ELEMENTS: Hg, cubic; K, tetragonal; Na, tetragonal; Sn*, rhombic, tetragonal; Bi, trigonal; Tl*; S*, rhombic, monoclinic; P*, cubic; I*, rhombic, monoclinic; As*, cubic, trigonal; Sb*, trigonal; Se*, monoclinic (2), trigonal.

The investigation of mercury was concerned primarily with the melting curve. Its freezing behavior is known to be abnormal, but since the freezing temperature is a much smaller fraction of the critical temperature than it is for most other substances, one would not perhaps expect the abnormality to result in polymorphism. Sodium and Potassium show nothing unusual. Perhaps, from the rather rapid approach of $\Delta v$ toward zero, Potassium might be regarded as a candidate for polymorphism at some pressure beyond the range of this work. Bismuth would be expected to have another form because its melting is of the ice type; it is a great surprise that it does not. Thallium has another known form; the change of volume is so small that I could not make accurate measurements.

The reason for trying Tin was that it is known to have a transition point at 20°, with a large change of volume, the low temperature phase (gray tin) being the less dense. A phase diagram like that of AgI might be expected. The transition from white to gray tin is known to be enormously viscous, so that one could not expect to observe this, but there was a possibility that the reaction to Tin III, if there is such, might run more readily. No such transition was found. It would be of interest to start with pure gray tin, and subject this to pressure at a low temperature.

Werner [38] has recently given a provisional location for a transition line between two varieties of white tin. The transition point at atmospheric pressure is at about 160°, and at 200°, it is 500 kgm. The change of volume is extraordinarily small, so small that I could hardly expect to detect it with my apparatus at a pressure so low as 500 kgm., where it is most insensitive.

The other elements have been described in a previous section.

MISCELLANEOUS SUBSTANCES.— $H_2O$*, trigonal; $CO_2$; $KNO_2$*; $K_4P_2O_7$; $K_4S_2O_7$; $K_2CO_3$; $KHCO_3$, monoclinic; Potassium Acid Tartrate, rhombic; Methyl Oxalate, monoclinic; Urethane*; Naphthaline, monoclinic; Cane Sugar, monoclinic.

$H_2O$ has a phase diagram exceeded in number of phases only by $NH_4NO_3$, and possibly by camphor. $CO_2$ was at first thought by Tammann, to have a triple point with the liquid near 4000, but he later withdrew this. I could find none to 12000. $KNO_2$ shows a transition line of rather large and unusual curvature; it is unfortunate that the substance was so impure.

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[38] M. Werner, ZS. Anorg. Chem., **83**, 275 (1913).

The reason for trying the six substances from $KNO_2$ to Potassium Acid Tartrate was merely that polymorphism seemed to be rather more common among the compounds of Potassium than of other elements, and it seemed worth while to try a number of examples. If this surmise is correct, these six substances were not such as to bear it out. $K_4P_2O_7$ was tried only at 20°, because it had some water in it. $K_4S_2O_7$ was tried at 20° and 200° as usual. Both were without result. Several runs were made with $K_2CO_3$. The first showed a small transition near 6500 kgm. at 200° and at a somewhat higher pressure at 185°. This was found unmistakably with two different fillings of the apparatus. But after standing in the apparatus for three days, the transition at 185° had entirely disappeared, and that at 200° had a smaller change of volume. An attempt to repeat the measurements after six months showed no trace of the transition. The result is hard to explain. I am inclined to think that the transition may be a genuine one, but that 200° is in the very viscous region, so that sometimes the transition will run, and sometimes not. The effect may, however, be due to moisture; the first specimen may possibly have been a trifle moist, but the second was carefully dried in vacuum.

Methyl Oxalate was tried because Tammann [39] has announced two modifications. I could find no other form; the matter has been discussed at considerable length in the paper on melting.

There was no particular reason for trying Urethane, except possibly its rather interesting method of decomposition on heating, and its polymorphism seems to have no suggestive connection with that of other substances. Naphthaline and Sugar offered no special promise of polymorphism; they are simply substances readily available for miscellaneous exploration.

## DISCUSSION.

A compact summary of the nature of the effects for all the polymorphic transitions investigated to date is given in Figures 30 and 31. These diagrams show the location of the transition lines between the several phases, which are indicated by Roman numerals or by L for the liquid; the arrows on the lines show the direction in which $\Delta v$

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[39] G. Tammann, Kristallisieren und Schmelzen, p. 265.

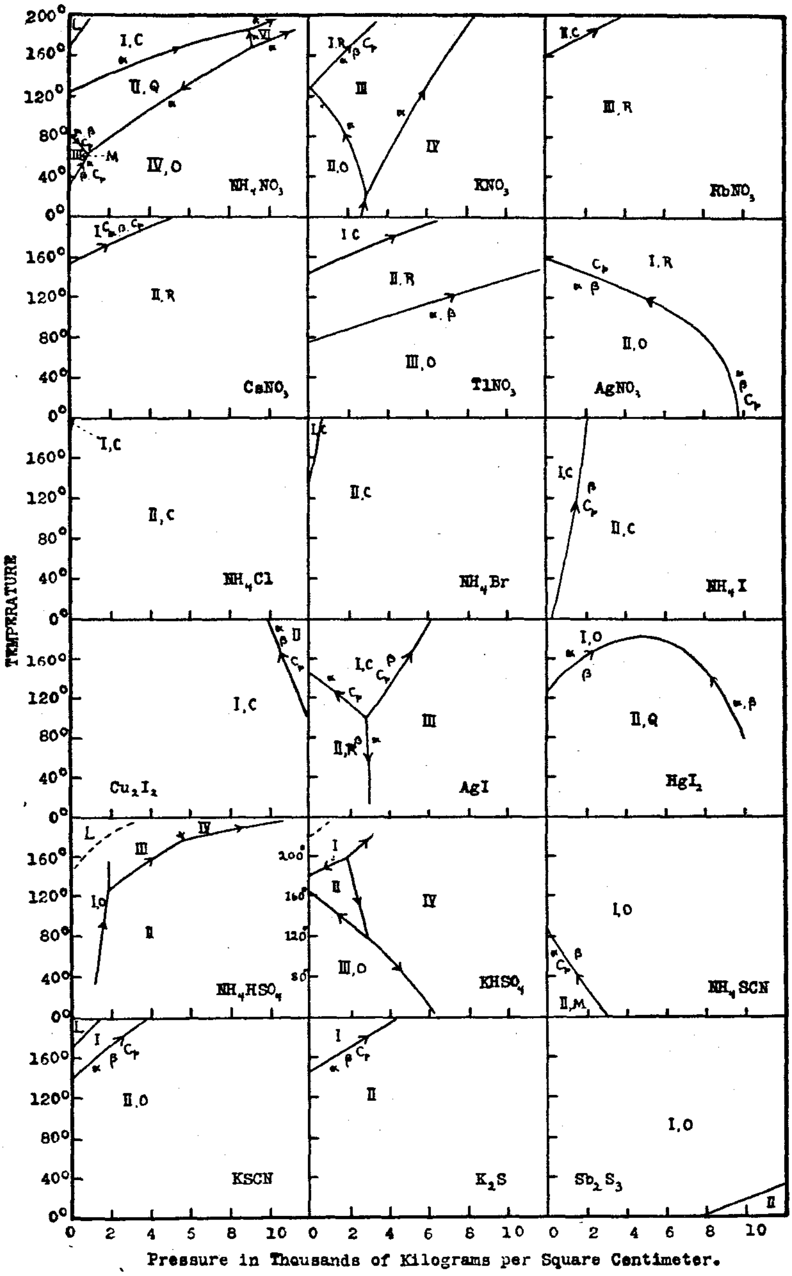

FIGURE 30. Collection of Results.

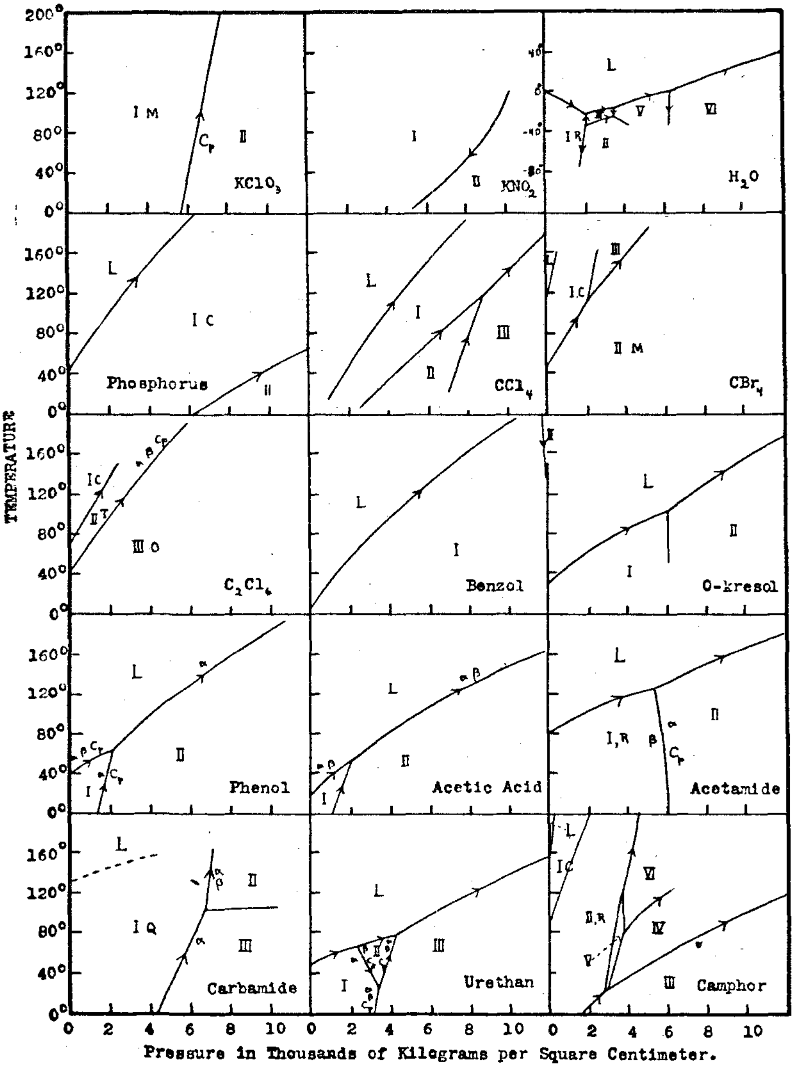

Figure 31. Collection of Results.

decreases numerically, an $\alpha$, $\beta$, or $C_p$ to one side of the line indicates that on that side of the line the compressibility or the thermal expansion or the specific heat is the greater, and the crystalline systems have been indicated in those cases where known by letters. The abbreviations used for the crystalline forms are: C, cubic; Q, tetragonal or quadratic; R, rhombohedric or trigonal, including hexagonal; O, orthorhombic, or rhombic; M, monoclinic; T, triclinic.

The diagrams bring out the fact, which was also brought out by the general survey, that mere chemical similarity is not sufficient to ensure similarity of phase diagram. A polymorphic change cannot be regarded, except from certain restricted thermodynamic view points, as a special case of a chemical reaction, but involves different and undoubtedly more mechanism. Similarity of the phase diagram of two substances involves a much more far-reaching correspondence of mechanism than similarity of chemical behavior. There are, as a matter of fact, only two groups of phase diagrams in the collection above which are equivalent. They are $RbNO_3$, $CsNO_3$, $TlNO_3$, with a remote possibility of $KNO_3$ on the one hand, and $NH_4Cl$, $NH_4Br$, $NH_4I$ on the other. In these cases it seems to be a first prerequisite for similar phase diagrams that the corresponding phases form complete series of mixed crystals in the range of temperature in which corresponding modifications of both pure components are stable. It would seem, however, that thorough going correspondence of phase diagrams demands a higher order of identity than simply ability to form a continuous series of mixed crystals at some one temperature and pressure. A probable example of this is $KHSO_4$ and $NH_4HSO_4$. It is very likely, although not definitely proved, that the ordinary rhombic form of $NH_4HSO_4$ is identical with the form of $KHSO_4$ above 180°, and that at pressures and temperatures in the region of stability of $KHSO_4$(I) the two salts form a continuous series of mixed crystals, unless indeed it should chance that one of the melting points is too low. On the other hand, it is conceivable that two substances should have corresponding phases with corresponding diagrams, but, because of special relations of the transition temperatures, no region of continuous mixed crystals. For instance, if the entire phase diagram of $TlNO_3$ were lifted to higher temperatures so that the transition from trigonal to rhombic takes place at say 160°, there would be no region in which the trigonal forms of $CsNO_3$ and $TlNO_3$ are completely miscible, but nevertheless the two phase diagrams would be closely corresponding. Whether cases of such large displacements of corresponding transition points occur in nature is a matter for experiment.

It is evidently useless to try to generalize from only the two groups found here. This conclusion does seem justified, however; similarity of phase diagrams between corresponding phases is evidence of identity of structure of a higher order than is concerned in the ordinary run of chemical or crystallographical phenomena. In general, the identity of structure must be complete enough to allow continuous series of mixed crystals, but this identity is not necessarily far reaching enough. Instances of such far reaching identity must be rare; it is all the more important to investigate other cases.

Similarity of phase diagrams means not only an identity of structure so complete that similarly arranged edifices are possible, but also means that the fields of force surrounding the elements are so similar that corresponding edifices are *stable*. This has a bearing on the custom of many crystallographers of classifying a substance as dimorphic if it can crystallize in limited proportions with another substance of different symmetry. Such a "dimorphism" cannot be of broad significance; it merely means that the similarity of the building stones of the two substances is great enough so that if sufficient compulsion is applied to the one set they may combine in limited proportions in the edifice appropriate to the other. A very wide range of similarity or dissimilarity is evidently included in a classification so elastic as this.

It is also the custom, or rather a matter of definition, to class a substance as polymorphic if it has more than one modification, stable or not. This again, has no well defined significance. Given a number of identical building blocks, it would evidently be possible to build these with our hands into a large variety of assemblages corresponding to different crystalline systems. Most of these arrangements would be very unstable, but all would persist for a small interval of time. This means that in this sense every substance is polymorphic in a very complicated way. In practise, however, not very many substances under ordinary conditions happen to form assemblages that are comparatively stable. But it is conceivable, and likely, that under different conditions of inoculation or subcooling the number of substances with unstable polymorphic forms (monotropic polymorphy) should be very largely increased. The point is that this kind of polymorphism is not of absolute significance, and the more we extend the list of polymorphic substances by increased skill in manipulation, the less significant does it become. There is no denying, of course, that the easy and persistent polymorphism shown by phosphorus, for example, is significant — it is merely impossible to draw a sharp line. Furthermore, many cases now classified as monotropically polymorphic would turn out to have larger significance if certain lines of conjecture prove to be justifiable. Many investigators have thought that forms unstable under atmospheric conditions would become stable at higher pressures. In this case such substances would become as significant as those enantiotropically polymorphic under ordinary conditions.

The experiments above have shown, however, that in none of the cases examined have forms ordinarily unstable become stable at higher pressures. This is, after all, not surprising, because the majority of unstable forms are less dense than the stable forms. Their region of stability, if it exists, is at negative pressures. The example of $MgSO_47H_2O$, mentioned above, is a case on the other hand where the unstable form is more dense, but is not formed at high pressures, even when inoculated. The dense unstable form of $MgSO_47H_2O$ may be obtained in the pure state by crystallizing from the supersaturated solution on inoculation with a crystal of $FeSO_47H_2O$.

Cases have been found, however, in which a phase unstable at high pressures becomes stable at relatively higher pressures. Water below 0° and near 6000 kgm. crystallizes most readily in the form of ice VI, which is unstable, in preference to the stable form, ice V. Ice VI is the denser form and becomes stable at higher pressures. Acetamide is another similar example. The case of o-kresol is of the opposite kind. Above the triple point the unstable modification I is very much more likely to crystallize from the melt than the stable form II. I is the less dense form, and it has a domain of stability at lower pressures, corresponding to negative pressures for ordinary substances. Although there are these examples showing the possibility in some cases of an unstable modification acquiring a region of stability, it is likely that in the majority of cases the unstable forms have no region of stability within experimental reach.

It seems preferable on the whole, therefore, in this discussion to confine the use of the word polymorphic to those substances with two or more phases which are capable of reversible transitions. It is of interest to inquire what is the frequency of occurrence of polymorphism. I have already emphasized that an examination like that above of many substances cannot possibly disclose all cases of polymorphism; a number of stable forms will not appear because of viscous resistance. The phase diagrams afford several cases where the new phase would not have been discovered if the exploration had been confined to the low temperatures. Examples are $KHSO_4$, $HgI_2$, and o-kresol. It is not possible to give any general rule that will show in what region a transition will become so viscous that it cannot be observed, because the behavior of different substances is very different. As a general rule, however, the viscous resistance to transition becomes greater at greater distances from the liquid phase. Substances with high melting points would be expected to show less frequent polymorphism. Now in all the above list of substances with polymorphic forms, the highest melting point is 628°. Not one of the substances examined which has a higher melting point shows polymorphism in my range. The only example among substances which I did not examine is $Cu_2S$ with a melting point of 1100°. There are, however, numerous examples known of polymorphism at higher temperatures; several of the substances examined above belong here. It is therefore likely that many of the substances would show polymorphism if examined over a wider range. It is, furthermore, significant that the nitrates, among which polymorphism is widely prevalent, are all low melting, as are also the iodides. The organic compounds all have low melting points; in the above list there are 39 organic compounds, of which 11 are polymorphic. This does not include substances with unstable forms. Of the inorganic substances with known melting points below 650°, 25 out of 42 are polymorphic. Polymorphism seems of more frequent occurrence with inorganic compounds. As a general average, perhaps one out of three substances are polymorphic.

We next examine the relative frequency of occurrence of the different crystalline systems. I have not yet been able to determine the system of any of the new forms stable at higher pressures; we cannot yet tell whether all substances tend to any one simple type under high pressures. The known forms include 17 cubic, 3 (or 4) tetragonal, 8 trigonal, 11 rhombic, 4 monoclinic, and 1 triclinic. The relatively high frequency of the rhombic system is perhaps surprising. The number of cases in which the cubic form, which has the highest symmetry, is of greater volume than a neighboring more unsymmetrical form is striking. It would perhaps be natural to expect that the forms stable at the higher temperatures, with the greater energy of temperature agitation, and in many cases the greater volume, would have fewer elements of symmetry. However, in 10 of the above 17 cases, the cubic crystal may be transformed by proper change of pressure and temperature to a phase of smaller volume and also of lower symmetry. It is evident that the cubic arrangement in these cases cannot be the arrangement of closest packing. There is, of course, no especial reason to expect it when the crystal is built up of different kinds of atoms. Out of the five cases above in which a trigonal form adjoins a known form, four are cases in which the trigonal form adjoins another either of smaller volume and lower symmetry, or larger volume and higher symmetry. The general rule seems to be the reverse of what we would expect, the phase of higher symmetry in the majority of cases has the larger volume.

In this connection it is also interesting to note that there are several cases in which the same crystalline system occurs in more than one phase of the same substance. $NH_4NO_3$ has two tetragonal forms (possibly these are identical), $RbNO_3$ has two trigonal forms, probably $KHSO_4$ two rhombic, and of course $NH_4I$, $NH_4Br$, and $NH_4Cl$ are a striking series in which the different modifications belong to the same sub-group. This shows that there is no restriction placed on the total number of possible polymorphic forms of any one substance by considerations of this character. As far as this goes, we might have more than 32 modifications.

It would seem that in our present state of knowledge the specification of the crystalline system of different polymorphs is without special significance, but is of value chiefly as a means of identification. And probably when we are able to give a more detailed description of the structure, specification of the crystalline system will be superfluous.

We now turn from crystallographical considerations, and discuss the general thermodynamic aspects of the phase diagrams. The enormous complexity of the phase diagrams of solids as contrasted with the melting diagrams is apparent. There are only two known melting curves that fall in temperature with rising pressure, those of water and bismuth; all others rise. The rising melting curves rise indefinitely, with no suggestion of a maximum or a critical point. All the melting curves, whether rising or falling, are concave toward the pressure axis. On every one of the curves $\Delta v$ decreases with rising temperature, and on every curve where accurate enough measurements can be made, the curve plotting $\Delta v$ against pressure is convex toward the pressure axis. Furthermore, the liquid is universally more compressible, and of a higher specific heat than the solid, and only one case is known in which the liquid has a smaller thermal expansion, that of water over a restricted range.

None of these uniformities hold for polymorphic changes. Retrogressive transition lines are of fairly common occurrence. Wallerant makes the statement in his Cristallographie that there are only four known transitions of this type; AgI, $NH_4NO_3$, Boracite, and Calcium Chloraluminate. In all, sixteen such transitions have been examined above, fourteen of them not known to Wallerant. It appears, then, that nearly one quarter of all the transition curves are of the ice type. The persistence of the curves for $AgNO_3$ and $HgI_2$ suggests that an ice type of transition may be as capable of continued stability over a wide range of pressure and temperature as an ordinary transition. In the early stages of this work I was inclined to regard the existence of an ice type of transition as *à priori* evidence that there must be at higher pressures a normal transition to supplant it, as on the melting curve of ice I. This surmise did not prove fruitful.

A summary of all the transition lines examined is shown in Table XIV. This shows the number of various classes of lines grouped according to important characteristics. Thus, for example, out of 69 lines examined, there are 3 rising curves whose direction of curvature is abnormal, and whose direction of variation of $\Delta v$ is also abnormal. In drawing up this table, normal behavior has been called that which is like that on the melting curve. In detail, normal curvature is concavity downward, normal variation of $\Delta v$ is decrease with rising temperature, $\alpha$ and $\beta$ are normal if the phase of larger volume is the more compressible or expansible, and $C_p$ is normal if the phase stable at the higher temperature has the higher specific heat. In drawing up the table all those lines which are sensibly straight, 33 out of 69, were not tabulated as of either normal or abnormal curvature, but their other properties were tabulated under the normal branch. The results for $HgI_2$ have not been included at all, because its curve both rises and falls.

In general the normal type of behavior preponderates, but the possibilities that have been discovered are so numerous that one would be prepared to admit that after extensive search probably representatives of every one of the divisions could be found. It is certainly evident that the mechanism of polymorphic transitions in different substances does not possess any one notable characteristic which expresses itself in a common type of behavior on all the transition lines, as is the case for melting.

Two significant features of the table call for comment. In the first place, abnormal curvature means that the factor by which the change of volume is multiplied to give the change of internal energy becomes smaller at higher pressures. This factor is called by some writers the internal pressure, and is taken as a measure of the internal cohesion. It is at first surprising that this cohesion should become less as the substance is compressed so as to occupy less volume. It is difficult to imagine the possibility of such an effect in a substance composed of spherical molecules. The effect must be due to the configuration of the molecules and the location of the centers of force — when the new modification is formed the centers of force are torn farther apart, but the geometrical centers come closer together. In the second place, the number of cases in which the compressibility is abnormal cannot but be significant; there are 17 abnormal cases against 10

TABLE XIV.

Summary of Behavior of Transition Lines.

| | | | | $\alpha$ | $\beta$ | $C_p$ |
|---|---|---|---|---|---|---|
| Rising 53 | Normal Curvature 18 | N. Δv 34 | N. | 8 | 4 | 4 |
| | | | Ab. | 7 | 4 | 5 |
| | | Ab. Δv 4 | N. | | | |
| | | | Ab. | 1 | | |
| | Abnormal Curvature 9 | N. Δv 4 | N. | | 1 | 1 |
| | | | Ab. | 2 | 2 | 2 |
| | | Ab. Δv 3 | N. | | | |
| | | | Ab. | | | |
| Falling 16 | Normal Curvature 8 | N. Δv 6 | N. | | | 3 |
| | | | Ab. | 4 | 1 | |
| | | Ab. Δv 8 | N. | 1 | 2 | 3 |
| | | | Ab. | 3 | 2 | |
| | Abnormal Curvature 1 | N. Δv 1 | N. | 1 | | |
| | | | Ab. | | 1 | 1 |
| | | Ab. Δv | N. | | | |
| | | | Ab. | | | |

normal cases. As a rule, therefore, the phase of larger volume is the less compressible. A possible way of understanding this is described later.

In connection with the multiplicity of type of transition lines Roozeboom's classification of triple points may be mentioned. He recognized eight different varieties, according to the relative location of the lines in different quadrants. Figure 32 shows the groups. At the time that he wrote examples of only one group were known, his group 4. The data above include examples of groups 1, 2, 3, 6, and 7. The only cases now missing are 5 and 8. The number of cases in the respective groups are 1, 1; 2, 4; 3, 5; 4, 8; 6, 2; and 7, 1.

Although the transition lines present great complexity of shape, there are two special types of which no examples have been found, lines with a minimum temperature or a critical point. The minimum temperature does not seem essentially different from a maximum, and

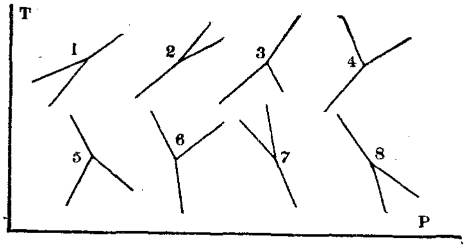

FIGURE 32. Reproduction of Roozeboom's classification of triple points.

there is no reason to suspect that such cases may not exist. But it is different with a critical point. If such exists, it means that by going around this point we can pass by continual gradations from one crystalline system to another. Whether such a possibility exists is open to grave question. It is at least significant that no case of it has been found above. The closest approach to it was on the IV–VI line of camphor, which could not be studied much further because of the extreme stickiness of the transition. Perhaps theoretically there is no objection to such a transition; we can imagine for instance that after a certain stage in the uniform compression of a cubic lattice one of the dimensions begins to change differently from the others, and the crystal becomes tetragonal. But although we can imagine the kinetics of such a change of system, we cannot conceive any adequate physical reason for such a change, and it is certain that any such change of system would be fundamentally opposed to all our present experience of crystals. It is easy to prove that if Neumann's law holds, that is, if all the other physical properties possess all the elements of symmetry of the crystalline form, then a uniform hydrostatic pressure or a uniform change of temperature cannot so change the dimensions of the crystal as to alter its crystalline system. The existence of a critical point is not compatible with Neumann's law. If Neumann's law expresses some fundamental fact of crystal structure, as it probably does, then we are morally certain that a critical point does not exist, and if a critical point should be discovered, this alone would be sufficient to dethrone Neumann's law from a position of vitally fundamental importance.

There is another possibility. Instead of suddenly stopping at a critical point, where the volumes of the two phases become equal, the transition line might suddenly stop at a point where the volumes were different, and spread out into a fan shaped region occupied by a continuous series of mixed crystals of the two phases as shown in Figure 33. But although this sort of thing is possible thermodynamically, such a phenomenon would be even more foreign to our experience than a critical point, and in all probability does not exist. Such an effect would be detected experimentally by a continuous change of volume throughout the shaded region. Never in any of my work have I found anything to suggest such an effect.

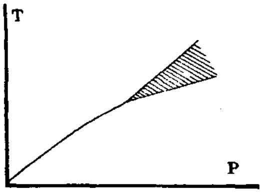

FIGURE 33. Showing a conceivable degeneration of a transition line into a region of mixed crystals.

It is interesting to inquire whether the theory of solids derived from quantum hypothesis has any restrictions to impose on the shape of the transition lines. The quantum hypothesis when applied to solids demands that in the neighborhood of the absolute zero $C_v$ and the thermal expansion are proportional to the third power of the temperature, and that the compressibility is constant. That is, $\left(\frac{\partial v}{\partial p}\right)_\tau = a$, $\left(\frac{\partial v}{\partial \tau}\right)_p = \beta\tau^3$, and $C_v = \gamma\tau^3$, where $a$, $\beta$, and $\gamma$ are constants. An easy thermodynamic transformation gives the relation $C_p = \gamma\tau^3 - \frac{\beta^2}{a}\tau^7$. Now if we assume that each phase separately satisfies these conditions, except with different values of the constants, and if we write down the condition of equality of the thermodynamic potentials of the two phases, we shall find for the equation of the transition line near the absolute zero,

$$ap^2 + bp\tau^4 + c\tau^8 + dp + e\tau^4 + f = 0,$$

a conic in $p$ and $\tau^4$, where,

$$a = \frac{\alpha_1 - \alpha_2}{2},\ b = \frac{\beta_1 - \beta_2}{4},\ c = \frac{1}{56}\left[\frac{\beta_1^2}{\alpha_1} - \frac{\beta_2^2}{\alpha_2}\right],\ d = v_1 - v_2,$$

$$e = -\frac{\gamma_1 - \gamma^2}{12},$$

$$f = E_1 - E_2.$$

$E_1$ and $E_2$ are the internal energies of the two phases at 0° *abs.* and zero pressure.

It may be shown immediately by differentiation that the curve approaches the pressure axis perpendicularly, a fact which, of course, has been known for some time. It is easy to see by solving the equation for $p$ at $\tau = 0$, that the following conditions must hold if there is a transition at absolute zero at some positive pressure. The existence of such a transition point is compatible with either a positive or a negative value for $a$. If the phase of larger volume is more compressible, no restriction is placed thereby on $b$, and no restriction placed on $E_1 - E_2$ $(= f)$ as to sign, but there is a restriction as to magnitude to insure that $p$ shall be real. If the phase of larger volume is less compressible, no restriction is thereby placed on $b$, but $E_1 - E_2$ must be negative and not too large. That is, the internal energy of the phase of smaller volume must be greater in this case. Whether the transition line, which is vertical at the axis, bends to the right or the left at higher temperatures depends on the sign of $b$; it curves toward the temperature axis if the phase of larger volume becomes more expansible at higher temperatures, and conversely.

The only result of this analysis of immediate interest is that quantum hypothesis still leaves open the possibility of various types of behavior at absolute zero, and in so far does not demand phenomena different in kind from those which we have found within our range.

There is another interesting point in connection with polymorphic transitions at the absolute zero. If at the absolute zero a transition runs irreversibly, in the metastable region, it is accompanied by evolution of heat, according to the equation $\Delta H = (v_1 - v_2)\Delta p$. This is demanded by the law of the conservation of energy. The idea of heat in finite quantities playing any essential or necessary part in phenomena at absolute zero seems at first a little strange. It means that in the mechanism of polymorphism there is something that during the change converts potential into kinetic energy.

At higher temperatures there is an interesting suggestion for the quantum theory of solids in the fact that there are polymorphic phases which are stable at the higher temperature, but have the lower specific heat. That is, the modification of higher specific heat absorbs heat and passes to a modification of lower specific heat. If the total heat (kinetic energy) content of either modification is equal to the heat absorbed in warming from absolute zero $(=\int c_p d_\tau)$ then certainly the specific heat curves of the two modifications cannot be of the same character all the way down to the absolute zero, but one must cross the other. Therefore the expression for the energy of the two phases cannot be of a universal type, differing only by the numerical value of a characteristic constant, as are the ordinary expressions of quantum theory. This means that substances with polymorphs cannot be treated like monatomic crystals, which is not surprising; but it also means that the characteristic function of more complicated salts is a function not only of the kinds of atoms, but also of their arrangement. Of course we have applied considerations above to $C_p$ which properly apply only to $C_v$, but this can usually be done without sensible error.

It is interesting to note in this connection that from the values which we have given above for $\Delta\alpha$, $\Delta\beta$, and $\Delta C_p$, we cannot calculate the value of $\Delta C_v$. $\Delta C_v$ cannot be obtained from the equation of the transition line and the difference of compressibility or $C_p$; to calculate it we must know the absolute value of at least either the compressibility, expansion, or $C_p$.

The last part of this discussion is to be occupied with considerations intended to make understandable how it is that there are polymorphic changes, and that the phenomena are as complicated as we find them. This will not be a theory of polymorphism in the proper sense of the word.

Brief mention should be first made of the recent theory of polymorphism of Smits[40]. His theory is that any substance which shows polymorphism forms at least two different kinds of molecules, and that these two kinds of molecules are present in both phases, but in different proportions. Furthermore, the ratio between the two kinds of molecules in a single phase varies continuously with temperature. It must be frankly recognized that this theory has succeeded in correlating a formidable array of chemical facts, many of them new, but nevertheless it seems to me that the picture it presents of the mechanism of a crystal involves considerable physical difficulties. It is hard to see how components of varying proportion can be arranged on definite space lattices, as we know they are. The modern conception of the crystal is one for which the molecule has lost its significance. We know that when we build up a crystal from its elements out of solution or the melt, these elements are added as entire molecules; or when a crystal is taken apart, as by melting, the crystal comes apart in molecules. Inside the crystal, however, the molecular bonds lose their individuality and fuse together. The molecular bond makes its appearance only when we try to remove a part from the crystal. Certainly as a thermodynamic entity, concerned with specific heats, the molecule has little significance for crystals.

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[40] A. Smits, Proc. Amst. Acad., numerous papers (1910–1915).

It is significant that none of Smits' results were obtained with single crystals, but with aggregates of small crystals. Between the crystal grains there must be transition layers more or less amorphous in character, and the idea of association may be applicable within these layers. It is not unreasonable to suppose that the phenomena which Smits finds are consistent with the idea of a varying association are connected with the transition layers. At any rate it seems to me that at present we should withhold acceptance of Smits' theory, in spite of the chemical facts on its side, until he has shown by actual X-ray photographs that in an individual crystal, of $HgI_2$ for example, there is the continuously varying constitution demanded by his theory.

This discussion is to be guided by the same idea as that underlying a previous discussion of the thermodynamic behavior of liquids under high pressures. It turned out that the experimental facts were of an unexpected complication, a complication so great that previous pictures of the atoms as smooth spheres were powerless to provide a sufficient range of possibilities. It was the purpose of that discussion to show that if we made the next degree of refinement in our representation of the atoms and considered them as having characteristic shapes, that we had thereby opened up at least the possibility of explaining all the effects. The purpose of this discussion is the same; to show that if we go to the next stage of refinement and consider the effect of the shape of the atoms, we have in our hands at least a possibility of explaining the complicated facts of polymorphism. As a matter of scientific economy, we are bound to push as far as we can the possibilities of the next stage of refinement. The discussion will in part take the form of showing in detail how some of the unexpected effects which we have found may be exhibited by aggregates of building blocks with definite shapes.

The crystal is supposed to be composed of units, atoms or molecules as the case may be, which remain the same in different polymorphic forms. Polymorphism is to be regarded in its most general aspect as due to regrouping of these units in different arrangements. This does not rule out at all the possibility of such special groupings as are considered in the association theories in which larger related groups may be distinguished. Each one of the units is to be thought of as terminated by rigid boundaries, that is, each unit has a shape as definitely as a brick has shape. Furthermore, at different localities on the surface of the units there are localized centers of force (attractive usually) so that two units, if free, will tend to come together with a definite orientation. A crystal is to be regarded as a system in which a compromise has been affected between the arrangement which the units would take in virtue of the action of the localized centers of force, the arrangement into which the units would be urged by the external pressure or the mean internal pressure so as to occupy the smallest possible volume, and the chaotic disarray which temperature agitation tends to produce.

This conception of an atom or molecule as a hunk of matter of definite shape and with localized centers of force is no doubt crude from certain points of view. We require a more detailed picture to account for the scattering of $\alpha$ particles, for example. But it does seem to contain enough of the essentials of the situation to make it a suggestive tool of thought in dealing with polymorphic changes. Doubtless a more valid picture of the atom is as a field of force. But in the last analysis this comes down to much the same thing as saying that the atom has shape. All that we mean when we say that any object of our experience has shape or boundaries is that as we approach the object the force with which it acts on us changes at a certain stage very rapidly into an intense repulsion. Our idea of shape is only a qualitative one, depending on how fast the repulsive force increases. It is worthy of remark that if we regard the atom as essentially a field of force we shall find that in order to account for the observed facts this field of force must become very rapidly a repulsion beyond a certain critical distance. Thus Grüneisen,[41] who has succeeded in correlating many of the thermodynamic properties of solids by assuming the atoms to be the centers of attractive and repulsive forces uniform in all directions, finds that the repulsive force must vary at least as the inverse twelfth power of the distance, while the attraction varies as the square. So rapid an increase of the repulsive force means a close approach to the phenomena of a definite boundary. As for regarding the atom (or unit) as possessing a definite geometrical shape other than spherical, we cannot well do otherwise, for the very existence of crystals shows that the field of force about an atom is not uniform in every direction. It has perhaps been more usual to regard the crystal unit as a rigid sphere with localized centers of attractive force. But this seems to be an unnecessary and unjustifiable restriction. From the point of view of the field of force, this is equivalent to saying that the attractive forces are unsymmetrically deposed, whereas the repulsive forces are symmetrically directed toward all sides.

The atom or molecule need not be thought of as absolutely rigid. It must of course be deformable to a certain extent, and there is experimental evidence (the increase of energy of a liquid if isothermal compression is carried beyond a certain point[42]) that such an effect is actually of importance. Furthermore, the temperature agitation in a solid is doubtless largely a matter of internal agitation of the molecules, rather than of motion of the molecules as wholes.

One of the puzzles of polymorphic change is offered by substances like benzol, in which the transition has no latent heat, but the internal energy of the phase of smaller volume is the greater. If the forces between the molecules are on the whole attractive, then we should expect the potential energy to be decreased when the molecules are brought closer together, instead of increased. An increase of energy would be expected only if the approach of the molecules took place against an average repulsive force. But if the force is on the average repulsive, it is difficult to see why the molecules become unstable and assume a new arrangement. However, if the atoms have properties like those above, this becomes understandable. Figure 34 shows schematically what may be the arrangement in the phase of larger volume. The molecules assume such an arrangement that the localized centers of force approach as closely as possible. This is the arrangement of minimum potential energy. But this is not the arrangement of smallest possible volume. As pressure is increased the molecules may be forced to turn into the positions of Figure 35, occupying the minimum volume. In this arrangement the attractive centers have been torn apart, and the potential energy has been increased.

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[41] E. Grüneisen, Ann. d. Phys., **39**, 257 (1912).
[42] G, p. 95.

This point is one of wide importance. If we suppose that the energy of temperature agitation is the same in two different phases at the same temperature, the mere existence of transitions for which there is no latent heat and all the work done by the external pressure is stored up as an *increase* of internal energy, affords conclusive evidence that the forces with which the atoms act on each other cannot be effectively situated at the centers of the atoms. The assumption of central forces is one that has been very widely used, from the early writings of Poisson on theoretical elasticity to the recent speculations of Grüneisen. These considerations show that this is not an adequate method of representing the inter-atomic forces, at least when the atoms are close together as in a solid. The effective centers of atomic attraction are not situated at the geometrical centers of the atoms, but must be nearer the surface. It is furthermore probable that the centers of attraction are *very* near the surface, because the nearer they are to the surface, the easier it is to conceive, without doing violence to our other conceptions of the atom, how it is that pushing the geometrical centers closer together may pull the centers of attraction further apart.

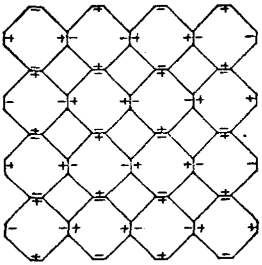

Fig. 34.

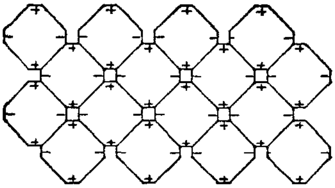

Fig. 35.

FIGURE 34. Hypothetical substance composed of approximately square atoms.

FIGURE 35. The atoms of Figure 34 in another arrangement. The arrangement of largest volume is that of minimum potential energy.

Another puzzling question is how we are to account for the phase at the higher temperature having the smaller volume. The diagrams above show how this may be. The localized centers of force are situated on projections some distance from the center. At low temperatures, when the energy of agitation is small, the molecules arrange themselves in a form in which the centers of force neutralize each other, producing a crystal with large open framework. But with increased temperature agitation, the forces at the apexes can no longer withstand the disrupting effect, and above a certain temperature the points are shaken loose, and the molecules settle down to the arrangement of smaller volume. Evidently high pressure also tends to produce the phase of smaller volume, so that at high pressure the temperature need not be raised so high to shake the crystal into the phase of smaller volume. In other words, when the volume of the high temperature phase is smaller, increased pressure lowers the transition point.

In the summary of Table XIV we have found on falling transition curves seven cases of abnormal compressibility, and only two of normal. (By "abnormal" compressibility we mean that the phase of larger volume is less compressible). The model of Figures 34 and 35 also suggests the reason for this. In general, two effects contribute to the apparent compressibility of a substance; the actual change of volume of the molecules under pressure, and the closing up of the free spaces which provide some of the possibility of temperature agitation. Now evidently in Figure 34 there can be very little free space for temperature agitation, because if the centers of the molecules are separated by only a slight distance from the position of tight packing a very small angular displacement suffices to carry the corners out of register with each other, and the structure becomes unstable. In the structure of Figure 35, however, much greater separation of the centers from the position of actual contact is possible before a given angular displacement carries the corners past each other. The phase of smaller volume is more compressible, therefore, because in it there is more free space from which the temperature agitation may be excluded. One does not care to speculate much about the behavior of the specific heats, now that the theorem of equipartition is known not to be valid, but it would seem in general as if the phase of smaller volume would have the most freedom, and so the greater specific heat, although it is conceivable that if the energy of temperature agitation were chiefly energy of the nucleus, that the nucleus might have less freedom at the smaller volume, and so the specific heat of the phase of smaller volume be less.

All the preceding considerations apply to transitions of the ice type. The ordinary type of transition, in which the form stable at the higher temperature has the larger volume, may be thought of as one in which projections on one molecule fit into depressions on others, the centers of force being on the projections or in the depressions. The projections or depressions are such that the molecules must be fitted together rather precisely to secure proper alignment. Temperature throws

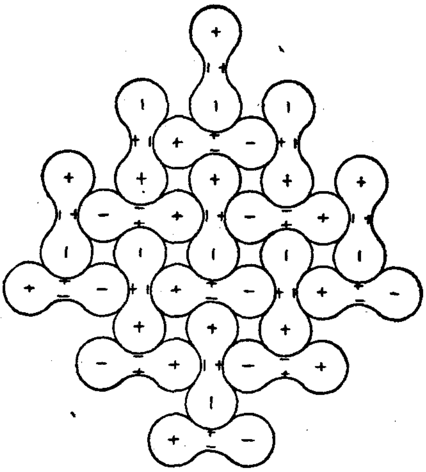

FIGURE 36. Hypothetical substance. This arrangement does not allow temperature agitation of much violence.

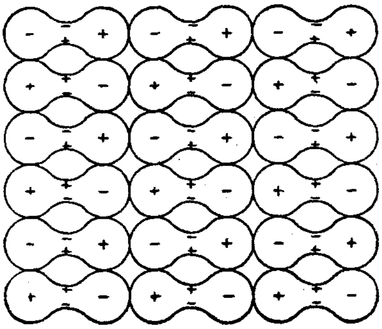

FIGURE 37. The atoms of Figure 36 in another arrangement occupying more volume and allowing greater violence of agitation.

them out of these nicely fitted positions into other positions further apart, of less mean internal pressure, and greater freedom for temperature agitation. Figures 36 and 37 show a possible scheme for the low and high temperature modifications. The effect of increased pressure is obviously to compact the phase of smaller volume, so that increased temperature is needed to pull the molecules apart. As a general rule, we would expect the phase of smaller volume to be more incompressible, but if the molecule is unequally compressible in different directions, as it must be, it is evident that under the proper conditions the phase of larger volume may be more incompressible.

The models of Figures 36 and 37 are suggestive in several other particulars. It may be mentioned in the first place that these units may be built up into several other structures, two at least of which have a still smaller volume than that of Figure 36, thus showing the possibility of a number of polymorphic forms. The arrangement of Figure 36 is evidently one which will show considerable persistence; it will stand a good deal of superheating and the transition velocity will be low, whereas that of Figure 37 will stand relatively slight superheating (or subcooling) and the transition velocity will be high. Figure 36 also illustrates a point made in a previous paper,[43] namely that if an atom passes from one modification to another it must rise from its position of equilibrium and pass through an intermediate position of greater potential energy. It is evident that some initial work will be required to pull an atom from the position of Figure 36, even if this work is more than returned when it settles down into the final position of equilibrium. In this way the band of indifference may be accounted for. It is also evident that if the two phases of Figures 36 and 37 are in contact there will be a field of force over the surface of each phase in which the atoms will tend to orient themselves in the appropriate positions. The tendency to pass from one phase to another is not an affair of absolute instability of one phase, but is a relative instability shown only in the presence of the other phase; this point was also mentioned in the paper just cited.

One implication of the view that regards crystals as built up from blocks of definite shape is to be especially insisted on. Only in exceptional cases will the edifice constructed from the blocks be such that there are no unfilled crevices around the corners, and in no case where there are two possible structures of different volumes will such empty spaces be absent in at least one of the structures. These empty spaces are to be thought of as playing an essential part in the phenomena of polymorphism. In the face of such changes as that from ice I to III with a change of volume of 17% and $NH_4Cl$, I to II, with a change of volume of 15%, it is most difficult to resist the conviction that there are empty spaces of relatively large size and that they are essentially concerned in polymorphic changes. Facts like these I find it most difficult to reconcile with Professor T. W. Richards' point of view that in a solid or a liquid nearly all the available space is completely filled by the atoms, and that the changes of volume produced by pressure or changes of temperature are the result of changes of volume of the atoms themselves. That the atoms are compressible, and that the compressibility enters essentially into many phenomena there can be no question, but it seems to me just as unquestionable that there must be vacant spaces around the corners which also play an important part in many phenomena, including preëminently the phenomena of polymorphism. The difficulty of Professor Richards' view seems to me increased by the fact that in many cases the phase of smaller volume is the more compressible.

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[43] I. p. 86.

These suggestions are no more than very rough indications of what may be the nature of the effects. The actual molecules are three dimensional instead of two; this alone will cause profound differences in the way in which a uniformly filled space can be built up from uniform elements. Furthermore, the elements in the diagrams have been chosen so simply that the crystalline framework of both polymorphs is the same, whereas only in isolated cases is this true in nature. It is of interest, however, that we have here a suggestion as to why it is that different phases may belong to the same system. We have also considered only one kind of element, whereas the majority of crystals are built up from different kinds of atoms. This alone will allow possibilities of enormously greater complications. The actual shape of the molecules are probably much more complicated and not so exaggerated as those shown above, and there must be a greater multiplicity of arrangements in which they can be piled. But the diagrams do illustrate the fertility of the fundamental idea; that by ascribing to the molecules definite shapes as well as localized centers of force, systems may be built up which show many of the complications of actual polymorphic forms.

The bewildering complexity shown by various polymorphic transition throws light on a group of phenomena of another kind. In a previous paper (G), I have described the effect of high pressure on the thermodynamic properties of a number of liquids. It appeared that under the more moderate pressures all the liquids tend to a more or less common type of behavior, but that at higher pressures the most varied abnormalities make their appearance, each liquid behaving in its own characteristic way. These abnormalities are doubtless to be explained by the increasing approach to some sort of order brought about when the molecules are squeezed closer together by high pressure. As pressure and temperature shift the predominating type of order in the relation of the molecules may change also, and we get the varied effects which we know occur when the arrangement of the atoms is altered as it is in a polymorphic transition. It seems most natural to suppose that the same sort of considerations and the same mechanism that will some day be discovered to account for polymorphic changes will also account for the complicated phenomena in liquids at high pressures.

## SUMMARY.

New data are presented for the polymorphic transitions of a number of substances between 0° and 200° and to 12000 kgm. All of the substances which I have examined which have shown no polymorphic changes in this same range are enumerated. This enumeration includes also all the substances with polymorphic forms, and classifies them according to chemical structure. The total number of substances examined to date is nearly 150. The discussion deals with all the phase diagrams examined in this series of papers. There is no simple type toward which all polymorphic diagrams tend at high pressures, nor are there a few common types. The complication is very great; probably solid transitions of any type whatever exist, except those involving a critical point. It is suggested that in order to explain these complicated phenomena the effect of the shape of the atoms is the next factor to be considered, and it is shown in detail by considering several artificial examples that assemblages of units of definite shape provided with localized centers of force are capable of exhibiting the variety of behavior which actual polymorphic substances show.

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# *THEORETICAL INVESTIGATION OF THE RADIATION CHARACTERISTICS OF AN ANTENNA.*

By George W. Pierce.

# THEORETICAL INVESTIGATION OF THE RADIATION CHARACTERISTICS OF AN ANTENNA.

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1. **Introduction.**— For the proper design of a radiotelegraphic transmitting station it is important to know the radiation characteristics of different types of antenna.

For example, if a flat-top antenna is to be employed, the question arises as to what is the best relation of the length of the horizontal part to the length of the vertical part, when the excitation is to be produced by a given type of generator. It may be known in a general way that the greater the vertical length, the greater the radiation resistance; it may also be known that the greater the horizontal length of the flat-top the greater the capacity of the antenna will be, and the greater will be the amount of current that can be made to flow from certain types of generator. Now these two quantities, radiation resistance and applied current, are both factors in determining the out-put from the antenna.

For a given generator, with known characteristics, the problem of getting the greatest output of high frequency energy is a problem in the determination of the maximum value of the product of current square and radiation resistance of the antenna.

But this is not the whole problem, for there comes also into consideration the question as to how much of the radiated energy is radiated by the horizontal flat-top in what may be a useless direction.

Again, of the energy radiated from the vertical part of the antenna, how much of it contributes to the electric and magnetic forces on the horizon, where the receiving station is situated?

For the solution of these various problems it is important to know the radiation characteristics of the antenna in the form of certain functional relations. These relations should be known even when inductance is added at the base of the antenna for providing coupling or for increasing the wavelength to adapt it to the generator. These quantities should be known theoretically, since the ordinary measurements of these quantities do not permit us to distinguish radiation that is useful from the useless radiation as heat losses and from the radiation in useless directions.

It is the purpose of this paper to give a treatment of this problem. Such a treatment is, so far as I know, up to the present entirely lacking, but the method here employed is that developed by Abraham [1] in a very remarkable paper entitled *Funkentelegraphie und Elektrodynamik*. In that paper, Abraham obtained theoretically the characteristics of a straight oscillator vibrating with its natural fundamental and harmonic frequencies. The present work is an extension of Abraham's method to the much more difficult problem of an antenna with a flat-top and with added inductance at the base.

**2. Inadequacy of the Conception of an Antenna as a Doublet.**— Apart from the brilliant investigation by Abraham, all other attempts at the treatment of the radiation from an antenna assume that the antenna is a Hertzian Doublet. This is only a very crude approximation to the facts, for *the derivation of the electromagnetic field about a doublet assumes that the length of the doublet is negligible in comparison with a quantity that is itself negligible in comparison with the wavelength.*

Hence, the doublet theory will apply in all of its essentials to an antenna, only provided the length of the antenna is not greater than one ten thousandth of the wavelength emitted. Of course, it may be that at great distances from the oscillator, the theory that it is a doublet may not introduce any large errors into certain problems such as the propagation over the surface of the earth; but the present treatment shows that the doublet theory does introduce large errors into computations of such quantities as the electric and magnetic field intensities and the radiation resistance of an antenna. It seems probable that other problems also should be revised in such a way as to replace the conception of the antenna as a doublet by the view of it as an oscillator that has a length comparable with one quarter of the wavelength.

**3. Method of the Present Investigation.**— In the present investigation, a doublet of infinitesimal length is assumed *at each point of the antenna.* This is the device used by Abraham. These elementary doublets are free from the objection regarding their lengths, as they are of infinitesimal lengths, while the wavelength is that due to the whole antenna and therefore is enormously large in comparison with the lengths of the elemental doublets. The electric and magnetic forces due to each of the doublets is determined at a distance point and is summed up for all of the doublets of the antenna, *with strict regard to the difference of phase due to the different locations of the different doublets.* Such a process performed for all points of a distant sphere surrounding the antenna gives the total electric and magnetic forces at all points on the sphere. Then by integrating Poynting's Vector over the entire sphere, we obtain the total power radiated, and from this we compute the radiation resistance and other characteristics of the antenna.

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[1] M. Abraham: Physikalische Zeitschrift, **2**, 329–334 (1901).

The effect due to the vertical portion of the antenna and to the horizontal flat-top portion are computed separately, so as to give information as to how much energy is radiated with its electric force vertical to the horizon and how much parallel to the horizon.

In deciding as to the proper distribution of the elemental doublets along the antenna, the form of the current curve from point to point of the antenna is assumed independently. This process is not entirely above reproach, because Maxwell's equations, if they could be properly applied to the problem, would themselves give the distribution that is consistent with the applied electromotive force at the base of the antenna and with the shape and form of the antenna. This step of accurately deriving the distribution is, however, at the present time not possible of mathematical execution.

The distribution here assumed for the current in the antenna, as a function of the time and of the position along the antenna, is a generalization of the distribution assumed by Abraham, and is given in the next section.

**4. Assumed Current Distribution.**—The form of antenna to which the whole discussion is devoted is illustrated in Figure 1, and consists of a vertical portion of length $a$ and a horizontal flat-top portion of length $b$. These quantities $a$ and $b$ may have any relative values whatever.

At the base of the antenna is an arbitrary inductance L for varying the wavelength.

The current at any point P′ of the antenna is assumed to be given by the equation

$$i = I \sin \frac{2\pi c}{\lambda} t \cdot \sin \frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - l\right), \qquad (1)$$

where

c = velocity of light,
$\lambda_0$ = natural wavelength of the antenna without inductance,
$\lambda$ = the wavelength with the inductance,
$i$ = the current at the point P′,
$l$ = length measured along the antenna from the inductance to the point P′.

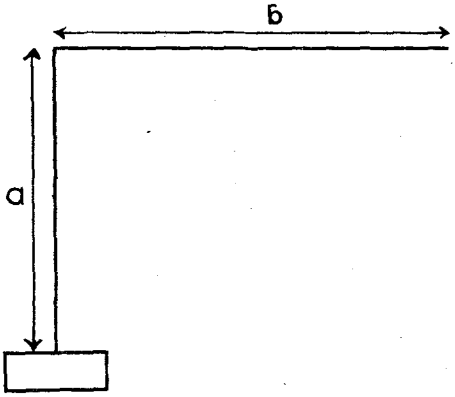

FIGURE 1.

The character of the assumed distribution is as follows: The factor $\sin\frac{2\pi c}{\lambda}t$ means that the current is sinusoidal in time at every point of the antenna, with the angular velocity

$$\omega = \frac{2\pi}{T} = \frac{2\pi c}{cT} = \frac{2\pi c}{\lambda}. \tag{2}$$

The meaning of the other factor

$$I \sin\frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - l\right) = J \text{ (say)} \tag{3}$$

is illustrated in the diagrams (a), (b) and (c) of Figure 2.

If there is no inductance, $\lambda = \lambda_0$, and the factor becomes

$$J = I \cos \frac{2\pi l}{\lambda}. \tag{4}$$

This is illustrated in (a).

In the case with added inductance, $\lambda \neq \lambda_0$, and we must keep the general form of J given in equation (3). This equation for positive values of $l$ gives the upper half of the diagram (b). When $l$ is supposed negative the curves obtained continue along the dotted lines of (b) and do not give a figure symmetrical with the upper half. *To produce proper symmetry the absolute value of l must be employed in equation (1) when it is applied to the distribution of the image to take account of reflection.*

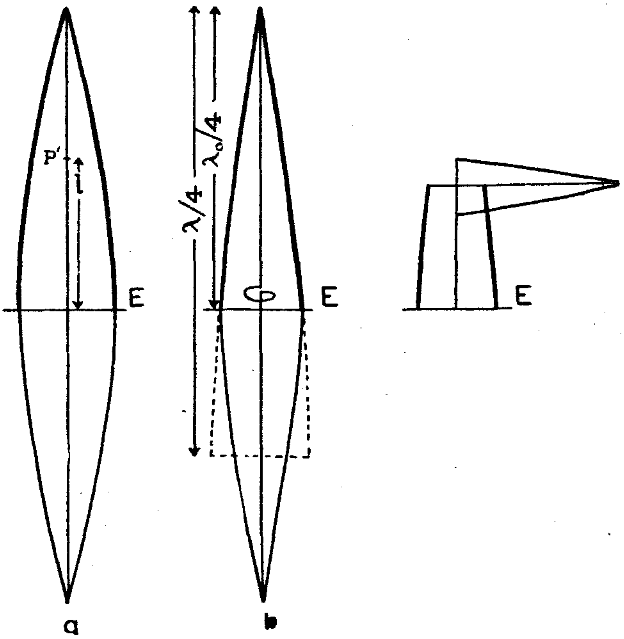

FIGURE 2.

It is also to be carefully noted that when $l = 0$, equation (1) becomes

$$i_0 = I \sin \frac{\pi\lambda_0}{2\lambda} \sin \frac{2\pi c}{\lambda} t, \tag{5}$$

so the amplitude at the base of the antenna is

$$I_0 = I \sin \frac{\pi\lambda_0}{2\lambda} \tag{6}$$

Now, finally, when the antenna has a flat-top it is assumed that the top part of the antenna is bent over without any significant change in the magnitude of the current at the various points.

When the equation (1) is to be applied to the vertical portion of the antenna, we shall call

$$l = z', \tag{7}$$

where

$z'$ = vertical distance from the ground of the point P′ on the antenna.

When the equation is to be applied to the horizontal part of the antenna, we shall call

$$l = a + x', \tag{8}$$

where

$x'$ = distance along the horizontal part of the antenna to any point P″ on the flat-top.

The discussion will now be divided into several Parts: Part I. Electromagnetic Field Due to Vertical Portion of the Antenna; Part II. Field due to Horizontal Portion of the Antenna; Part III. The Mutual Term in Power Determination. Part IV. Computations of Radiation Resistance. Part V. Field Intensities and Summary.

# PART I.

## Field due to vertical portion of Antenna.

5. **Coördinates.**— Let the origin of coördinates be at the point of connection of the antenna to the ground. Let the $z$-axis be vertical. About this vertical axis as polar diameter, let us construct a system

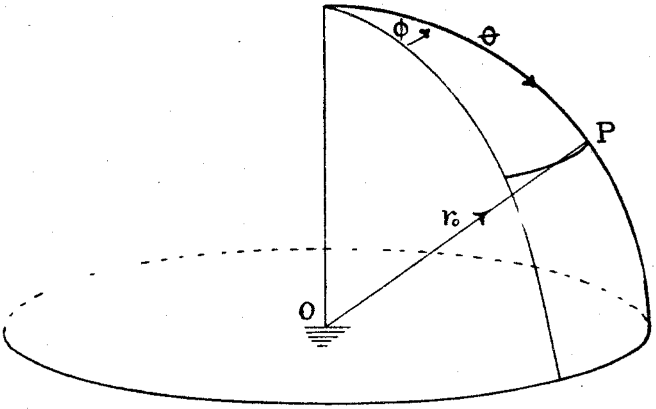

Figure 3.

of polar coördinates in which the position of any point P is given by its distance $r_0$ from the origin, and the angles $\theta$ and $\phi$.

$\theta$ = the angle along meridional lines from the pole,
$\phi$ = the angle along parallels of latitude from a vertical plane of reference whose position is at present immaterial.

This system of coördinates with the positive directions of the angles indicated is given in Figure 3.

If $z'$ is the vertical ordinate of any point P′ on the vertical portion of the antenna, and $r$ the distance from P′ to P, and if the distance OP is large in comparison with $z'$, we may write (see Figure 4)

$$r = r_0 - z' \cos \theta. \tag{9}$$

**6. Field Due to a Doublet at P′.**— At a distant point P the electric and magnetic intensities due to a doublet of length $dz'$ and charges $e$ and $-e$ at $P'$ is, by Hertz's theory,

$$dE_\theta = dH_\varphi = \frac{\sin\theta}{c^2 r_0}\ddot{f}\,(t - r/c), \tag{10}$$

where

$f(t)$ = the moment of the doublet

$\quad = e\,dz'$, (11)

$dE_\theta$ = the electric intensity in electrostatic units, which is entirely in the direction of $\theta$; that is, of the meridional lines;

$dH\phi$ = the magnetic intensity in electromagnetic units, which is entirely in the direction of the parallels of latitude.

$r$ = distance P′P in centimeters,

$c$ = velocity of light in centimeters per second.

The two dots over the $f$ in (10) indicate the second time derivative.

In writing equation (10), the slight difference in the direction of the perpendicular to $r$ from the direction of the perpendicular to $r_0$ is neglected in view of the largeness of $r_0$ in comparison with the length $z'$ measured on the antenna.

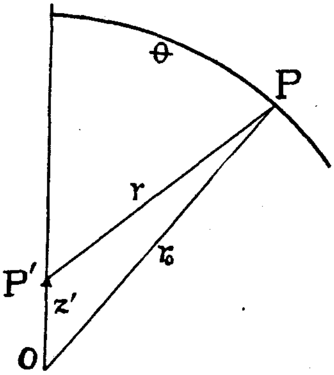

FIGURE 4.

Also the $r$ which should occur in the denominator of (10) has been replaced by $r_0$, which can be done without appreciable error for large values of $r$. The same substitution cannot be made in the argument of $f$ in (10), for there $r$ determines the phase of the oscillation, and this phase changes through an angle of $\pi$ for a half wavelength, independent of the distance from the origin.

**7. Expression of the Field in Terms of Current.**— We shall next express the moment of the doublet and the intensities of the field in terms of the current $i$ at the point $z'$. To do this we shall think of the current as delivering a charge $+ e$ to one end of the element of length $dz'$ and a change $- e$ to the other end of $dz'$ in a certain time. A neighboring doublet has a different current and delivers different charges $+ e_1$ and $- e_1$ partly counteracting the charges of the given doublet, and leaving just the charge $e - e_1$ that actually occurs on the wire. This is represented in Figure 5.

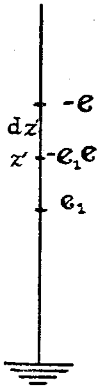

FIGURE 5.

With this view of the case

$$i = \dot{e},$$

and

$$\ddot{f}\,(t) = \ddot{e}\,dz' = \frac{\partial i}{\partial t}\,dz'. \qquad (12)$$

Whence, by substituting the value of $i$ from equation (1) into equation (12) we shall have, in view of (7) and (9)

$$dE_\theta = dH_\phi = \frac{2\pi I \sin\theta}{\lambda c r_0} \cos\frac{2\pi}{\lambda}\left(ct - r_0 - z'\cos\theta\right)\cdot \sin\frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - z'\right)dz'. \qquad (13)$$

By integrating this expression from $z' = 0$ to $z' = a$, we obtain the electric and magnetic intensities at the point P due to direct transmission from the vertical portion of the antenna. Indicating this integration, we have

$$E_\theta = H_\phi = \frac{2\pi I \sin\theta}{\lambda c r_0} \int_o^a \cos\frac{2\pi}{\lambda}(ct - r_0 - z'\cos\theta) \cdot \sin\frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - z'\right) dz'. \quad (14)$$

By reflection from the earth, which we shall regard as a perfect reflector, we have intensities that must be added to the above. These intensities may be obtained by considering the radiation to come from an image point at a distance $z'$ below the surface. The effect of this is obtained by changing the sign of the $z'$ in the cosine term of equation (14), but as was pointed out in section **4** the sign of $z'$ in the sine term must remain. We obtain thus for the intensities due to the reflected wave emitted by the vertical portion of the antenna the value

$$E_\theta = H_\phi = \frac{2\pi I \sin\theta}{\lambda c r_0} \int_o^a \cos\frac{2\pi}{\lambda}(ct - r_0 + z'\cos\theta) \cdot \sin\frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - z'\right) dz'. \quad (15)$$

Adding the equation (15) for the reflected intensities to the direct intensities of (14), remembering that if A and B are any two angles

$$\cos(A - B) + \cos(A + B) = 2\cos A \cos A, \quad (16)$$

we obtain for the total intensities at P the equation

$$E_\theta = H_\phi = \frac{4\pi I \sin\theta}{\lambda c r_0} \cos\frac{2\pi}{\lambda}(ct - r_0) \int_o^a \cos\left(\frac{2\pi z'}{\lambda}\cos\theta\right) \cdot \sin\frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - z'\right) dz', \quad (17)$$

which resolves into

$$E_\theta = H_\phi = \frac{4\pi I \sin\theta}{\lambda c r_0} \cos\frac{2\pi}{\lambda}(ct - r_0) \Bigg[ \sin\frac{\pi\lambda_0}{2\lambda} \int_o^a \cos\frac{2\pi z'\cos\theta}{\lambda} \cos\frac{2\pi z'}{\lambda} dz' - \cos\frac{\pi\lambda_0}{2\lambda} \int_o^a \cos\frac{2\pi z'\cos\theta}{\lambda} \sin\frac{2\pi z'}{\lambda} dz' \Bigg]. \quad (18)$$

This expression may be integrated by the formulas 360 and 361 of B. O. Pierce's *Short Table of Integrals* and gives

$$E_\theta = H_\phi = \frac{2\,I}{cr_0 \sin\theta} \cos\frac{2\pi}{\lambda}(ct - r_0) \left\{ \cos B \cos(A\cos\theta) - \sin B \cos\theta \sin(A\cos\theta) - \cos G \right\} \qquad (19)$$

where

$$\left.\begin{aligned} B &= \frac{2\pi b}{\lambda} \\ A &= \frac{2\pi a}{\lambda} \\ G &= \frac{\pi\lambda_0}{2\lambda} = A + B \end{aligned}\right\} \qquad (20)$$

The quantity $b$, which is the length of the flat top, gets into (20) and (19) by reason of the fact that $a + b =$ the whole length of the antenna, so that

$$\lambda_0 = 4\,(a + b). \qquad (21)$$

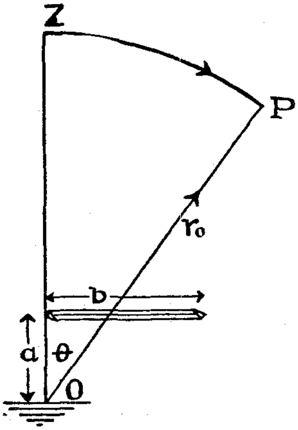

FIGURE 6.

*Equation (19), with the notation of equations (20) and (21) is the general equation for the electric and magnetic Intensities at any point P,* *due to the whole vertical part of the antenna.* In this formula, referring to Figure 6,

$r_0$ = the distance OP in cm.,
$\theta$ = the zenith angle ZOP,
$b$ = length of the horizontal flat top in meters,
$a$ = length of vertical part of antenna, in meters,
$\lambda_0 = 4(a + b)$ = natural wave length in meters,
$\lambda$ = wave length in meters actually emitted, and differing from $\lambda_0$ by virtue of the added inductance,
$I_0$ = amplitude of current in absolute electrostatic units at the base of the antenna and related to I by the equation,

$$I_0 = I \sin \frac{\pi\lambda_0}{2\lambda}\,.$$

We shall reserve comment on this equation until after investigation of other characteristics of the radiation. See Part IV.

**8. Total Power Radiated from the Vertical Part of the Antenna.** — Having obtained in equation (19) the electric and magnetic intensities at any required point at a distance from the antenna, we shall next compute the total power radiated from the vertical part of the antenna, and shall then obtain its radiation resistance.

Since $E_\theta$ and $H_\phi$ are perpendicular to one another and perpendicular to $r_0$, we have, according to Poynting's theorem for the power radiated in the direction of $r_0$ through an element of surface $dS$ perpendicular to $r_0$ the quantity

$$dp = \frac{c}{4\pi} E_\theta H_\phi \, dS. \tag{22}$$

Let the element of surface be an elemental zone on the surface of the sphere, then

$$dS = 2\pi r_0^2 \sin \theta d\theta \tag{23}$$

This quantity, together with the values of $E_\theta$ and $H_\phi$ from (19), substituted in (22) and properly integrated, gives for the total power radiated through the whole hemisphere above the earth's surface, the value in ergs per second following:

$$\begin{aligned}
p = \frac{2I^2}{c}\cos^2\left\{\frac{2\pi}{\lambda}(ct - r_0)\right\}\Bigg[&\cos^2 B\int_0^{\pi/2}\frac{\cos^2(A\cos\theta)\,d\theta}{\sin\theta}\\
&+ \sin^2 B\int_0^{\pi/2}\frac{\cos^2\theta\,\sin^2(A\cos\theta)\,d\theta}{\sin\theta} + \cos^2 G\int_0^{\pi/2}\frac{d\theta}{\sin\theta}\\
&- 2\sin B\cos B\int_0^{\pi/2}\frac{\cos\theta\,\sin(A\cos\theta)\cos(A\cos\theta)\,d\theta}{\sin\theta}\\
&- 2\cos B\cos G\int_0^{\pi/2}\frac{\cos(A\cos\theta)\,d\theta}{\sin\theta}\\
&+ 2\sin B\cos G\int_0^{\pi/2}\frac{\cos\theta\,\sin(A\cos\theta)\,d\theta}{\sin\theta}\Bigg].
\end{aligned}\tag{24}$$

This equation when integrated gives the power radiated from the vertical part of the antenna. The integration is a tedious operation, and is given in the next section, which may be omitted by readers not interested in the mathematical processes involved. The result of the integration is found in Section 10.

**9. The Integration of Equation (24).** — By the use of such trigonometric equations as

$$\cos^2 x = \frac{1 + \cos 2x}{2},$$

$$\sin^2 x = \frac{1 - \cos 2x}{2},$$

the squares of sines and cosines in the integrands of (24) may be avoided, and equation (24) written

$$\begin{aligned}
p = \frac{2\,I^2}{c}\cos^2\left\{\frac{2\pi}{\lambda}(ct - r_0)\right\}\Bigg[&\left(\frac{1}{2} + \cos^2 G\right)\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sin\theta}\\
&+ \frac{\cos 2\,B}{2}\int_0^{\pi/2}\frac{\cos(2\,A\cos\theta)\,d\theta}{\sin\theta} - \frac{\sin^2 B}{2}\int_0^{\pi/2}\sin\theta\,d\theta\\
&+ \frac{\sin^2 B}{2}\int_0^{\pi/2}\sin\theta\cos(2\,A\cos\theta)\,d\theta\\
&- \frac{\sin 2\,B}{2}\int_0^{\pi/2}\frac{\cos\theta\,\sin(2\,A\cos\theta)\,d\theta}{\sin\theta}\\
&- 2\cos B\cos G\int_0^{\pi/2}\frac{\cos(A\cos\theta)\,d\theta}{\sin\theta}\\
&+ 2\sin B\cos G\int_0^{\pi/2}\frac{\cos\theta\,\sin(A\cos\theta)\,d\theta}{\sin\theta}\Bigg].
\end{aligned}\tag{25}$$

The third and fourth terms may be integrated directly. In the other terms let us introduce a change of variable as follows:

Let

$$u = \cos\,\theta$$
$$d\theta = \frac{-\,du}{\sin\theta},$$

then

$$\int_o^{\pi/2} \frac{d\theta}{\sin\theta} = \int_1^o \frac{-\,du}{1-u^2} = \frac{1}{2}\int_o^1 \left(\frac{1}{1+u}+\frac{1}{1-u}\right)du$$
$$= \frac{1}{2}\int_o^1 \frac{du}{1+u} + \frac{1}{2}\int_{-1}^o \frac{du}{1+u} = \frac{1}{2}\int_{-1}^{+1} \frac{du}{1+u}. \quad (26)$$

With this operation as a model, two of the other integrals of (25) may be written, respectively

$$\int_o^{\pi/2} \frac{\cos\,(2\,A\,\cos\theta)\,d\theta}{\sin\theta} = \frac{1}{2}\int_{-1}^{+1} \frac{\cos\,(2\,Au)\,du}{1+u}, \quad (27)$$
$$\int_o^{\pi/2} \frac{\cos\,(A\,\cos\theta)\,d\theta}{\sin\theta} = \frac{1}{2}\int_{-1}^{+1} \frac{\cos\,(Au)\,du}{1+u}. \quad (28)$$

Another of the integrals, examined in more detail, gives

$$\int_o^{\pi/2} \frac{\cos\theta\,\sin\,(2\,A\,\cos\theta)\,d\theta}{\sin\theta}$$
$$= \int_o^1 \frac{u\,\sin\,(2\,Au)\,du}{1-u^2}$$
$$= \frac{1}{2}\int_o^{+1} \left(\frac{1}{1-u}-\frac{1}{1+u}\right)\sin\,(2\,Au)\,du$$
$$= -\frac{1}{2}\int_o^1 \frac{\sin\,(2\,Au)\,du}{1+u} + \frac{1}{2}\int_o^{-1} \frac{\sin\,(2\,Au)\,du}{1+u}$$
$$= -\frac{1}{2}\int_{-1}^{+1} \frac{\sin\,(2\,Au)\,du}{1+u}. \quad (29)$$

Similarly, the remaining integral becomes

$$\int_o^{\pi/2} \frac{\cos\theta\,\sin\,(A\,\cos\theta)\,d\theta}{\sin\theta} = -\frac{1}{2}\int_{-1}^{+1} \frac{\sin\,(Au)\,du}{1+u}. \quad (30)$$

Returning now to equation (25), we shall integrate the third and fourth terms, setting them first, and shall substitute (26) to (30) for the other terms, obtaining

$$
\begin{aligned}
p = \frac{2\,I^2}{c}\cos^2\left\{\frac{2\pi}{\lambda}(ct - r_0)\right\}\Bigg[ & -\frac{\sin^2 B}{2} + \frac{\sin^2 B}{2}\frac{\sin 2\,A}{2\,A} \\
& + \left(\frac{1}{2} + \cos^2 G\right)\frac{1}{2}\int_{-1}^{+1}\frac{du}{1+u} \\
& + \frac{\cos 2\,B}{4}\int_{-1}^{+1}\frac{\cos(2\,Au)\,du}{1+u} + \frac{\sin 2\,B}{4}\int_{-1}^{+1}\frac{\sin(2\,Au)\,du}{1+u} \\
- \cos G\left\{\cos B\int_{-1}^{+1}\frac{\cos(Au)\,du}{1+u} + \sin B\int_{-1}^{+1}\frac{\sin(Au)\,du}{1+u}\right\}\Bigg]. & \qquad (31)
\end{aligned}
$$

Let us now write

$$
\begin{aligned}
\gamma &= 2\,A\,(1+u), \\
2\,Au &= \gamma - 2A, \\
du &= \frac{d\gamma}{2\,A}, \\
\frac{du}{1+u} &= \frac{d\gamma}{\gamma};
\end{aligned}
$$

then the second and third integrals of (31) become

$$
\begin{aligned}
& \frac{\cos 2\,B}{4}\int_{-1}^{+1}\frac{\cos(2\,Au)\,du}{1+u} + \frac{\sin 2\,B}{4}\int_{-1}^{+1}\frac{\sin(2\,Au)\,du}{1+u} \\
&= \frac{\cos 2\,B}{4}\int_o^{4A}\{\cos\gamma\cos 2\,A + \sin\gamma\sin 2\,A\}\frac{d\gamma}{\gamma} \\
&\qquad + \frac{\sin 2\,B}{4}\int_o^{4A}\{\sin\gamma\cos 2\,A - \cos\gamma\sin 2\,A\}\frac{d\,\gamma}{\gamma} \\
&= \frac{\cos(2\,A + 2\,B)}{4}\int_o^{4A}\frac{\cos\gamma\,d\gamma}{\gamma} + \frac{\sin(2\,A + 2\,B)}{4}\int_o^{4A}\frac{\sin\gamma\,d\gamma}{\gamma} \\
&= \frac{\cos 2\,G}{4}\int_o^{4A}\frac{\cos\gamma}{\gamma}d\gamma + \frac{\sin 2\,G}{4}\int_o^{4A}\frac{\sin\gamma}{\gamma}d\gamma.
\end{aligned}
$$

In like manner, the last line of (31) becomes

$$
-\cos^2 G\int^{2A}\frac{\cos\gamma}{\gamma}d\gamma - \cos G\sin G\int_o^{2A}\frac{\sin\gamma}{\gamma}d\gamma. \qquad (33)
$$

Let us now decompose the coefficient of the first integral of (31) as follows:

$$
\begin{aligned}
\frac{1}{4}+\frac{\cos^2 G}{2} &= \frac{1}{4}+\cos^2 G-\frac{\cos^2 G}{2}\\
&= \frac{1}{4}-\frac{1+\cos 2\,G}{4}+\cos^2 G\\
&= -\frac{\cos 2\,G}{4}+\cos^2 G.
\end{aligned}
$$

Then the whole equation (31) may be written

$$
\begin{aligned}
p = \frac{2\,I^2}{c}\cos^2\left\{\frac{2\pi}{\lambda}(ct-r_0)\right\}\Bigg[&-\frac{\sin^2 B}{2}+\frac{\sin^2 B\sin 2\,A}{4\,A}\\
&-\frac{\cos 2\,G}{4}\int_o^{4A}\frac{(1-\cos\gamma)\,d\gamma}{\gamma}+\frac{\sin 2\,G}{4}\int_o^{4A}\frac{\sin\gamma}{\gamma}\,d\gamma\\
&+\cos^2 G\int_o^{2A}\frac{(1-\cos\gamma)\,d\gamma}{\gamma}-\frac{\sin 2\,G}{2}\int_o^{2A}\frac{\sin\gamma}{\gamma}\,d\gamma\Bigg]. \qquad (34)
\end{aligned}
$$

The various integrals may now be obtained by expanding in series and integrating term by term. This gives

$$
\begin{aligned}
p = \frac{2\,I^2}{c}\cos^2\left\{\frac{2\pi}{\lambda}(ct-r_0)\right\}\Bigg[&\frac{\sin^2 B}{2}\left(\frac{\sin 2A}{2A}-1\right)\\
&-\frac{\cos 2G}{4}\left\{\frac{(4A)^2}{2!2}-\frac{(4A)^4}{4!4}+\frac{(4A)^6}{6!6}-\cdots\right\}\\
&+\frac{1+\cos 2G}{2}\left\{\frac{(2A)^2}{2!2}-\frac{(2A)^4}{4!4}+\frac{(2A)^6}{6!6}-\cdots\right\}\\
&+\frac{\sin 2G}{4}\left\{4A-\frac{(4A)^3}{3!3}+\frac{(4A)^5}{5!5}-\cdots\right\}\\
&-\frac{\sin 2G}{2}\left\{2A-\frac{(2A)^3}{3!3}+\frac{(2A)^5}{5!5}-\cdots\right\}\Bigg]. \qquad (35)
\end{aligned}
$$

Let us now eliminate $B$ from the first terms of this equation, by substituting $B = G - A$, obtaining

$$
\begin{aligned}
\frac{\sin^2 B}{2}\left(\frac{\sin 2A}{2A}-1\right) &= \frac{1-\cos 2B}{4}\left(\frac{\sin 2A}{2A}-1\right) \\
&= \left\{\frac{1}{4}-\frac{\cos(2G-2A)}{4}\right\}\left(\frac{\sin 2A}{2A}-1\right) \\
&= -\frac{1}{4}+\frac{\cos 2G\cos 2A}{4}+\frac{\sin 2G\sin 2A}{4} \\
&\quad +\frac{\sin 2A}{8A}-\frac{\cos 2G\sin 4A}{16A} \\
&\quad -\frac{\sin 2G}{4}\,\frac{1-\cos 4A}{4A}. \qquad (36)
\end{aligned}
$$

If now we expand in series the quantities involving $A$ in (36) and substitute in (35), we obtain, if

$$
\left.\begin{aligned}
k &= 2A \\
q &= 2G
\end{aligned}\right\} \qquad (37)
$$

$$
\begin{aligned}
p = \frac{2I^2}{c}\cos^2\left\{\frac{2\pi}{\lambda}(\mathrm{ct}-r_0)\right\}\Bigg[&\frac{1}{4}\left\{-\frac{k^2}{3!}+\frac{k^4}{5!}-\frac{k^6}{7!}+\cdots\right\} \\
&+\frac{1}{4}\left\{\frac{2k^2}{2!2}-\frac{2k^4}{4!4}+\frac{2k^6}{6!6}-\cdots\right\} \\
&+\frac{\cos q}{4}\left\{1-\frac{k^2}{2!}+\frac{k^4}{4!}-\frac{k^6}{6!}+\cdots\right\} \\
&+\frac{\sin q}{4}\left\{k-\frac{k^3}{3!}+\frac{k^5}{5!}-\frac{k^7}{7!}+\cdots\right\} \\
&-\frac{\cos q}{4}\left\{1-\frac{(2k)^2}{3!}+\frac{(2k)^4}{5!}+\cdots\right\} \\
&-\frac{\sin q}{4}\left\{\frac{2k}{2!}-\frac{(2k)^3}{4!}+\frac{(2k)^5}{6!}-\cdots\right\} \\
&-\frac{\cos q}{4}\left\{\frac{(2k)^2}{2!2}-\frac{(2k)^4}{4!4}+\frac{(2k)^6}{6!6}-\cdots\right\} \\
&+\frac{\cos q}{2}\left\{\frac{k^2}{2!2}-\frac{k^4}{4!4}+\frac{k^6}{6!6}-\cdots\right\} \\
&+\frac{\sin q}{4}\left\{2k-\frac{(2k)^3}{3!3}+\frac{(2k)^5}{5!5}-\cdots\right\} \\
&-\frac{\sin q}{2}\left\{k-\frac{k^3}{3!3}+\frac{k^5}{5!5}-\cdots\right\}\Bigg]. \qquad (38)
\end{aligned}
$$

If now we add together the terms multiplied by sin $q$ and those multiplied by cos $q$, and those not involving $q$, we have (on factoring out the $\frac{1}{4}$)

$$p = \frac{I^2}{2c}\cos^2\left\{\frac{2\pi}{\lambda}(ct - r_0)\right\}\Bigg[\left\{\frac{2+2}{3!2}k^2 - \frac{4+2}{5!4}k^4 + \frac{6+2}{7!6}k^6 - \cdots\right\}$$

$$+ \cos q\left\{-\frac{2^2+2^2-4}{3!2}k^2 + \frac{4^2+2^4-6}{5!4}k^4 - \frac{6^2+2^6-8}{7!6}k^6 + \frac{8^2+2^8-10}{9!8}k^8 - \cdots\right\}$$

$$+ \sin q\left\{-\frac{3^2+2^3-5}{4!3}k^3 + \frac{5^2+2^5-7}{6!5}k^5 - \frac{7^2+2^7-9}{8!7}k^7 + \frac{9^2+2^9-11}{10!9}k^9 - \cdots\right\}. \tag{39}$$

*Equation (39) gives the total power radiated by the vertical portion of the antenna into the hemisphere above the earth's surface.* In this equation, the current factor $I$ is in absolute c. g. s. electrostatic units, and the power $p$ is in ergs per second.

It is convenient to change the current factor into amperes and the radiated power into watts, which can be done by multiplying the right hand side of (39) by 30 c. This is done, and the equation is rewritten in the next section.

**10. Result of the Integration for Power Radiated from the Vertical Part of the Antenna.** — By equation (39), when reduced to practical units, the total power radiated into the aërial hemisphere from the vertical part of the antenna may be written

$$p = I^2\cos^2\left\{\frac{2\pi}{\lambda}(ct - r_0)\right\}\left[R_1 - R_2\cos q - R_3\sin q\right], \tag{40}$$

where

$$\left.\begin{aligned}
R_1 &= 15\left\{\frac{2+2}{3!2}k^2 - \frac{4+2}{5!4}k^4 + \frac{6+2}{7!6}k^6 - \dots\right\}\\
R_2 &= 15\left\{\frac{2^2+2^2-4}{3!2}k^2 - \frac{4^2+2^4-6}{5!4}k^4 + \frac{6^2+2^6-8}{7!6}k^6 - \dots\right\}\\
R_3 &= 15\left\{\frac{3^2+2^3-5}{4!3}k^3 - \frac{5^2+2^5-7}{6!5}k^5 + \frac{7^2+2^7-9}{8!7}k^7 - \dots\right\}
\end{aligned}\right\} \tag{41}$$

$$\left.\begin{aligned}
q &= \frac{\pi\lambda_0}{\lambda}\\
k &= \frac{4\pi a}{\lambda}
\end{aligned}\right\} \tag{42}$$

$a$ = length of vertical part of antenna in same unit as $\lambda$ (e. g. meters),

$p$ = radiated power in watts instantaneous value,

$$I = \frac{I_0}{\sin q/2}, \tag{43}$$

where

$I_0$ = amplitude of current at the base of antenna in amperes.

**11. Radiation Resistance of Vertical Part of the Antenna.** — In equation (40) is given the power radiated from the vertical part of the antenna, on the assumption that radiation from the horizontal part of the antenna does not interfere with it. It will be shown later in §14 et seq. how this interference is computed and allowed for. Accepting for the present the assumption of non-interference, we may obtain the radiation resistance of the vertical part of the antenna.

The radiation resistance is defined as the *time average of radiated power divided by the time average of the square of the current at the base of the antenna.*

In taking the time average of the power (40), it is to be noted that the time average of $\cos^2 \left\{ \frac{2\pi}{\lambda} (ct - r_0) \right\}$ is 1/2. The time average of current square at the base of the antenna, by (1) is $\frac{1}{2} I^2 \sin^2 \frac{\pi \lambda_0}{2\lambda} = \frac{1}{2} I^2 \sin^2 \left(\frac{q}{2}\right)$ Whence the radiation resistance becomes in ohms

$$R_\Omega = \frac{1}{\sin^2 \left(\frac{q}{2}\right)} \left\{ R_1 - R_2 \cos q - R_3 \sin q \right\}, \tag{44}$$

in which $R_1$, $R_2$, $R_3$ and $q$ have the values given in (41) and (42).

We shall later give tables of $R_1$, $R_2$, and $R_3$, that will reduce the calculations of R to very simple operations, and shall compare the results with calculations on the doublet hypothesis and with observations.

We shall, however, first investigate theoretically the radiation from the horizontal part of the antenna. This is a problem of considerable mathematical difficulty but is capable of solution.

# PART II.

## Field due to Horizontal Portion of Antenna.

**12. Introductory Notions.** — To determine the electromagnetic field and radiation characteristics of the horizontal flat-top portion of the antenna, let the rectangular coördinates of any distant point P (Fig. 7) be $x$, $y$, $z$.

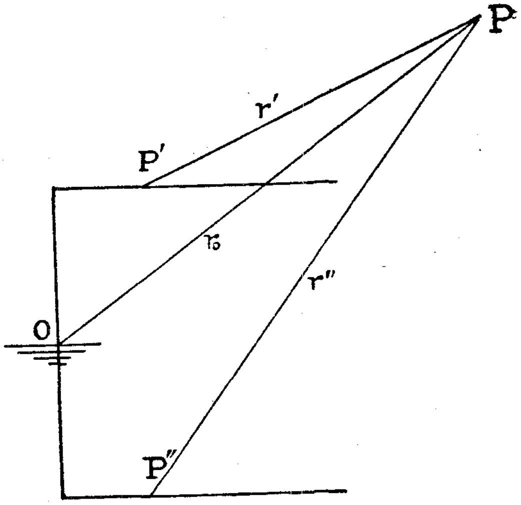

Figure 7.

And let the coördinates of any point $P'$ on the flat-top be $x'$, 0, $a$; the coördinates of the image point $P''$ be $x'$, 0, $-a$.

Then the distance from the origin of coördinates to the distant point is

$$OP = r_0 = \sqrt{x^2 + y^2 + z^2}$$

The distances of the distant point from the point on the flat-top and its image respectively are

$$P'\,P = r' = \sqrt{(x - x')^2 + y^2 + (z - a)^2}.$$

and $$P''\,P = r'' = \sqrt{(x - x')^2 + y^2 + (z + a)^2}.$$

Then $$r' - r_0 = \sqrt{(x - x')^2 + y^2 + (z - a)^2} - \sqrt{x^2 + y^2 + z^2}.$$

As an approximation, let us multiply by the sum of these radicals and divide by the approximate value of this sum for large values of $r_0$; namely, by $2r_0$, obtaining

$$r' = r_0 + \frac{x'^2 - 2xx' - 2za + a^2}{2r_0}, \tag{45}$$

$$r'' = r_0 + \frac{x'^2 - 2xx' + 2za + a^2}{2r_0}. \tag{46}$$

**13. Determination of Electric and Magnetic Intensities due to Flat-top.** — The values of $r'$ and $r''$ in (45) and (46) may be replaced by $r_0$ in intensity factors, but not in phase terms, and give for the sum of the effects of a doublet at $P'$ and another at $P''$ (the image doublet) the electric and magnetic intensities

$$dE_\psi = dH_\Sigma = \frac{\sin\psi}{r_0c^2}\left\{\ddot{f}_1\,(t - r'/c) + \ddot{f}_2\,(t - r''/c)\right\}, \tag{47}$$

where $f_1(t)$ and $f_2(t)$ are the moments of the two doublets respectively.

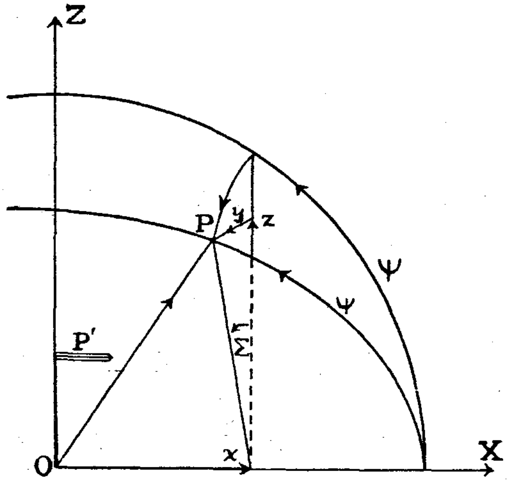

Figure 8.

The angles $\varphi$ and $\Sigma$ correspond to the angles $\theta$ and $\phi$ of Figure 3, except that the figure is turned on its side, so as to put the polar diameter along the $x$-axis instead of the $z$-axis. This arrangement is shown in Figure 8. The plane of the zero value of $\Sigma$ is now to be fixed as the plane of the $x$ and $z$-axes.

Now using the current distribution of equation (1), we must replace $l$ by $a + x'$, which gives, when treated as (12) was treated,

$$\ddot{f}_1 = \frac{2\pi c I}{\lambda} \cos\left\{ \frac{2\pi}{\lambda}\left( (ct - r_0 - \frac{x'^2 - 2x'x - 2za + a^2}{2r_0} \right) \right\} \sin\left\{ \frac{2\pi}{\lambda}\left( \frac{\lambda_0}{4} - a - x' \right) \right\} dx'. \tag{48}$$

The fictitive current at $P''$ is just equal and opposite to that at $P'$, with, however, a different distance from the point $P$, so we may write

$$\ddot{f}_2 = -\frac{2c\pi I}{\lambda} \cos\left\{ \frac{2\pi}{\lambda}\left( ct - r_0 - \frac{x'^2 - 2xx' + 2za + a^2}{2r_0} \right) \right\} \sin\left\{ \frac{2\pi}{\lambda}\left( \frac{\lambda_0}{4} - a - x' \right) \right\} dx'. \tag{49}$$

Whence by addition, employing the trigonometric relation

$$\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2 \sin\alpha \sin\beta,$$

equation (47) becomes

$$dE_\psi = dH_\Sigma = -\frac{4\pi I \sin\psi}{r_0 c \lambda} \sin\left\{ \frac{2\pi}{\lambda}\left( ct - r_0 - \frac{x'^2 - 2xx' + a^2}{2r_0} \right) \right\} \left[ \sin\frac{2\pi a z}{\lambda r_0} \sin\left\{ \frac{2\pi}{\lambda}\left( \frac{\lambda_0}{4} - a - x' \right) \right\} dx' \right].$$

In this equation we may as usual replace $\frac{2\pi a}{\lambda}$ by A. Also we may make an approximation as follows: For large values of $r_0$

$$\frac{x'^2 - 2xx' + a^2}{2r_0} = -\frac{xx'}{r_0} = -x' \cos\psi.$$

In making this approximation the neglected term is $\frac{x'^2 + a^2}{2r_0}$, and this is to be neglected even in the phase angle, because its value is absolutely small. We have then

$$dE_{\psi} = dH_{\Sigma} = -\frac{4\pi I \sin\psi}{r_0 c\lambda} \sin\frac{Az}{r_0} \sin\left\{\frac{2\pi}{\lambda}(ct - r_0 + x'\cos\psi)\right\} \left[\sin\left\{\frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - a - x'\right)\right\} dx'\right]. \qquad (50)$$

This equation may be shortened up by writing

$$\tau = \frac{2\pi}{\lambda}(ct - r_0) \qquad (51)$$

and

$$B = \frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - a\right) = \frac{2\pi b}{\lambda}. \qquad (52)$$

To obtain the total electric and magnetic intensities due to the flat-top, the equation (50) must be integrated for all the doublets and their images between the limits

$$x' = 0 \text{ and } x' = b$$

where $b$ is the length of the flat-top. This integration is expressed in the following equation.

$$E_{\psi} = H_{\Sigma} = -\frac{4\pi I \sin\psi}{r_0 c\lambda} \sin\frac{Az}{r_0} \int^{b} \sin\left(\tau + \frac{2\pi x'}{\lambda}\cos\psi\right) \sin\left(B - \frac{2\pi x'}{\lambda}\right) dx'. \qquad (53)$$

To perform the integration let us introduce a change of variable by putting

$$s = B - \frac{2\pi x'}{\lambda} \qquad \text{then } dx = -\frac{\lambda}{2\pi} ds$$

and the limits of integration become

for $x' = 0, \quad s = B,$ for $x' = b, \quad s = 0.$

Equation (53) then becomes

$$
\begin{aligned}
E_\psi = H_\Sigma &= \frac{2I\sin\psi}{r_0 c}\sin\frac{Az}{r_0}\int_B^o \sin(\tau + B\cos\psi - s\cos\psi)\sin s\,ds \\
&= \frac{2\,I\sin\psi}{r_0 c}\sin\frac{Az}{r_0}\Big[\sin(\tau + B\cos\psi)\int_B^o \cos(s\cos\psi)\sin s\,ds \\
&\qquad - \cos(\tau + B\cos\psi)\int_B^o \sin(s\cos\psi)\sin s\,ds\Big]. \qquad (54)
\end{aligned}
$$

The expressions of this equation may be integrated by the use of formulas 360 and 359 of B. O. Peirce's Tables and give

$$
\begin{aligned}
E_\psi = \mathrm{H}_\Sigma &= \frac{2\,I}{r_0 c\sin\psi}\sin\frac{Az}{r_0}\Big\{\sin(\tau + B\cos\psi)\Big[-\cos s\cos(s\cos\psi) \\
&\qquad - \cos\psi\sin s\sin(s\cos\psi)\Big]_B^0 \\
&\qquad - \cos(\tau + B\cos\psi)\Big[\cos\psi\sin s\cos(s\cos\psi) \\
&\qquad - \cos s\sin(s\cos\psi)\Big]_B^0 \\
&= \frac{2\,I}{r_0 c\sin\psi}\sin\frac{Az}{r_0}\Big[\sin(\tau + B\cos\psi)\Big\{-1 + \cos B\cos \\
&\qquad \Big(B(\cos\psi)\Big) \\
&\qquad + \cos\psi\sin B\sin(B\cos\psi)\Big\} \\
&\qquad + \cos(\tau + B\cos\psi)\Big\{\cos\psi\sin B\cos(B\cos\psi) \\
&\qquad - \cos B\sin(B\cos\psi)\Big\}\Big] \\
&= \frac{2\,I}{r_0 c\sin\psi}\sin\frac{Az}{r_0}\Big[\sin\tau\Big\{\cos B - \cos(B\cos\psi)\Big\} \\
&\qquad + \cos\tau\Big\{\cos\psi\sin B - \sin(B\cos\psi)\Big\}\Big]. \qquad (55)
\end{aligned}
$$

*Equation (55) gives the electric and magnetic intensities due to the flat-top at any distant point whose coördinates are*

$r_0$ = distance of the point from the origin,
$z$ = vertical height of the point above the earth's surface,
$\psi$ = angle between $r_0$ and the $x$-axis; this $x$-axis being parallel to the flat-top.

The quantities A, B, and $\tau$ are defines by equations (20) and (51). We shall next discuss the total power radiated from the antenna.

**14. Concerning Power Radiated from the Total Antenna.**—It is to be noticed that the electric and magnetic intensities due to the flat-top of the antenna and those intensities due to the vertical portions of the antenna are directed along the meridional and latitudinal lines of two systems of polar coördinates with their poles one quadrant apart. This does not make the respective intensities perpendicular to each other, and it becomes necessary to resolve one set of these intensities along and perpendicular to the other set of intensities. At a given point on the sphere about the origin of coordinates, the quantities $\phi$, $\theta$, $\Sigma$, and $\psi$ are oriented in a manner represented in Figure 9.

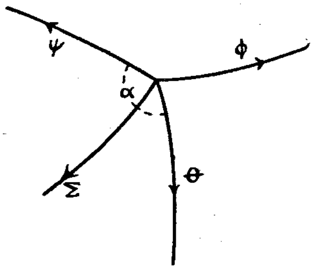

FIGURE 9.

If we let

$$\alpha = \text{angle between } \psi \text{ and } \theta$$

then also

$$\alpha = \text{angle between } \phi \text{ and } \Sigma.$$

It is also apparent that

$$\text{Angle between } \Sigma \text{ and } \theta = \alpha - \frac{\pi}{2}$$

$$\text{Angle between } \psi \text{ and } \phi = \frac{3\pi}{2} - \alpha$$

Let us now resolve $E_\psi$ and $H_\Sigma$ into components along $\theta$ and perpendicular thereto (that is, along $\phi$) obtaining for the $\theta$-components

$$E_{\psi,\theta} = E_\psi \cos \alpha$$

$$H_{\Sigma,\theta} = H_\Sigma \cos\left(\alpha - \frac{\pi}{2}\right) = \mathrm{H}_\Sigma \sin \alpha.$$

and for the $\phi$-components

$$E_{\psi,\varphi} = E_\psi \cos\left(\frac{3\pi}{2} - \alpha\right) = -\,E_\psi \sin \alpha$$

$$H_{\Sigma,\varphi} = H_\Sigma \cos \alpha.$$

Adding these quantities to the corresponding components of the intensities due to the vertical part of the antenna, we obtain for the total intensities, which are designated by primes, the values

$$\begin{aligned} E'_\theta &= E_\theta + E_\psi \cos \alpha, \\ E'_\varphi &= -\,E_\psi \sin \alpha, \\ H'_\theta &= \quad H_\Sigma \sin \alpha \\ H'_\varphi &= H_\varphi + H_\Sigma \cos \alpha. \end{aligned}$$

All of these intensities are perpendicular to $r_0$. To get the power radiated through an element of surface $dS$ perpendicular to $r_0$, we may make use of Poynting's vector, in the form

$$dp = \frac{c}{4\pi}\left(\mathbf{E}' \times \mathbf{H}'\right) dS$$

where the *cross* between the vectors means the vector-product. This vector-product, expanded, gives

$$\begin{aligned} dp &= \frac{c}{4\pi}\left(E'_\theta H'_\varphi - E'_\varphi H'_\theta\right) dS \\ &= \frac{c}{4\pi}\Big(E_\theta H_\varphi + E_\psi H_\Sigma \cos^2 \alpha + H_\varphi E_\psi \cos \alpha + E_\theta H_\Sigma \cos \alpha + \\ &\qquad\qquad E_\psi H_\Sigma \sin \alpha\Big) dS \\ &= \frac{c}{4\pi}\left(E_\theta H_\varphi + E_\psi H_\Sigma + 2 \cos \alpha\, E_\theta H_\psi\right) dS. \qquad (56) \end{aligned}$$

We have already found the first term of this power and have obtained its integral all over the aërial hemisphere. This integral we have called *the power radiated from the vertical part* of the antenna. We shall call the second term above (56), when properly integrated, *the power radiated from the flat-top.* The third term, since it contains both sets of coördinates, may be called *power radiated mutually.* These designations are merely for convenience in paragraphing the mathematics involved.

**15. Power Radiated from the Flat-top.**—Let us now enter upon a determination of the power contributed by the second term of the right hand side of equation (56), and integrate this term over the aërial hemisphere; that is, the hemisphere above the surface of the earth regarded as a plane.

The element of area of this hemisphere is

$$dS = r_0^2 \sin \psi \, d\psi \, d\Sigma. \tag{57}$$

This is to be substituted in the required term involving $E_\psi$ and $H_\Sigma$; but these quantities involve the coördinate $z$, which must be replaced by its value in polar coördinates

$$z = r_0 \sin \psi \cos \Sigma. \tag{58}$$

Besides (57) and (58) we are also to substitute the values of $E_\psi$ and $H_\Sigma$ from (55) into the term

$$dp = \frac{c}{4\pi}\left(E_\psi H_\Sigma\right) dS. \tag{59}$$

$E_\psi$ and $H_\Sigma$ are identical, by (55); the product will give certain terms involving $\sin^2\tau$, other terms involving $\cos^2\tau$, and still other terms involving $\sin\tau\cos\tau$; where $\tau$ has the value given in (51). If we take the time average for a complete cycle, or, if we prefer, for a time that is large in comparison with a complete period, we have

$$\text{av.} \sin^2\tau = \text{av.} \cos^2\tau = \tfrac{1}{2};$$

while the average of the product

$$\text{av.} \sin\tau\cos\tau = 0.$$

The integral form of (59) then becomes, if $\bar{p}$ = the time average of radiated power,

$$\bar{p} = \frac{I^2}{2\pi c}\int_0^{\pi}\frac{d\psi}{\sin\psi}\Bigg[\Bigg\{\cos^2 B + \cos^2\psi\sin^2 B + 1 - 2\cos B\cos(B\cos\psi)$$

$$- 2\cos\psi\sin B\sin(B\cos\psi)\int_{-\pi/2}^{+\pi/2} d\Sigma\,[\sin^2(A\sin\psi\cos\Sigma)]\Bigg\}\Bigg]. \qquad (60)$$

We shall first perform the integration with respect to Σ

$$\int_{-\pi/2}^{\pi/2} d\Sigma\,\{\sin^2(A\sin\psi\cos\Sigma)\} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\Sigma\left\{\frac{1-\cos(2A\sin\psi\cos\Sigma)}{2}\right\}$$

$$= \frac{\pi}{2} - \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(2A\sin\psi\cos\Sigma)\,d\Sigma.$$

$$= \frac{\pi}{2} - \frac{1}{2}\int_{-\frac{\pi}{2}}^{0}\cos(2A\sin\psi\cos\Sigma)\,d\Sigma - \frac{1}{2}\int_0^{\pi/2}\cos(2A\sin\psi\cos\Sigma)d\Sigma \qquad (61)$$

$$= \frac{\pi}{2} - \frac{1}{2}\int_0^{\pi}\cos(2A\sin\psi\cos\Sigma)\,d\Sigma. \qquad (62)$$

This last step consists in changing the variable of the first integral of the right-hand side of (61) by putting

$$\Sigma' = \pi + \Sigma,$$

which makes the limits $\frac{\pi}{2}$ and $\pi$ without any other change, except the change of Σ to Σ′. But since this is the variable of integration, the prime may be omitted, and the terms of (61) added, giving (62).

Equation (62) may now be integrated by Formula (11) Art. 121 of Byerly's *Fourier's Series and Spherical Harmonics* giving for the integral of (62)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\Sigma \,\{\sin^2 (A \sin\psi \cos\Sigma)\} = \frac{\pi}{2} - \frac{\pi}{2} J_0 (2\,A \sin\psi), \tag{63}$$

where $J_0$ is the Bessel's Function of the zeroth order, with a development of the form

$$J_0 (x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\,4^2} - \frac{x^6}{2^2\,4^2\,6^2} + \cdots\cdots \tag{64}$$

Before substituting in (60) let us simplify the general trigonometric factor in the brace of (60) by placing $\cos^2\psi$ by $1 - \sin^2\psi$, and letting $k = 2A$, as in (42), we then obtain

$$\begin{aligned}
\bar{p} = \frac{I^2}{4c} \int_o^{\pi} & \left\{ \frac{1 - J_0 (k \sin\psi)}{\sin\psi} \right\} \Big\{ 2 - \sin^2\psi \sin^2 B \\
& - 2 \cos B \cos (B \cos\psi) - 2 \cos\psi \sin B \sin (B \cos\psi) \Big\} \, d\psi \\
= \frac{I^2}{4c} \int_o^{\pi} & \left\{ \frac{k^2 \sin^2\psi}{2^2} - \frac{k^4 \sin^3\psi}{2^2\,4^2} + \frac{k^6 \sin^5\psi}{2^2\,4^2\,6^2} - \cdots\cdots \right\} \\
& \Big\{ 2 - \sin^2\psi \sin^2 B - 2 \cos B \cos (B \cos\psi) \\
& - 2 \cos\psi \sin B \sin (B \cos\psi) \Big\} \, d\psi.
\end{aligned} \tag{65}$$

or

$$\begin{aligned}
\bar{p} = \frac{I^2}{4c} \Bigg[ & - 2 \sum (-1)^{\frac{n}{2}} \frac{k^n}{2^2 4^2 6^2 \cdots n^2} \int_o^{\pi} \sin^{n-1}\psi \, d\psi \\
& + \sin^2 B \sum (-1)^{\frac{n}{2}} \frac{k^n}{2^2 4^2 6^2 \cdot\cdot n^2} \int_o^{\pi} \sin^{n+1}\psi \, d\psi \\
& + 2 \cos B \sum (-1)^{\frac{n}{2}} \frac{k^n}{2^2 4^2 6^2 \cdots n^2} \int_o^{\pi} \sin^{n-1}\psi \cos (B \cos\psi) \, d\psi \\
& + 2 \sin B \sum (-1)^{\frac{n}{2}} \frac{k^n}{2^2 4^2 6^2 \cdots n^2} \int_o^{\pi} \sin^{n-1}\psi \cos\psi \sin \\
& \qquad (B \cos\psi) \, d\psi \Bigg]
\end{aligned}$$

$$n = 2,\ 4,\ 6, \ldots\ldots \tag{66}$$

Treating these several integrals separately, we have

$$\begin{aligned}\int_o^\pi \sin^{n-1}\psi d\psi &= \int_o^{\frac{\pi}{2}} \sin^{n-1}\psi d\psi + \int_{\frac{\pi}{2}}^{\pi} \sin^{n-1}\psi d\psi \\ &= \int_o^{\frac{\pi}{2}} \sin^{n-1}\psi d\psi + \int_o^{\frac{\pi}{2}} \cos^{n-1}\psi d\psi \\ &= 2\left\{\frac{2\cdot 4\cdot 6\cdots\cdot n-2}{1\cdot 3\cdot 5\cdots\cdot n-1}\right\} \qquad (67)\end{aligned}$$

by B. O. Peirce's Tables, Formula No. 483.

Likewise

$$\int_o^\pi \sin^{n+1}\psi d\psi = 2\left\{\frac{2\cdot 4\cdot 6\cdots\cdot n}{1\cdot 3\cdot 5\cdots\cdot n+1}\right\}. \qquad (68)$$

Now by Byerly's *Fourier's Series and Spherical Harmonics* Equation (9), Art. 122,

$$\int_o^\pi \sin^{n-1}\psi\cos(B\cos\psi)\,d\psi = \frac{2^{\frac{n-1}{2}}\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)}{B^{\frac{n-1}{2}}} J_{\frac{n-1}{2}}(B) \qquad (69)$$

where $J_{\frac{n-1}{2}}(B)$ is a Bessel's Function of the order $(n-1)/2$, and $\Gamma\left(\frac{n}{2}\right)$ is the Gamma Function of $\frac{n}{2}$.

For the last integral of (66), we have by Problem 2 and Equation (9) of the same article of Byerly's *Fourier's Series*

$$\begin{aligned}&\int_o^\pi \sin^{n-1}\psi\cos\psi\sin(B\cos\psi)\,d\psi \\ &\qquad = \frac{B}{n}\int_o^\pi \sin^{n+1}\psi\cos(B\cos\psi)\,d\psi \\ &\qquad = \frac{B}{n}\cdot\frac{2^{\frac{n+1}{2}}\sqrt{\pi}\,\Gamma\left(\frac{n}{2}+1\right)}{B^{\frac{n+1}{2}}} J_{\frac{n+1}{2}}(B). \qquad (70)\end{aligned}$$

Substituting these various integrations (67), (68), (69), and (70) in (66), we have

$$\bar{p} = \frac{1^2}{4c}\Bigg[ -4\sum(-1)^{\frac{n}{2}}\frac{k^n}{n!n} + 2\sin^2 B\sum(-1)^{\frac{n}{2}}\frac{k^n}{n+1!}$$

$$+\,2\cos B\sum(-1)^{\frac{n}{2}}\frac{k^n}{2^2\,4^2\cdots n^2}\;\frac{2^{\frac{n-1}{2}}\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)}{B^{\frac{n-1}{2}}}\,J_{\frac{n-1}{2}}(B)$$

$$+\,2\sin B\sum(-1)^{\frac{n}{2}}\frac{k^n}{2^2 4^2\cdots n^2}\,\frac{B}{n}\;\frac{2^{\frac{n+1}{2}}\sqrt{\pi}\,\Gamma\left(\frac{n}{2}+1\right)}{B^{\frac{n+1}{2}}}\,J_{\frac{n+1}{2}}(B)\cdot \tag{71}$$

$n = 2, 4, 6, \cdots \infty$ . $B = \frac{2\pi b}{\lambda}$ is between 0 and $\frac{\pi}{2}$.

This result may be expessed in a power series by expanding the Bessel's Functions by equation (6), Art. 120 of Byerly's *Fourier's Series*, giving

$$J_{\frac{n-1}{2}}(B) = \frac{B^{\frac{n-1}{2}}}{2^{\frac{n-1}{2}}\,\Gamma\left(\frac{n+1}{2}\right)}\Bigg[1 - \frac{B^2}{2^2\left(\frac{n+1}{2}\right)}$$

$$+\frac{B^4}{2!2^4\left(\frac{n+1}{2}\right)\left(\frac{n+3}{2}\right)} - \frac{B^6}{3!2^6\left(\frac{n+1}{2}\right)\left(\frac{n+3}{2}\right)\left(\frac{n+5}{2}\right)} + \cdots\Bigg] \tag{72}$$

$$J_{\frac{n+1}{2}}(B) = \frac{B^{\frac{n+1}{2}}}{2^{\frac{n+1}{2}}\,\Gamma\left(\frac{n+3}{2}\right)}\Bigg[1 - \frac{B^2}{2^2\left(\frac{n+3}{2}\right)}$$

$$+\frac{B^4}{2!2^4\left(\frac{n+3}{2}\right)\left(\frac{n+5}{2}\right)} - \frac{B^6}{3!2^6\left(\frac{n+3}{2}\right)\left(\frac{n+5}{2}\right)\left(\frac{n+7}{2}\right)} + \cdots\Bigg]. \tag{73}$$

Note that

$$\sqrt{\pi}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} = 2\frac{2\cdot 4\cdot 6\cdots n-2}{1\cdot 3\cdot 5\cdots n-1}, \tag{74}$$

and

$$\sqrt{\pi}\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+3}{2}\right)} = 2\frac{2\cdot 4\cdot 6\cdots n}{1\cdot 3\cdot 5\cdots n+1}. \tag{75}$$

Putting these values in equation (71) we obtain

$$\begin{aligned}\bar{p} = \frac{I^2}{c}\sum(-1)^{\frac{n}{2}}\frac{k^n}{n!}\Bigg[&-\frac{1}{n}+\frac{\sin^2 B}{2(n+1)} \\ &+\cos B\,\frac{1}{n}\left\{1-\frac{B^2}{2(n+1)}+\frac{B^4}{2!2^2}\frac{1}{(n+1)(n+3)}\right. \\ &\qquad\left.-\frac{B^6}{3!2^3}\frac{1}{(n+1)(n+3)(n+5)}+\cdots\cdots\right\} \\ &+B\sin B\,\frac{1}{n}\left\{\frac{1}{n+1}-\frac{B^2}{2}\frac{1}{(n+1)(n+3)}\right. \\ &\qquad+\frac{B^4}{2^2 2}\frac{1}{(n+1)(n+3)(n+5)} \\ &\qquad\left.-\frac{B^6}{2^3 3}\frac{1}{(n+1)(n+3)(n+5)(n+7)}+\cdots\cdots\right\}\Bigg],\end{aligned}$$

where

$$n = 2, 4, 6, \ldots. \tag{76}$$

Equation (76) may be further improved for purposes of calculation by expanding the trigonometric functions in power series and collecting the terms. For this purpose

$$\frac{\sin^2 B}{2} = \frac{1-\cos 2B}{4} = \frac{B^2}{2!}-\frac{2^2B^4}{4!}+\frac{2^4B^6}{6!}-\frac{2^6B^8}{8!}+\cdots\cdots, \tag{77}$$

$$\cos B = 1 - \frac{B^2}{2!} + \frac{B^4}{4!} - \frac{B^6}{6!} + \cdots\cdots, \tag{78}$$

$$B \sin B = B^2 - \frac{B^4}{3!} + \frac{B^6}{5!} - \cdots\cdots \tag{79}$$

Equations (77), (78) and (79) substituted in (76) will give

$$\bar{p} = \frac{I^2}{c} \sum (-1)^{\frac{n}{2}} \frac{k^n}{n!} \cdot F_n(B), \tag{80}$$

where $F_n(B)$ is a polynomial in $B^0, B^2, B^4$, etc., where the coefficients of the several powers of $B$ are contained in the table of page 225.

In this table the bottom row of terms gives the coefficients of the powers of $B$, when the summation indicated in (80) is performed with $n = 2, 4, 6 \cdots \infty$. The various terms in the columns were employed in obtaining the last row by addition.

The coefficient of $B^{10}$ is not contained in the table, because of its numerous terms, but its value when summed up is

$$\frac{255n^4 + 6084n^3 + 51396n^2 + 177264n + 193536}{10!\,(n+1)\,(n+3)\,(n+5)\,(n+7)\,(n+9)}$$

Substituting the values of the coefficients multiplied by the corresponding powers of $B$ and summing up as indicated in equation (80), we obtain for the power the expression

$$\begin{aligned} \bar{p} = \frac{I^2}{c} \Bigg[ & k^2 \left\{ \frac{B^4}{60} - \frac{11B^6}{3780} + \frac{13B^8}{56700} - \frac{B^{10}}{93555} + \cdots\cdots \right\} \\ & - k^4 \left\{ \frac{B^4}{1120} - \frac{B^6}{6480} + \frac{B^8}{83160} - \frac{B^{10}}{77395500} + \cdots\cdots \right\} \\ & + k^6 \left\{ \frac{B^4}{45360} - \frac{B^6}{24960960} + \frac{7B^8}{6134720} - \cdots\cdots \right\} \end{aligned} \tag{81}$$

*This equation gives the average power radiated in the aërial hemisphere from the flat-top of the antenna regarded as a separate radiator with the distribution that it has under the fundamental assumptions of the problem. The current is to be measured in absolute electrostatic units, and the power is in ergs per second.*

| $B^0$ | $B^2$ | $B^4$ | $B^6$ | $B^8$ |
|---|---|---|---|---|
| $-\frac{1}{n}$ | $+\frac{1}{2!\,(n+1)}$ | $-\frac{2^2}{4!\,(n+1)}$ | $+\frac{2^4}{6!\,(n+1)}$ | $-\frac{2^6}{8!\,(n+1)}$ |
| $+\frac{1}{n}$ | $-\frac{1}{2\,n(n+1)}$ | $+\frac{1}{2!2^2\,n(n+1)\,(n+3)}$ | $-\frac{1}{3!2^3\,n(n+1)\,(n+3)\,(n+5)}$ | $+\frac{1}{4!2^4\,n(n+1)\,(n+3)\,(n+5)\,(n+7)}$ |
| | $-\frac{1}{2!\,n}$ | $+\frac{1}{2!2\,n(n+1)}$ | $-\frac{1}{2^2 2!2!\,n(n+1)\,(n+3)}$ | $+\frac{1}{2!3!2^3\,n(n+1)\,(n+3)\,(n+5)}$ |
| | | $+\frac{1}{4!\,n}$ | $-\frac{1}{4!2\,n(n+1)}$ | $+\frac{1}{2!2^2 4!\,n(n+1)\,(n+3)}$ |
| | | | $-\frac{1}{6!\,n}$ | $+\frac{1}{6!2\,n(n+1)}$ |
| | $+\frac{1}{n(n+1)}$ | $-\frac{1}{2\,n(n+1)\,(n+3)}$ | $+\frac{1}{2!2^2\,n(n+1)\,(n+3)\,(n+5)}$ | $+\frac{1}{8!\,n}$ |
| | | $-\frac{1}{3!\,n(n+1)}$ | $+\frac{1}{3!2\;n(n+1)\,(n+3)}$ | $-\frac{1}{3!2^3\,n(n+1)\,(n+3)\,(n+5)\,(n+7)}$ |
| | | | $+\frac{1}{5!\,n(n+1)}$ | $-\frac{1}{3!2!2^2\,n(n+1)\,(n+3)\,(n+5)}$ |
| | | | | $-\frac{1}{5!2\,n(n+1)\,(n+3)}$ |
| | | | | $-\frac{1}{7!\,n(n+1)}$ |
| 0 | 0 | $-\frac{n+2}{4\cdot 2\,(n+1)\,(n+3)}$ | $+\frac{3n^2+22n+32}{4!6\,(n+1)\,(n+3)\,(n+5)}$ | $-\frac{3n^3+44n^2+196n+240}{5!2^4\,(n+1)\,(n+3)\,(n+5)\,(n+7)}$ |

In this equation

$$B = \frac{2\pi b}{\lambda}$$

$$k = 2A = \frac{4\pi a}{\lambda}.$$

It remains to find how this power is modified by the mutual effect consisting of the interference between the waves emitted from the vertical portion of the antenna and the waves emitted from the horizontal part. This is the subject matter of Part III.

## PART III.

### THE MUTUAL TERM IN POWER DETERMINATION.

**16. The Trigonometric Relations.** — In Section **14,** equation (56), it has been shown that the power radiated through an element of surface consists of three terms in the form

$$dp = \frac{c}{4\pi} \left( E_\theta H_\phi + E_\psi H_\Sigma + 2 \cos \alpha \, E_\theta H_\psi \right) dS.$$

The first two of these terms we have already discussed. Putting in the values of $E_\theta$ and $H_\psi$ from equations (19) and (55) the remaining power term, which we have for convenience called *mutual power*, becomes in the time average

$$d\bar{p} = \frac{I^2 \cos \alpha \, dS}{\pi c \, r_0^2 \sin \theta \sin \psi} \sin \frac{Az}{r_0} \left\{ \cos \psi \sin B - \sin \left( B \cos \psi \right) \right\}$$
$$\left\{ \cos B \cos \left( A \cos \theta \right) - \sin B \cos \theta \sin \left( A \cos \theta \right) - \cos G \right\}. \quad (82)$$

In forming this equation we have multiplied the expression for $E_\theta$ of eq. (19) by the expression for $H_\psi$, eq. (55). The product so obtained contains terms involving $\sin \tau \cos \tau$ plus terms involving $\cos^2 \tau$. The time average of the $\sin \tau \cos \tau$ terms is zero; while the time average of $\cos^2 \tau$ is $\frac{1}{2}$; these facts have been used in forming (82).

To be able to integrate equation (82) we must replace $\alpha$, $z$, $\psi$ and $dS$ by their values in terms of $\theta$, $\phi$ and $r_0$. By Fig. 3,

$$z = r_0 \cos \theta, \quad (83)$$

$$dS = r_0^2 \sin \theta \, d\theta \, d\phi. \quad (84)$$

In the spherical triangle of Figure 10, $\alpha$ is represented, as defined, as the angle between $\theta$ and $\psi$, while opposite to $\alpha$ the side is $\pi/2$. The important trigonometric relation in a spherical triangle is as follows:

I. The cosine of any side is equal to the product of the cosines of the two other sides plus the continued product of the sines of these sides and the cosine of the included angle.

By this proposition, referring to Figure 10, we see that

$$\cos\psi = \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta\cos\phi$$
$$= \sin\ \theta\cos\phi. \tag{85}$$

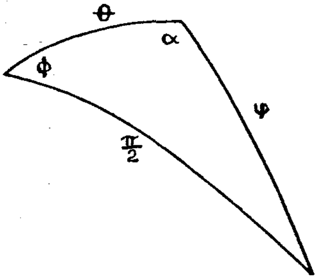

FIGURE 10.

By the same proposition

$$\cos\frac{\pi}{2} = \cos\theta\cos\psi + \sin\theta\sin\psi\cos\alpha;$$

∴

$$\cos\alpha = -\frac{\cos\theta\cos\psi}{\sin\theta\sin^2\psi}, \tag{86}$$

or

$$\frac{\cos\alpha}{\sin\psi} = -\frac{\cos\theta\cos\psi}{\sin\theta\sin^2\psi}; \tag{87}$$

and by (85) this becomes

$$\frac{\cos\alpha}{\sin\psi} = -\frac{\cos\theta\cos\phi}{1-\sin^2\theta\cos^2\phi}. \tag{88}$$

**17. Integration for Mutual Power.** — Now substituting the trigonometric relations (83), (84), (85), (88) into equation (82), we obtain the following integral expression for the time average of the mutual power radiated through the *aërial hemisphere*:

$$\bar{p} = \frac{I^2}{c\pi}\int_0^{\pi/2} d\theta \sin(A\cos\theta)\left\{\cos B\cos(A\cos\theta) - \sin B\cos\theta\sin(A\cos\theta) - \cos G\right\}$$

$$\left[\cos\theta\int_0^{2\pi}\frac{\cos\phi\sin(B\sin\theta\cos\phi)\,d\phi}{1-\sin^2\theta\cos^2\phi} - \cos\theta\sin\theta\sin B\int_0^{2\pi}\frac{\cos\phi\,d\phi}{1-\sin^2\theta\cos^2\phi}\right]. \tag{89}$$

This is a very complicated expression involving the integral of an integral.

We shall first proceed to perform the integration with respect to $\phi$.

$$\text{Let } V = \int_0^{2\pi}\frac{\cos\phi\sin(B\sin\theta\cos\phi)\,d\phi}{1-\sin^2\theta\cos^2\phi} \tag{90}$$

and break the integral into the sum of two integrals thus:

$$V = \int_0^{\pi} + \int_{\pi}^{2\pi}.$$

. By a change of variable in the second of these two integrals by replacing $\phi$ by $\phi' + \pi$, we find that the integrand is unchanged, while the limits become 0 and $\pi$, so we may write

$$V = 2\int_0^{\pi}\frac{\cos\phi\sin(B\sin\theta\cos\phi)\,d\phi}{1-\sin^2\theta\cos^2\phi}. \tag{91}$$

Again decomposing this into the sum of two integrals we have

$$V = 2\left\{\int_0^{\pi/2} + \int_{\pi/2}^{\pi}\right\} \tag{92}$$

and changing the variable in the second integral by putting $\phi = \pi - \phi'$, the second integral becomes

$$\int_{\pi/2}^{\pi} = \int_{\pi/2}^{0}\frac{-d\phi'(-\cos\phi')(-\sin(B\sin\theta\cos\phi'))}{1-\sin^2\theta\cos^2\phi'},$$

which by dropping the primes and substituting in (92) and (91) gives

$$V = 4\int_0^{\frac{\pi}{2}} \frac{\cos\phi \sin(B\sin\theta\cos\phi)\,d\phi}{1-\sin^2\theta\cos^2\phi}. \tag{93}$$

Now expanding in series as follows:

$$\sin(B\sin\theta\cos\phi) = B\sin\theta\cos\phi - \frac{B^3\sin^3\theta\cos^3\phi}{3!} + \frac{B^5\sin^5\theta\cos^5\phi}{5!} - \dots,$$

and

$$\frac{1}{1-\sin^2\theta\cos^2\phi} = 1 + \sin^2\theta\cos^2\phi + \sin^4\theta\cos^4\phi + \dots\dots; \tag{93a}$$

and by taking the product of these two series we obtain

$$\begin{aligned} V = 4\int_0^{\pi/2} d\phi \Big[ & B\sin\theta\cos^2\phi \\ & + \left\{ B - \frac{B^3}{3!} \right\} \sin^3\theta\cos^4\phi \\ & + \left\{ B - \frac{B^3}{3!} + \frac{B^5}{5!} \right\} \sin^5\theta\cos^6\phi \\ & + \dots\dots \Big]. \end{aligned} \tag{94}$$

Integrating (94) by formula 483 of B. O. Peirce's Tables, we obtain

$$\begin{aligned} V = 2\pi\Big[ & \frac{1}{2} B\sin\theta \\ & + \frac{1\cdot3}{2\cdot4}\left\{ B - \frac{B^3}{3!} \right\} \sin^3\theta \\ & + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left\{ B - \frac{B^3}{3!} + \frac{B^5}{5!} \right\} \sin^5\theta \\ & + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\left\{ B - \frac{B^3}{3!} + \frac{B^5}{5!} - \frac{B^7}{7!} \right\} \sin^7\theta \\ & \dots\dots\dots \end{aligned} \tag{95}$$

We shall next proceed to perform the second integration with respect to $\phi$ indicated in (89). For abbreviation let us write

$$W = \int_o^{2\pi} \frac{\cos^2\phi\, d\phi}{1 - \sin^2\theta\cos^2\phi} = 4\int_o^{\pi/2} \frac{\cos^2\phi\, d\phi}{1 - \sin^2\theta\cos^2\phi}$$

by reasoning similar to the above. Expanding the denominator by (93a), we have

$$\begin{aligned} W &= 4\int_o^{\pi/2} d\phi\cos^2\phi \left\{ 1 + \sin^2\theta\cos^2\phi + \sin^4\theta\cos^4\phi + \cdots\cdot \right\} \\ &= 2\pi\left\{ \frac{1}{2} + \frac{1\cdot 3}{2\cdot 4}\sin^2\theta + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\sin^4\theta + \text{-------} \right\}. \qquad (96) \end{aligned}$$

(If we need it, this integral can be obtained by direct integration in the form

$$W = 2\pi\left\{ \frac{1}{\cos\theta\,(1 + \cos\theta)} \right\}$$

but the expanded form is more useful for our purpose).

Now substituting (95) and (96) in (89) we obtain

$$\begin{aligned} \bar{p} = \frac{2\,I^2}{c}\int_o^{\pi/2} & d\theta\sin\theta\sin(A\cos\theta)\left\{ \cos B\cos(A\cos\theta) \right. \\ & \left. - \sin B\cos\theta\sin(A\cos\theta) - \cos G \right\}. \\ & \left[ \frac{1}{2}(B - \sin B)\sin\theta \right. \\ & + \frac{1\cdot 3}{2\cdot 4}\left(B - \frac{B^3}{3!} - \sin B\right)\sin^3\theta \\ & + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(B - \frac{B^3}{3!} + \frac{B^5}{5!} - \sin B\right)\sin^5\theta \\ & \left. + \cdots\cdots\cdot \right]. \qquad (97) \end{aligned}$$

To evaluate this expression we must obtain the following integrals:

$$I_1 = \int_o^{\frac{\pi}{2}} d\theta\sin^n\theta\,\frac{\sin(2A\cos\theta)}{2}, \qquad (98)$$

$$I_2 = \int_o^{\pi/2} d\theta\sin^n\theta\cos\theta\sin^2(A\cos\theta), \qquad (99)$$

$$I_3 = \int_o^{\pi/2} d\theta \sin^n \theta \sin (A \cos \theta), \tag{100}$$

where $n = 2, 4, 6, 8, \ldots\ldots$

$I_3$ is the simplest of these integrals and will be considered first. By expanding sin (A cos $\theta$) in series we have

$$I_3 = \int_o^{\pi/2} d\theta \sin^n \theta \left\{ A \cos \theta - \frac{A^3 \cos^3 \theta}{3!} + \frac{A^5 \cos^5 \theta}{5!} - \ldots\ldots \right\}$$

which by Byerly Int. Calc., Art. 99, Ex. 2, may be integrated in Gamma Functions as follows:

$$\begin{aligned} I_3 = \frac{A}{n+1} &- \frac{A^3}{3!} \frac{\Gamma(2)\,\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{n+3}{2}+1\right)} \\ &+ \frac{A^5}{5!} \frac{\Gamma(3)\,\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{n+5}{2}+1\right)} \\ &- \frac{A^7}{7!} \frac{\Gamma(4)\,\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{n+7}{2}+1\right)} \\ &+ \ldots\ldots\ldots\ldots \end{aligned} \tag{101}$$

If we note that

$$\Gamma\left(\frac{n+3}{2}+1\right) = \frac{n+3}{2}\,\frac{n+1}{2}\,\Gamma\left(\frac{n+1}{2}\right)$$

$$\Gamma\left(\frac{n+5}{2}+1\right) = \frac{n+5}{2}\,\frac{n+3}{2}\,\frac{n+1}{2}\,\Gamma\left(\frac{n+1}{2}\right)$$

$$\Gamma(2) = 1$$

$$\Gamma(3) = 2!$$

$$\Gamma(4) = 3!$$

we obtain

$$I_3 = \frac{A}{n+1} - \frac{A^3}{3!}\,\frac{2}{(n+1)(n+3)} + \frac{A^5}{5!}\,\frac{2^2\,2!}{(n+1)(n+3)(n+5)} - \ldots\ldots$$

$$= \frac{A}{n+1}\left\{1 - \frac{A^2}{3\,(n+3)} + \frac{A^4}{5\cdot 3\,(n+3)\,(n+5)} - \frac{A^6}{7\cdot 5\cdot 3\,(n+3)\,(n+5)\,(n+7)} + \cdots\right\}. \qquad (102)$$

In like manner

$$I_1 = \frac{2A}{2\,(n+1)}\left\{1 - \frac{(2A)^2}{3\,(n+3)} + \frac{(2A)^4}{5\cdot 3\,(n+3)\,(n+5)} - \frac{(2A)^6}{7\cdot 5\cdot 3\,(n+3)\,(n+5)\,(n+7)} + \cdots\right\}. \qquad (103)$$

Now taking up integral $I_2$ from equation (99), let us write it

$$I_2 = \int_o^{\pi/2} d\phi \sin^n\theta \cos\theta \left\{\frac{1 - \cos\,(2A\cos\theta)}{2}\right\},$$

and expanding cos (2 A cos $\theta$) in series, obtain

$$I_2 = \frac{1}{2}\int d\phi \left[\sin^n\theta\cos\theta\left\{\frac{(2A)^2\cos^2\theta}{2!} - \frac{(2a)^4\cos^4\theta}{4!} + \cdots\cdots\right\}.\right.$$

This equation, integrated in Gamma Functions between the limits 0 and $\pi/2$ gives

$$I_2 = \frac{1}{2}\left[\frac{(2A)^2}{2!}\,\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\,(2)}{2\,\Gamma\left(\frac{n+3}{2}+1\right)} - \frac{(2A)^4}{4!}\,\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\,(3)}{2\Gamma\left(\frac{n+5}{2}+1\right)} + \cdots\cdots\cdot\right],$$

$$= \frac{2A^2}{(n+1)\,(n+3)}\left\{1 - \frac{2A^2}{3\,(n+5)} + \frac{2A^4}{5\cdot 3\,(n+5)\,(n+7)} - \cdots\cdot\right\}. \qquad (104)$$

Employing the values of $I_1$, $I_2$, $I_3$ found in equations (103), (104) and (102) we may write the expression for the mutual power in the integrated form

$$
\begin{aligned}
\bar{p} = \frac{2I^2}{c}\Bigg[\cos B \Bigg\{ & \frac{1}{2}(B - \sin B)\frac{A}{3}\Bigg[1 - \frac{(2A)^2}{3\cdot5} + \frac{(2A)^4}{5\cdot3\cdot5\cdot7} - \\
& \qquad \frac{(2A)^6}{7\cdot5\cdot3\cdot5\cdot7\cdot9} + \cdots \Bigg] \\
& + \frac{1\cdot3}{2\cdot4}\left(B - \frac{B^3}{3!} - \sin B\right)\frac{A}{5}\Bigg[1 - \frac{(2A)^2}{3\cdot7} + \frac{(2A)^4}{5\cdot3\cdot7\cdot9} - \\
& \qquad \frac{(2A)^6}{7\cdot5\cdot3\cdot7\cdot9\cdot11} + \cdots \Bigg] \\
& + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(B - \frac{B^3}{3!} + \frac{B^5}{5!} - \sin B\right)\frac{A}{7}\Bigg[1 - \frac{(2A)^2}{3\cdot9} + \frac{(2A)^4}{5\cdot3\cdot9\cdot11} - \\
& \qquad \frac{(2A)^6}{7\cdot5\cdot3\cdot9\cdot11\cdot13} + \cdots \Bigg] \\
& + \cdots\cdots\cdots\cdots\cdots\cdots \Bigg\} \\
- \sin B \Bigg\{ & \frac{1}{2}\left(B - \sin B\right)\frac{2A^2}{3\cdot5}\Bigg[1 - \frac{(2A)^2}{3\cdot7} + \frac{(2A)^4}{5\cdot3\cdot7\cdot9} - \\
& \qquad \frac{(2A)^6}{7\cdot5\cdot3\cdot7\cdot9\cdot11} + \cdots \Bigg] \\
& + \frac{1\cdot3}{2\cdot4}\left((B - \frac{B^3}{3!} - \sin B\right)\frac{2A^2}{5\cdot7}\Bigg[1 - \frac{(2A)^2}{3\cdot9} + \frac{(2A)^4}{5\cdot3\cdot9\cdot11} - \\
& \qquad \frac{(2A)^6}{7\cdot5\cdot3\cdot9\cdot11\cdot13} + \cdots \Bigg] \\
& + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(B - \frac{B^3}{3!} + \frac{B^5}{5!} - \sin B\right)\frac{2A^2}{7\cdot9}\Bigg[1 - \frac{(2A)^2}{3\cdot11} + \frac{(2A)^4}{5\cdot3\cdot11\cdot13} - \\
& \qquad \frac{(2A)^6}{7\cdot5\cdot3\cdot11\cdot13\cdot15} + \cdots \Bigg] \\
& + \cdots\cdots\cdots\cdots\cdots\cdots \Bigg\} \\
- \cos G \Bigg\{ & \frac{1}{2}\left(B - \sin B\right)\frac{A}{3}\Bigg[1 - \frac{A^2}{3\cdot5} + \frac{A^4}{5\cdot3\cdot5\cdot7} - \\
& \qquad \frac{A^6}{7\cdot5\cdot3\cdot5\cdot7\cdot9} + \cdots \Bigg] \\
& + \frac{1\cdot3}{2\cdot4}\left(B - \frac{B^3}{3!} - \sin B\right)\frac{A}{5}\Bigg[1 - \frac{A^2}{3\cdot7} + \frac{A^4}{5\cdot3\cdot7\cdot9} - \\
& \qquad \frac{A^6}{7\cdot5\cdot3\cdot7\cdot9\cdot11} + \cdots \Bigg] \\
& + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(B - \frac{B^3}{3!} + \frac{B^5}{5!} - \sin B\right)\frac{A}{7}\Bigg[1 - \frac{A^2}{3\cdot9} + \frac{A^4}{5\cdot3\cdot9\cdot11} - \\
& \qquad \frac{A^6}{7\cdot5\cdot3\cdot9\cdot11\cdot13} + \cdots \Bigg] \\
& + \cdots\cdots\cdots\cdots\cdots\cdots \Bigg\}\Bigg]. \qquad (104)
\end{aligned}
$$

If now we recall that $G = A + B$, it will be seen that the equation (104) is entirely in terms of $A$ and $B$ and $I$.

For purpose of computation it is found advisable to expand all of the trigonometrical expressions in power series and then perform with them the indicated operations. This was done with considerable labor and gave the following expression for mutual power:

$$\begin{aligned}
\bar{p} = \frac{2I^2}{c}\Bigg[ & A^2 \Big\{ .0166\,B^4 - .00404\,B^6 + .000390\,B^8 - \\
& \qquad\qquad .0000144\,B^{10} + \cdots \Big\} \\
& + A^3 \Big\{ .0083\,B^3 - .00480\,B^5 + .000729\,B^7 - \\
& \qquad\qquad .0000486\,B^9 + \cdots \Big\} \\
& + A^4 \Big\{ - .00433\,B^4 + .00104\,B^6 - .000102\,B^8 + \\
& \qquad\qquad .0000051\,B^{10} - \cdots \Big\} \\
& + A^5 \Big\{ - .00127\,B^3 + .000741\,B^5 - .000106\,B^7 + \\
& \qquad\qquad .0000073\,B^9 - \cdots \Big\} \\
& + A^6 \Big\{ .000404\,B^4 - .000101\,B^6 + .0000101\,B^8 - \\
& \qquad\qquad .0000005\,B^{10} + \\
& + \cdots\cdots\cdots\cdots\cdots\cdots \Bigg]. \qquad (105)
\end{aligned}$$

*This equation gives the time average of the power radiated in the aërial hemisphere by the mutual effect of the fields from both parts of the antenna and is the correction to be added to the power radiated by the two parts, estimated as independent of one another. The current I is in absolute c.g.s. electrostatic units, and the power is in ergs per second.*

**18. Summation of Flat-top Power and Mutual Power.**—We have obtained in equation (81) the time average of flat-top radiated power, and in equation (105) the time average of mutual radiated power. If we replace the $k$ of (81) by its value in terms of $A$, the two expressions may be added together. At the time of the addition we shall reduce the units to the practical system by multiplying the right hand sides of both power equations by 30 times the velocity of light in centimeters per second (i. e. by 30 c), and obtain

$$\begin{aligned} p = 60\, I^2 \Big[ & A^2 \left\{ .05 \quad B^4 - .00985\, B^6 + .000849\, B^8 - .0000356\, B^{10} \cdots \right\} \\ & + A^3 \left\{ .0083\, B^3 - .00480\, B^5 + .000729\, B^7 - .0000486\, B^9 + \cdot\cdot \right\} \\ & - A^4 \left\{ .01148\, B^4 - .00227\, B^6 + .000198\, B^8 - .00000953\, B^{10} + \cdots \right\} \\ & - A^5 \left\{ .00127\, B^3 - .000741\, B^5 + .000106\, B^7 - .0000073\, B^9 + \cdots \right\} \\ & + A^6 \left\{ .00111\, B^4 - .00014\, B^6 + .000019\, B^8 - \cdots\cdots \right\} \\ & + \cdots\cdots \Big]. \end{aligned} \tag{106}$$

*This is the total power contribution of the flat top by virtue of its individual and mutual action. The power is in watts, and the current $I$ is in amperes.*

Certain Tables computed in the next Part of this communication make calculations with this series comparatively simple.

# PART IV.

## COMPUTATIONS OF RADIATION RESISTANCE.

**19. Equation for Radiation Resistance.** — If

$a$ = length of vertical part in meters,
$b$ = length of horizontal part in meters,
$\lambda_0$ = the natural wavelength of the antenna in meters,
$\lambda$ = the wavelength in meters of the antenna as loaded with inductance at its base,

$$A = \frac{2\pi a}{\lambda},$$

$$B = \frac{2\pi b}{\lambda},$$

$$q = \frac{\pi\lambda_0}{\lambda},$$

we may obtain the radiation resistance of the antenna by dividing the power radiated by the mean square of the current at the base of the antenna. This mean square current at the base of the antenna is by (5)

$$\overline{I_0^2} = \frac{I^2 \sin^2 (q/2)}{2}.$$

Performing this division as to the flat-top power employing equation (106) and adding the result to the radiation resistance for the vertical portion as given in equation (44) we obtain for the total radiation resistance of the antenna the equation

$$R_\Omega = \frac{1}{\sin^2 (q/2)} \left\{ R_1 - R_2 \cos q - R_3 \sin q + r_2 A^2 + r_3 A^3 - r_4 A^4 - r_5 A^5 + r_6 A^6 + \cdots \right\}. \qquad (107)$$

*This is Radiation Resistance in Ohms, where*

$$R_1 = 15 \left\{ \frac{2+2}{3!2}(2A)^2 - \frac{4+2}{5!4}(2A)^4 + \frac{6+2}{7!6}(2A)^6 - \cdots \right\}$$

$$R_2 = 15 \left\{ \frac{2^2+2^2-4}{3!2}(2A)^2 - \frac{4^2+2^4-6}{5!4}(2A)^4 + \right.$$

$$\left. \frac{6^2+2^6-8}{7!6}(2A)^6 - \cdots \right\}$$

$$R_3 = 15 \left\{ \frac{3^2+2^3-5}{4!3}(2A)^3 - \frac{5^2+2^5-7}{6!5}(2A)^5 + \right.$$

$$\left. \frac{7^2+2^7-9}{8!7}(2A)^7 - \cdots \right\}$$

$$r_2 = 120 \left\{ .05\,B^4 - .00985\,B^6 + .000849\,B^8 - .0000356\,B^{10} + \cdots \right\}$$

$$r_3 = 120 \left\{ .0083\,B^3 - .00480\,B^5 + .000729\,B^7 - \right.$$

$$\left. .0000486\,B^9 + \cdots \right\}$$

$$r_4 = 120 \left\{ .01148\,B^4 - .00227\,B^6 + .000198\,B^8 - \right.$$

$$\left. .00000953\,B^{10} + \cdots \right\}$$

$$r_5 = 120 \left\{ .00127\,B^3 - .000741\,B^5 + .000106\,B^7 - \right.$$

$$\left. .0000073\,B^9 + \cdots \right\}$$

$$r_6 = 120 \left\{ .00111\,B^4 - .00014\,B^6 + .000019\,B^8 - \cdots \right\}. \tag{108}$$

**20. Tables of Coefficients of Radiation Resistance.**— There follow in Tables I and II the values of the coefficients $R_1$, $R_2$, $R_3$, $r_2$, $r_3$, $r_4$, $r_5$, $r_6$ for various values of A and B respectively. These tables have been computed by the equations (108).

## TABLE I.

### COEFFICIENTS $R_1$, $R_2$, AND $R_3$.

| $2A$ | $\lambda/4a$ | $R_1$ | $R_2$ | $R_3$ |
|---|---|---|---|---|
| .1 | 31.416 | .04998 | .049919 | .002498 |
| .2 | 15.70 | .19971 | .19870 | .01994 |
| .3 | 10.47 | .44848 | .44344 | .06700 |
| .4 | 7.85 | .79521 | .78107 | .1579 |
| .5 | 6.28 | 1.2383 | 1.20634 | .3060 |
| .6 | 5.236 | 1.7759 | 1.6969 | .5241 |
| .7 | 4.488 | 2.4055 | 2.2602 | .8232 |
| .8 | 3.927 | 3.1240 | 2.8786 | 1.2137 |
| .9 | 3.491 | 3.9290 | 3.5403 | 1.696 |
| 1.0 | 3.141 | 4.8165 | 4.2315 | 2.300 |
| 1.1 | 2.854 | 5.7837 | 4.9383 | 3.009 |
| 1.2 | 2.616 | 6.8232 | 5.6442 | 3.823 |
| 1.4 | 2.241 | 9.150 | 7.000 | 5.90 |
| 1.5 | 2.092 | 10.3392 | 7.611 | 6.999 |
| 1.6 | 1.962 | 11.64 | 8.15 | 8.35 |
| 1.732 | 1.812 | 13.415 | 8.798 | 10.113 |
| 1.8 | 1.743 | 14.40 | 9.10 | 11.20 |
| 2.00 | 1.570 | 17.241 | 9.550 | 14.354 |
| 2.20 | 1.427 | 20.15 | 9.55 | 17.80 |
| 2.236 | 1.403 | 20.778 | 9.508 | 18.470 |
| 2.40 | 1.307 | 23.22 | 9.00 | 21.42 |
| 2.60 | 1.207 | 26.37 | 7.90 | 25.20 |
| 2.642 | 1.189 | 27.053 | 7.60 | 25.927 |
| 2.80 | 1.121 | 29.40 | 6.22 | 29.05 |
| 3.141 | 1.000 | 34.45 | 2.12 | 35.64 |

## TABLE II.

### COEFFICIENTS $r_2$, $r_3$, ETC.

| $B$ | $\lambda/4b$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ |
|---|---|---|---|---|---|---|
| 1.4 | 1.112 | 15.8 | .686 | 3.55 | .055 | .481 |
| 1.2 | 1.31 | 9.32 | .585 | 2.13 | .107 | .234 |
| 1.0 | 1.57 | 4.92 | .498 | 1.13 | .075 | .118 |
| .8 | 1.96 | 2.16 | .340 | .513 | .051 | .050 |
| .6 | 2.61 | .73 | .171 | .158 | .026 | .016 |
| .4 | 3.93 | .15 | .057 | .035 | .009 | .004 |
| .2 | 7.85 | .009 | .008 | .002 | .001 | .0002 |

**21. Curves of Resistance Due to Radiation from the Flat-top.** — We shall now proceed to discuss the curves of radiation resistance of variously proportioned antennae when employed at various wavelengths relative to the natural wavelength. As preliminary, the resistance due to radiation from the flat-topped portion of the antennae is first computed. The equation for this is the summation of terms in (107) containing the small $r$'s as factors; that is,

$$\underset{\substack{\text{due to}\\\text{flat-top}}}{R_\Omega} = \frac{1}{\sin^2(q/2)}\left\{ r_2A^2 + r_3A^3 - r_4A^4 - r_5A^5 + r_6A^6 + \cdots \right\} \tag{109}$$

in which

$$A = \frac{2\pi a}{\lambda}$$

$$B = \frac{2\pi b}{\lambda}$$

$$q = \frac{\pi\lambda_0}{\lambda} = 2(A + B)\cdot$$

Since the coefficients (small $r$'s) are functions of $B$ only, as given in Table II, it follows that when A and B are given, the value of the flat-top $R$ may be computed. The results of the computations for various values of A and B are plotted in Figure 11.

In this figure values of $B$ are the abscissae, while the flat-top resistances in ohms are ordinates. The separate curves numbered .1, .2, .3, etc. to .9 are for values of $A$ = .1, .2, .3, etc. to .9.

The outside end-points of these several curves, through which a limiting curve is drawn, are determined by the equality of the impressed wavelength $\lambda$ and the natural wavelength of the antenna $\lambda_0$; that is, by the value of $A + B = \pi/2$, which is the largest value $A + B$ can have for the fundamental oscillation of the antenna.

**22. Curves of Total Radiation Resistance.** — The next step consists in computing the radiation resistance of the vertical portion of the antenna, using the first three terms of equation (107), and employing a large number of values of $A$ and $B$. To these values of resistance due to the vertical portion of the antenna the corresponding resistance of the flat-top are added so as to give the total resistance of the antenna for various values of $A$ and $B$. Curves of resistance for various values of $A + B$ are then plotted in Figure 12, with values of B as abscissae and values of resistance as ordinates. Figure 13 is an enlarged view of some of the curves that are on too small a scale to read in Figure 12. Then to make the family of curves more useful for ready reference a series of curves are drawn through all

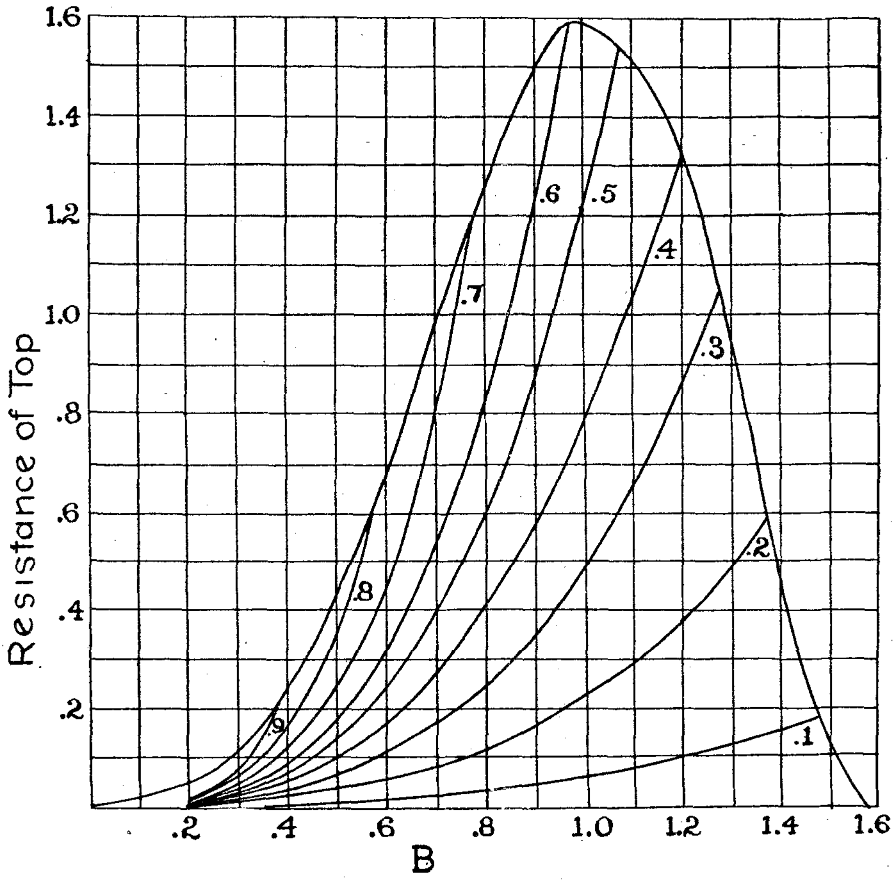

FIGURE 11. Resistance of horizontal top portion of antenna plotted against values of B. The separate curves numbered •1, •2, •3, etc. to •9 are for values of A = •1, •2, •3, etc. to •9.

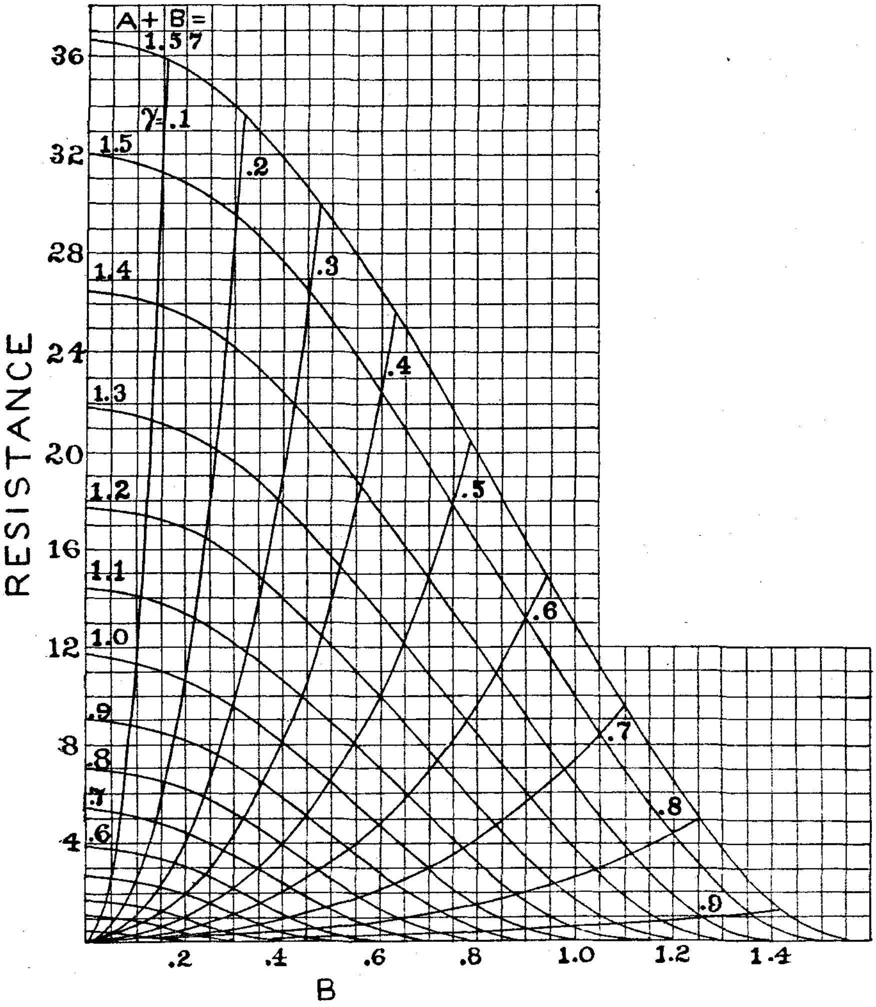

FIGURE 12. Total radiation resistance plotted against values of B. The separate curves through the origin are for designated values of $\gamma$. Separate curves not passing through origin are for different values of A + B.

the points which have a common ratio of length of flat-top to length of total antenna. This ratio is designated by $\gamma$, where

$$\gamma = \frac{B}{A+B} = \frac{b}{a+b} \tag{110}$$

with $b$ = length of flat-top

$a$ = length of vertical part of antenna.

These $\gamma$-curves all pass through the origin.

Next as a final step the curves of Figure 14 are taken from the curves

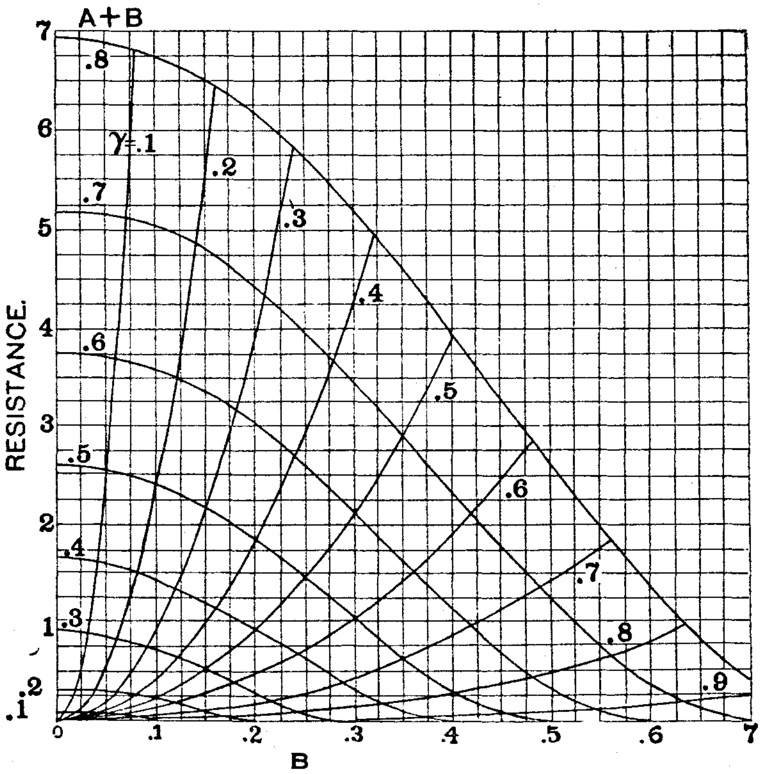

FIGURE 13. Enlarged view of some of the curves of Figure 12.

of Figures 12 and 13 with the new set of parameters. These curves of Figure 14 are the final curves of total radiation resistance, and are in

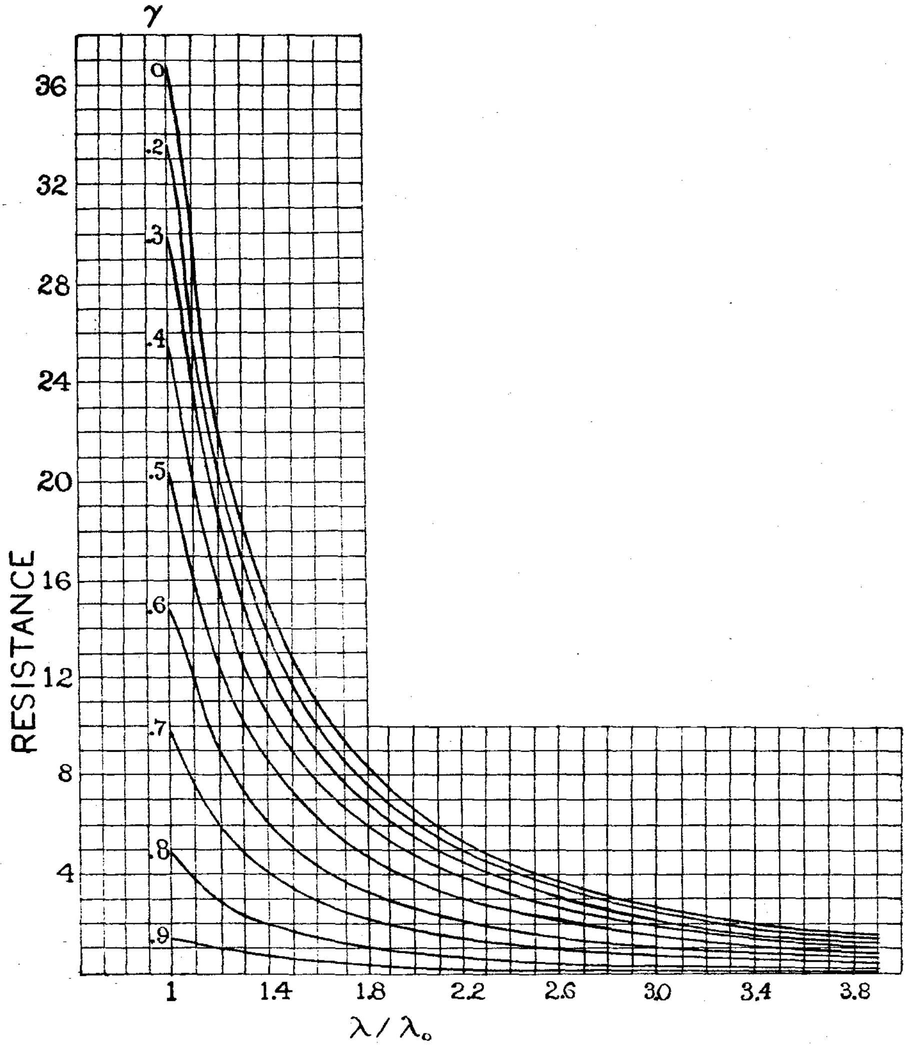

FIGURE 14. Total Radiation resistance plotted against $\lambda/\lambda_0$. The separate curves marked 0, ·2, ·3, etc. are for values of $\gamma$ = 0, ·2, ·3, etc.

terms of the ratio of the wavelength employed to the natural wavelength (that is $\lambda/\lambda_0$) and the ratio of the length of flat-top to total length of antenna (that is $\gamma$). Figure 15 is merely a magnified view of certain of the curves that are too small to read on Figure 14.

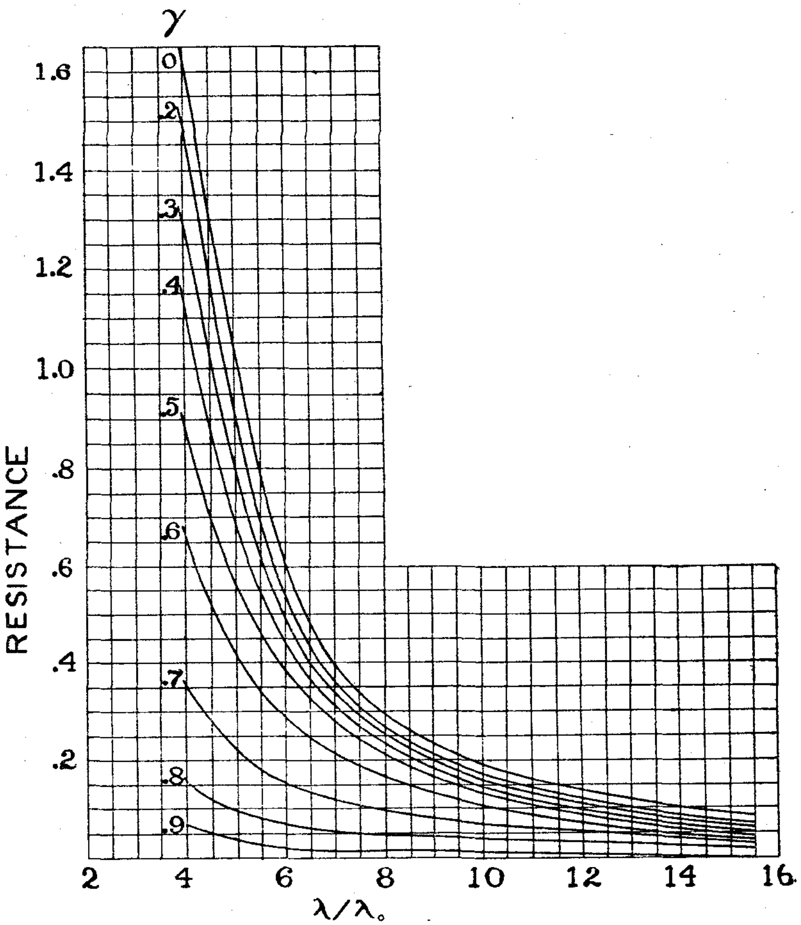

FIGURE 15. Extension of curves of Figure 14 to larger values of $\lambda/\lambda_0$.

**23. Total Radiation Resistance of a Straight Vertical Antenna at Various Wavelengths Obtained by Inductance at the Base.**—As an example, let it be required to find the total radiation

## TABLE III.

RESISTANCE OF A STRAIGHT VERTICAL ANTENNA FOR DIFFERENT VALUES OF WAVELENGTH OBTAINED BY INDUCTANCE AT THE BASE.

| $\lambda/\lambda_0$ Ratio of Wavelength to Natural Wavelength | $R$ Radiation Resistance in Ohms Computed by Present Theory | Radiation Resistance in Ohms Computed on Doublet Theory |
|---|---|---|
| 1.00 | 36.57 | 98.7 |
| 1.12 | 26.40 | 78.7 |
| 1.21 | 21.70 | 67.3 |
| 1.31 | 17.65 | 57.5 |
| 1.43 | 14.28 | 48.2 |
| 1.57 | 11.62 | 40.0 |
| 1.74 | 9.10 | 32.6 |
| 1.97 | 6.92 | 25.4 |
| 2.24 | 5.19 | 19.7 |
| 2.62 | 3.78 | 14.4 |
| 3.14 | 2.58 | 10.0 |
| 3.93 | 1.65 | 6.40 |
| 5.26 | .90 | 3.60 |
| 7.85 | .30 | 1.16 |
| 15.70 | .082 | .40 |
| 31.42 | .01 | .10 |

resistance of a straight vertical antenna for various wavelengths obtained by adding various inductances at the base. For this case $\gamma = 0$, and from the $\gamma = 0$ curve of Figures 14 and 15 $R$ may be directly read. The values which were used in plotting this curve are given in Table III, where they are compared with the corresponding values computed on the assumption that the oscillator is a Hertzian doublet. This latter assumption gives

$$R = 160 \frac{\pi^2 a^2}{\lambda^2}$$

It is seen that the departure of the present theory from the doublet theory is very large for the straight vertical antenna, as should be expected.

It should be noted that the first value in the column of resistances computed by the present theory agrees with the value for this case computed by Abraham in the work cited in the introduction. This one value is the only value arrived at by Abraham for the fundamental oscillation, and is the case of a straight vertical antenna oscillating with its natural frequency. Abraham's other computed values are for the harmonic vibrations with more than one loop of potential always without loading the antenna by inductance, and without any flat-top extension of the antenna.

**24. Comparison of Computations on the Present Theory with Dr. Austin's Values for the Battleship "Maine."**—Figure 16 gives the Radiation Resistance of the Antenna of the Battleship "Maine" as computed by the present Theory in comparison with Dr. Austin's measured values of the total resistance of this antenna, and in comparison with values computed on the doublet theory of Hertz. The black dots of Figure 16 are Dr. Austin's observed values. The heavy line was obtained by computation by the present theory, and the weaker line, by computation regarding the antenna as a doublet of half-length equal to the vertical height of the antenna.

It is seen that the departure between the present theory and the doublet theory is not so great as in the case of the straight vertical antenna, for the reason that the doublet theory becomes more nearly correct as the half-length of the oscillator becomes small in comparison with the wavelength.

Neither of the theories gives a rising value of the resistance with increase of wavelength, and, as Dr. Austin has pointed out, his rising values for long waves are probably not due to radiation from the antenna but possibly to dielectric hysteresis in the ground beneath the flat-top.

I do not give more extended comparisons with experimental values at the present time, because I am now making some experiments to see how much reliance may be placed in antenna resistance measurements made by buzzer methods of excitation in comparison with measurements made by excitation with gaseous oscillators and other methods of continuous excitation.

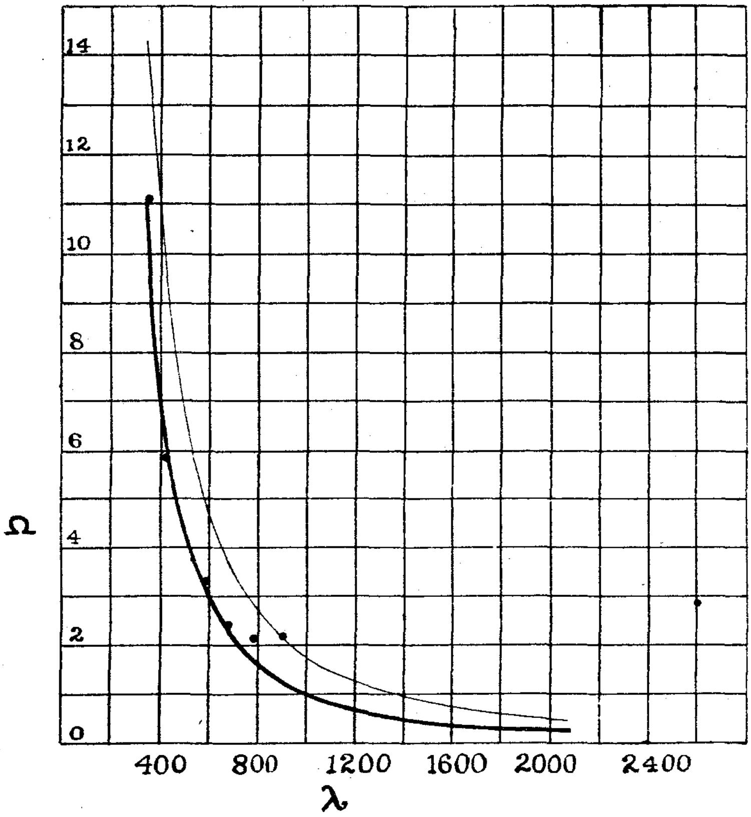

FIGURE 16. Total radiation resistance versus wavelength for the Antenna of the Battleship "Maine." Black dots are Dr. Austin's observed values; heavy line, computations by present theory; light line, computations by doublet theory.

**25. Example of Different Methods of Constructing an Antenna that Will Have a Specified Resistance for a Given Wavelength.**— Let it be required to construct an antenna that will have a given resistance (4 ohms, say) for a given wavelength (2000 meters, say). To solve this problem, it is only necessary to look up the four ohm point on the different $\gamma$-curves of Figures 14 or 15, and find the corresponding value of $\lambda/\lambda_0$. We can then find the $\lambda_0$ of the antenna, since $\lambda$ is given. Dividing the $\lambda_0$ by 4 we obtain the total length of antenna. The value of $\gamma$ gives the fractional part of this length which is to be horizontal. The complete result is tabulated in Table IV.

TABLE IV.

Constants of the Different Antennae that have 4 Ohms Resistance at 2000 Meters.

| $\gamma$ | $\lambda/\lambda_0$ | $\lambda_0$ | Total Length Meters | Vertical Length Meters | Horizontal Length Meters | Intensity Factor in Horizontal Plane |
|---|---|---|---|---|---|---|
| .8 | 1.075 | 1860 | 465 | 93.0 | 372 | .275 |
| .7 | 1.39 | 1435 | 359 | 107.7 | 251.3 | .300 |
| .6 | 1.67 | 1198 | 299 | 119.6 | 179.4 | .310 |
| .5 | 1.94 | 1030 | 258 | 129.0 | 129.0 | .312 |
| .4 | 2.18 | 916 | 229 | 137.4 | 91.6 | .313 |
| .3 | 2.32 | 861 | 215 | 150.5 | 64.5 | .314 |
| .2 | 2.44 | 820 | 205 | 164.0 | 41.0 | .315 |
| .0 | 2.52 | 793 | 198 | 198.0 | 00.0 | .320 |

The question as to which of these antenna to choose for the given purpose is now chiefly a problem in economics. The economic question is, which, for example, is cheaper: Two poles or towers 93 meters high and 372 meters apart, or one tower 198 meters high? This of course pre-supposes that it is designed to use a flat-top antenna instead of some other type, such as an umbrella.

The problem is, however, not wholly economic because the lower antenna would be preferable as a receiving antenna on account of its weaker response to atmospheric disturbances. There is also the further question as to which of the tabulated antennae will give the greatest vertical intensity of electric and magnetic force on the horizon at a distant receiving station. This is the subject matter of the next Part (Part V).

# PART V.

## FIELD INTENSITIES AND SUMMARY.

**26. The Electric and Magnetic Intensities at a Distant Point in the Horizontal Plane.** — Equation (19) gives the values of the electric and magnetic intensities at a distant point due to the vertical portion of the antenna. If we replace I of that equation by its value in terms of $I_0$ from equation (6), and make $\cos\ \theta = 0$, we have the intensities in the horizontal plane in terms of $I_0$, which is the amplitude of the current at the base of the antenna. This gives

$$E_\theta = H_\phi = \frac{2I_0}{cr_0} \cos \frac{2\pi}{\lambda}(ct - r_0)\left[\frac{\cos B - \cos G}{\sin \frac{\pi\lambda_0}{2\lambda}}\right]. \tag{111}$$

The quantities outside the square brackets are constant for a given distance $r_0$ and a given amplitude of transmitting current $I_0$. The *relative intensities* are therefore determined by the factor in the square brackets, which we may designate by

$$X = \frac{\cos B - \cos G}{\sin \frac{\pi\lambda_0}{2\lambda}}. \tag{112}$$

Using the values of $B$, $G$, given in equation (20) and the value of $\gamma$ in (110), this equation (112) becomes

$$X = \frac{\cos \gamma \left(\frac{\pi\lambda_0}{2\lambda}\right) - \cos \frac{\pi\lambda_0}{2\lambda}}{\sin \frac{\pi\lambda_0}{2\lambda}}. \tag{113}$$

This quantity X we shall call "The Intensity Factor in the Horizontal Plane." It is to be noted that the electric and magnetic intensities in the horizon plane are not effected by radiation from the flat-top; for, by equation (55), the field intensities from the flat-top are zero for $z = 0$; that is, all over the horizontal plane through the origin.

In Figure 17 the Intensity Factor in the Horizontal Plane is plotted for various values of $\gamma$ and various values of $\lambda/\lambda_0$. Taking from these curves the values of the intensity factors corresponding to the values of $\gamma$ and $\lambda/\lambda_0$ of Table IV we obtain the results in the last column of Table IV. It is seen that the intensity factor is slightly smaller for

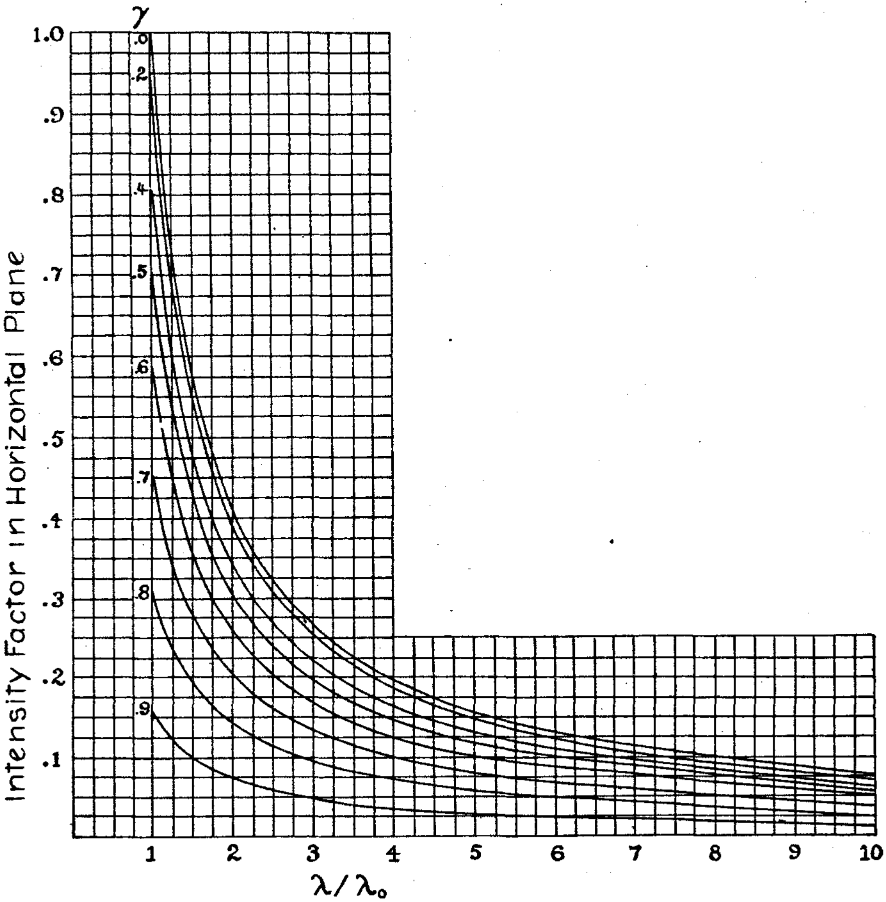

FIGURE 17. Relative intensity of vertical component of Electric Force in a horizontal plane at a given distance from various antennae and for a given amplitude of transmitting current.

the larger values of the relative length of flat-top. This diminished value of the intensity factor should be compensated by the use of a slightly larger transmitting current. The required increase of current may be easily computed by equation (111).

**27. Summary.** — This paper contains a mathematical theory of the flat-top antenna. The process employed consists in the integration of the effects of an aggregate of doublets assumed to be distributed along the antenna so as to give a current distribution described by equation (1) and illustrated in Figure 2. The electric and magnetic field intensity due to each of the doublets is determined by the Maxwell and Hertz Theories for all distant points in space. These field intensities are summed up for all the doublets with strict allowance for the differences of phase due to different doublets; the summation gives the resultant field intensities. Then by Poynting's theorem the power radiated from the antenna through a distant hemisphere (bounded by the earth's surface assumed plane) is computed by the integration of a number of intricate expressions. From the radiated power the radiation resistance is obtained by dividing by the mean square of the current at the base of the antenna. Tables of coefficients for computing radiation resistance are given, and curves are plotted of the calculated values of radiation resistance for different ratios of the length of the flat-top to the total length of the antenna and for different relative wavelengths obtained by loading the antenna with inductance. Curves are also given for determining the relative electric and magnetic field intensities in the horizontal plane for differently proportioned antennae variously loaded. Various equations developed in the treatment may find application to problems in the design of radiotelegraphic stations. Although this investigation was undertaken in ignorance of a simple case investigated by Professor Max Abraham, by a similar fundamental method, his work was discovered early in the course of the treatment and served as a check on one of the resistance values here given. This paper may be regarded as an extension of the remarkable work of Professor Abraham.

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CONTRIBUTIONS FROM THE ZOÖLOGICAL LABORATORY
OF THE MUSEUM OF COMPARATIVE ZOÖLOGY
AT HARVARD COLLEGE.— No. 280.

# *THE 'REFRACTIVE BODY' AND THE 'MITOCHONDRIA' OF ASCARIS CANIS WERNER.*

By A. C. Walton.

With Two Plates.

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# THE 'REFRACTIVE BODY' AND THE 'MITOCHONDRIA' OF ASCARIS CANIS WERNER.

By A. C. Walton.

Received June 2, 1916. Presented by E. L. Mark.

The work of Mayer ('08), Romieu ('11), Meves ('11), Fauré-Frémiet ('11), Wildman ('12) and Romeis ('12) on Ascaris megalocephala, and that of Marcus ('06) on Ascaris canis, has shown that the refractive body of the sperm is formed in the vas deferens shortly before the process of copulation takes place. Their work has also shown that the cytoplasmic granules which surround the nuclei of the nuclei of the spermatids are the so-called 'mitochondria.' Marcus, working on what he supposed to be A. canis, was the first to recognize the source of the refractive bodies. The present writer has elsewhere shown that the form worked on by Marcus was, however, not A. canis but another, as yet unplaced, nematode. It remained for Wildman to follow out more completely the history of the formation of the 'refractive body' and its relation to the 'mitochondria' in A. megalocephala and to show that Marcus's conjecture was an actual fact. Inasmuch as the work of Marcus was not, after all, on A. canis, a description of the conditions in that species may have some value.

The writer here wishes to express his thanks to Dr. S. I. Kornhauser of Northwestern University for assistance in his earlier work on A. canis, and also to Dr. E. L. Mark for his helpful advice and criticism in the completion of this paper.

## 'The Refractive Body.'

Marcus in A. canis, and Mayer, Romieu and Wildman in A. megalocephala, have shown that the refractive body is formed by the fusion of the 'refringent vesicles' of the spermatocytic stages. Wildman alone gives an account of the formation of the refringent vesicles, or granules; he thus completes the history of the formation of the refractive body. The writer has attempted to follow out the history of this body in the case of A. canis and to ascertain if in that species Wildman's conclusions are realized.

### *Technique.*

The material used for this work was fixed in Flemming's (strong) or in Carnoy's fluid. Some of the material was stained with iron haematoxylin-Bordeaux red, some with Ehrlich-Biondi stain, and some according to Benda's method. By these stains it was easy to distinguish between the materials of cytoplasmic and karyoplasmic origin, since they took decidedly different colors.

### *Origin of the 'refringent vesicles.'*

The cytoplasm of the early spermatogonia does not show any of the bodies which take the blue color in Benda's stain ('karyochondria' of Wildman), but the nucleus does show numerous such particles. No evidence was found, such as that given by Wildman, to show that this material covered the surface of the karyochromatin. These separate particles were clearly distinguishable from the karyosome masses and the plastosome (Fig. 1). In the older spermatogonia and in the youngest spermatocytes, bodies staining blue like those of the nucleus appeared in the cytoplasm and at the same time the number of such particles in the nucleus was greatly reduced (Fig. 2). No direct evidence of the actual migration of the particles from the nucleus into the cytoplasm, such as shown by Wildman, Figure 4, to be the case in A. megalocephala, was found in A. canis. Little doubt as to the identity of the particles in the nucleus and in the cytoplasm can be entertained, since the staining reactions of the two groups of granules are identical and the increase of one comes at the time of a great decrease in the numbers of the other. These small cytoplasmic bodies are the primitive 'refringent granules,' which are therefore nuclear in origin. These bodies correspond to the 'Trophochromatin' of Marcus and the 'karyochondria' of Wildman. They swell up by the accumulation of fat, thus forming the 'refringent vesicles' that are so prominent, even in the living material, during the late spermatocytic and early spermatid stages.

### *Fate of the 'refringent vesicles.'*

During the first maturation division, the 'vesicles' arrange themselves in radiating rows parallel to the astral rays and each assumes an oval form (Figs. 3 and 4). In the center of each of these granules, after careful destaining, can be seen short, deeply staining, granular rod-like bodies (Fig. 5) that run lengthwise of the vesicle. The beginning of the second maturation division finds the elongation of the refringent vesicles very pronounced, the vesicles being from three to four times as long as broad. As the second division goes on, the vesicles take a very marked peripheral position with their long axes perpendicular to the cell membrane (Figs. 6 and 7), leaving a clear perinuclear space filled with small granules. These are the 'microsomes' of Van Beneden and the 'plastochondria' of later writers. At the same time the rod-like granular bodies within the refringent vesicles disappear. Both Meves and Romeis have clearly shown that these bodies and the 'plastochondria' are identical structures in the case of A. megalocephala. They certainly have the same staining reactions in Benda's and Ehrlich-Biondi stains, and there can be little doubt but that they are also identical with the bodies seen in A. megalocephala.

The process of 'cytoplasmic reduction' takes place shortly after the second division has been completed, but no refringent vesicles are lost with the extruded cytoplasm, only a considerable number of the small granules derived from the vesicles at the time of the second division. The spermatid is thus reduced by at least one third of its original volume, the refringent vesicles becoming very closely packed together. The vesicles now fuse into irregularly shaped bodies that soon round up into highly refractive spheres (Figs. 8 and 9). These larger spheres again fuse with one another, forming a hollow sphere of globules surrounding the nucleus (Fig. 10). The globules, in undergoing further fusion, move towards one end of the spermatid, leaving the nucleus and its surrounding plastochondria lying free in the cytoplasm (Fig. 11). In Benda's stain these globules now begin to lose the blue color and to take on the yellow stain of yolk material, showing that the karyochondrial material is gradually being changed into food. Fusion continues until a single hemispherical body, slightly concave on the side next to the nucleus, is formed (Fig. 12). This body is the 'refractive body' and is fully formed in the mature spermatozoön just before copulation takes place. Its chemical make-up is almost entirely that of yolk material.

After copulation, the amoeboid spermatozoön makes its way up to the proximal end of the uterus before insemination can take place. Here, as Wildman and others have shown in A. megalocephala, it is found that in about 90 per cent of the spermatozoa the refractive bodies have degenerated, and often are entirely wanting (Figs. 13 and 14). As many spermatozoa completely lack the refractive body as have it in a fully developed condition. That this appearance is not due to a degeneration of the sperm is demonstrated by the fact that more eggs were found penetrated by sperms in this condition than by sperms having a fully developed refractive body. It seems to the writer that, as Marcus has hinted and Wildman definitely stated, the function of the refractive body is not one of a mechanical support for the sperm head in penetrating the egg, but rather that of a source of food during the long interval between copulation and insemination. If this view is correct, the food of the spermatozoön is provided by the extrusion of a nuclear substance in the early spermatocyte and a building up of a food supply through the action of this material upon the cytoplasm of the spermatocyte and of the spermatid.

## The 'Mitochondria.'

With iron haematoxylin-Bordeaux red stain the plastosome and other scattered nuclear granules take a red stain, while the karyosome and its derivatives take a blue stain. These granules later (early spermatocytes) disappear from the nucleus simultaneously with the disappearance of the plastosome. At the same time similar granules are found in the 'refringent vesicles' and continue to show quite plainly up to the time of the second division (Fig. 5). As Montgomery ('11) has shown in Euschistus, the plastosome is of karyosomal origin; and as the refringent vesicles of Ascaris are karyochondria, the statement of Wildman, that the granules of the refringent vesicles are also of karyosomal origin seems most probable.

As mentioned above, at the anaphase of the second maturation division, these granules are given off by the refringent vesicles as the plastochondria. The majority of these granules gather in the perinuclear space, but a large number of the smaller ones are also found scattered generally throughout the cytoplasm of the cell (Fig. 6) and hence are lost at the time of the 'cytoplasmic reduction.' Wildman states that these granules fuse to a slight extent, but no direct proof of this could be found in A. canis, although there are fewer and larger granules at the time of the completion of the refractive body than there were in the early spermatids. The greater number of these granules, or plastochondria, are clustered around the nucleus, although a few of the smaller ones are found scattered throughout the cytoplasmic sheath that covers the refractive body (Fig. 12). These are most plainly seen in spermatozoa in which the refractive body has been partially or entirely used up (Figs. 13 and 14). After having been carried into the egg with the sperm, the larger of the plastochondria can be distinguished for some time after the sperm nucleus has left them and set up its own vesicle preparatory to uniting with the female pronucleus (Fig. 15). These granules slowly fuse with, and become indistinguishable from, the granules of the egg cytoplasm by the time the two pronuclei have united, although occasional cells show a few of the plastochondria still distinguishable at this time (Fig. 16).

Meves believes that these plastochondria are the 'plasma bearers of heredity' because they fuse with similar granules in the egg cytoplasm. In order to prove that they may be such 'bearers,' satisfactory answers must be found to several perplexing questions that are as yet unanswered. One of these is the question as to the nature of the 'cytoplasmic reduction.' In this reduction a great many of the plastochondria are lost. To prevent the loss of parental qualities borne by these granules it must be that there either is a selective division,— so that only plastochondria bearing duplicate characters are cast off,— or else the larger granules, which remain near the nucleus and are never lost, are the only ones that carry hereditary qualities. The first supposition is not substantiated by facts thus far brought out and hence must be abandoned. The second supposition, that the hereditary qualities are borne by the larger plastochondria, has been invalidated by the work of Vejdovský, in which he has shown that, inasmuch as these plastochondria entirely lose their identity, they are in no way continuous from generation to generation and hence can not be bearers of hereditary qualities.

The work of Lillie on Nereis is also an argument against the functioning of the plastochondria as bearers of hereditary qualities, for he finds that the plastochondria of the sperm do not even enter the egg, but, with the tail-piece, remain outside. The classical work of Boveri on the fertilization of enucleated egg fragments also shows that the plastochondria of the female do not carry hereditary qualities. These granules unite with the corresponding ones of the male. Now, if the latter carry male characteristics, we should be justified in supposing that the female granules would similarly carry female characters, since the offspring of normal unions have qualities of both parents. If, however, the female plastochondria do not carry such determinants, as Boveri's work shows, then we are not justified in supposing that the male plastochondria do.

From this evidence it seems clear that the plastochondria, or the so-called 'mitochondria,' can not be true 'plasma bearers of heredity,' at least in the nematodes Ascaris canis and Ascaris megalocephala, as well as in the cases studied by Lillie and by Boveri.

## SUMMARY.

(1) The 'refractive body' is formed by the fusion of the 'refringent vesicles.'

(2) The 'refringent vesicles' are formed from the cytoplasm of the spermatocytes through the action of small extruded granules of karyochromatin, the 'karyochondria' of Wildman.

(3) The 'refractive body' in Ascaris canis Werner takes no part in the fertilization of the egg other than as a source of food supply to the spermatozoön between the time of copulation and the time of insemination.

(4) The 'plastochondria' are partly of plastosomal and partly of karyochondrial origin through their formation in the 'refringent granules.'

(5) The 'plastochondria' (mitochondria) are not 'plasma bearers of heredity' in Ascaris canis Werner.

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## DESCRIPTION OF PLATES.

All figures are camera drawings, made with a Spencer microscope; achromatic, homog. immersion lens, 1.8 mm., N. A. 1.30; tube length 165 mm.; and projection distance 420 mm.

Figures 1–14 are reproduced at an enlargement of 4,000 diameters (compensating ocular 18x, Zeiss); Figures 15 and 16 an enlargement of 1700 diameters (compensating ocular 9x, Zeiss). In reproducing Figures 15 and 16 they were reduced to ⅔ original size, so that their magnification on the plate is 1133 diameters.

### *Abbreviations.*

| | |
|---|---|
| *cp. rfr.* | refractive body |
| *kchnd.* | karyochondria |
| *plstchnd.* | plastochondria |
| *vs. rfr.* | refringent vesicles |

## PLATE 1.

| | |
|---|---|
| FIGURE 1. | Spermatogonium showing intra-nuclear karyochondria (*kchnd.*). |
| FIGURE 2. | Early spermatocyte showing extra-nuclear distribution of karyochondria. |
| FIGURE 3. | Late prophase, first division, showing refringent granules (*vs.rfr.*) developed from karyochondria. |
| FIGURE 4. | Anaphase, first division, showing radial arrangement of 'refringent vesicles' (*vs. rfr.*). |
| FIGURE 5. | Prophase, second division, showing extreme elongation of 'refringent vesicles' and their contained plastochondrial, granular rods (*plstchnd.*). |
| FIGURE 6. | Anaphase, second division, showing peripheral position of 'refringent vesicles' and also the plastochondria, now lying free in the cytoplasm. |
| FIGURE 7. | Early spermatid, showing peripheral position of 'refringent vesicles' and the large plastochondria in the perinuclear space. |

## PLATE 2.

**Figure 8.** Spermatid, showing fusion of the 'refringent vesicles.'
**Figure 9.** Spermatid, showing advanced fusion of the 'refringent vesicles.'
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**Figure 13.** Mature spermatozoön after remaining for some time in oviduct of female, showing shrinkage in size of the nutritive refractive body (*cp. rfr.*).
**Figure 14.** Mature spermatozoön, at time of insemination, which has entirely used up refractive body. Plastochondria are scattered all through the cytoplasm.
**Figure 15.** Fertilized egg with pronuclei not as yet united, showing persistence of male plastochondria.
**Figure 16.** Fertilized egg with pronuclei uniting, showing last remnants of male plastochondria, the most of which are scattered throughout the cell.

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# *DIFFERENTIAL GEOMETRY OF TWO DIMENSIONAL SURFACES IN HYPERSPACE.*

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**1. Introduction.** There are several ways of generalizing the ordinary differential theory of surfaces. The one most extensively treated is that which deals with varieties of $n-1$ dimensions in a Euclidean space of $n$ dimensions.[1] A second method is to investigate properties of two-dimensional varieties in a space of four or indeed of $n$ dimensions.[2] A third and more general extension of the theory would be to study varieties of $k$ dimensions in a space of $n$ dimensions, and under this head a very interesting species can arise[3] when $n = 2k-1$. The recent contributions to this third have dealt with the projective differential properties and thus have afforded only a partial generalization of the general theory.

We propose here to study the theory of two-dimensional varieties in space of $n$ dimensions and to exhibit the way in which the ordinary theory arises through specialization. The generalization in this case is not so immediately obvious as in the first case and perhaps throws more light on the ordinary theory of surfaces than does that.

---

[1] See, for instance, Killing, *Die Nicht-Euklidischen Raumformen;* Bianchi, *Lezioni di Geometria Differenziale*, Vol. I, Chaps. 11, 14; Shaw, *Amer. J. Math.*, **35**, No. 4, 395–406.

[2] K. Kommerell, Die Krümmung der Zweidimensionalen Gebilde, in ebenen Raum von vier Dimensionen, Dissertation, Tübingen, 1897, 53 pp.; and Riemannsche Flächen in ebenen Raum von vier Dimensionen, *Math. Ann.*, **60**, in which the dissertation is also summarized; E. E. Levi, Saggio sulla Teoria delle Superficie a due Dimensioni immersi in un Iperspazio, *Ann. R. Scu. Norm.*, Pisa, **10**, 99 pp.; C. L. E. Moore, *Ann. Math.* (2) **16**, 89–96 (1915).

[3] C. Segre, Su una Classe di Superficie ecc., *Att. Torino*, **42** (1907), and *Rend. Circ. Mat.*, **30**, 87–121 (1910); and further developments by Bompiani and Terracini.

**2. Methods of attack.** When attacking the theory of the two-surface in $S_n$, the method of attack is of fundamental importance. That followed by Kommerell consists in starting with the finite equations of the surface and in trying by geometric intuition to find what sort of properties lend themselves most readily to generalization. This method has the disadvantage that it is somewhat lacking in system, and one is never confident that he is not overlooking things that are perhaps the most vital to the subject. Levi starts with the finite parametric equations of the surface and determines the invariants $I$ of orthogonal transformations and the elements $J$ covariant under a change of parameters. This is more systematic and safer.

It seems clear that the safest and most systematic method of attack is to discuss the surface entirely from the point of view of the differential quadratic form or better of the set of differential quadratic forms which define the surface. In following this method we have the advantage that Ricci, in his *Lezioni sulla Teoria delle Superficie,*[4] has pursued more consistently than any one else the same method with regard to surfaces in ordinary space. In his work those properties which depend on the first fundamental form are first developed, and then those which follow from the first and second forms together. Now the first fundamental form defines a surface in so far and only so far as that surface may one of the infinite class of surfaces applicable upon it. Thus the first fundamental form determines a surface as a perfectly flexible inextensible membrane. There is no restriction to a rigid surface in space and none upon the number of dimensions in which the surface may lie; we work entirely on the surface itself. Hence all the results which Ricci obtained in Part I of his *Lezioni* are true without any modification of the proofs or the interpretation in any number of dimensions.

---

[4] Padova, Drucker, 1898 (Lithographed). The contents of this book is as follows:—*Introduzione:* I. Delle equazioni lineari ed omogenee a derivate parziali di I. ordine e dei sistemi completi, p. 1. II. Nozioni generali sulle forme differenziali quadratiche, p. 36. III. Del calcolo differenziale assoluto ad $n$ variabili, p. 45. IV. Della classificazione delle forme differenziali quadratiche positive, p. 73. V. Degli invarianti assoluti comuni ad una forma fondamentale ed ai sistemi associati, p. 91. VI. Del calcolo differenziali assoluto a due variabili independenti, p. 105. *Parte Prima: Della proprietà delle superficie considerate come veli flessibili ed inestendibili.* I. Dei sistemi di coordinate sopra una superficie qualunque, p. 134. II. Generalità sulle congruenze di linee tracciate sopra una superficie, p. 148. III. Considerazioni generali sugli invarianti differenziali ecc., p. 163. IV. Delle congruenze di linee geodetiche e di linee parallele, p. 176. V. Fascii e sistemi isotermi, e rappresentazioni conformi, p. 202. VI. Sulla integrazione della equazione delle congruenze geodetiche, p. 223. VII. Delle congruenze isoterme di Liouville, p. 248. *Parte Seconda: Teoria delle superficie considerate come dotate di forma rigida nello spazio.* I. Equazioni generali della teoria delle superficie, p. 270. II. Delle linee di curvatura e delle linee asintotiche, p. 287. III. Della rappresentazione sferica di Gauss, p. 309. IV. Di alcune classi speciali di superficie, p. 322. V. Evolute e superficie di Weingarten, p. 350. VI. Delle superficie di secondo grado, p. 366. VII. Della applicabilità delle superficie, p. 385.

**3. Quadratic differential forms.** When we wish to interpret a manifold defined by a binary quadratic differential form as a surface in space we have to introduce a set of variables such that

$$ds^2 = \Sigma_{ij}\, a_{ij} dx_i dx_j = dy_1^2 + dy_2^2 + dy_3^2, \qquad i,j = 1, 2;$$

and it is the determination of this set of variables which leads to the second fundamental form. It is a fundamental proposition in the theory of binary quadratic forms that such a form may be written as the sum of three squares. Hence for the interpretation of a binary differential form as a surface, three dimensions are sufficient. When the theory of the ternary differential quadratic form is studied with reference to its reduction to a sum of squares, it is found that in general *six* variables are needed. Hence to interpret the theory of the ternary form we must in general go to a spread $V_3$ of three dimensions in $S_6$. It is clear from this that the theory of the $V_3$ in $S_4$ does not correspond with the theory of any but a very special class of ternary forms. Hence from the point of view of the quadratic form the theory of surfaces does not generalize very simply. In general for a quadratic differential form in $k$ variables the reduction to the sum of squares may require $k(k-1)/2$ variables, and the minimum number of additional variables required, above $k$, is called the *class* of the form.[5]

When in our special case of a binary quadratic form, we wish to interpret the form as a surface in $S_n$, we have to determine $n$ variables $y$ so that $ds^2 = \Sigma_i\, dy_i^2$, $i = 1, \cdots, n$. The determination is made by the properties of systems of partial differential equations, in particular complete systems. This has been accomplished by Ricci in the general case and his result is stated in a theorem.[6]

---

[5] Ricci, *Lezioni*, Introduction, Chap. 4.

[6] Ricci, *Lezioni*, pp. 90–91. Ricci has also treated the more general question of a variety of $n$ dimensions immersed in a variety (not a Euclidean space) of $n + m$ dimensions and the transformation of $\Sigma a_{rs} dx_r dx_s$, $r, s = 1, \ldots, n$, into $\Sigma c_{uv} dy_u dy_v$, $u, v = 1, \ldots, n + m$. *Rend. R. Acc. Lincei*, (5) **11**, 355–362 (1902).

We shall not make a direct use of that theorem but shall give an independent demonstration of the special case in which we are interested. For in order properly to understand the statement of this theorem it would be necessary to explain the technical language of Ricci's absolute differential calculus, and we consider it better to explain this piece by piece as we need it, and to give at length demonstrations of theorems which are special cases of his, in order that we may make the work less abstract.

## CHAPTER I. RICCI'S METHOD.[7]

**4. Two types of transformations.** If by a change of variable,

$$x_1 = x_1(y_1, y_2), \quad x_2 = x_2(y_1, y_2),$$

we transform the differential $X_1dx_1 + X_2dx_2$ into a new differential in the new variables so that

$$X_1dx_1 + X_2dx_2 = Y_1dy_1 + Y_2dy_2,$$

we find

$$Y_1 = X_1\frac{\partial x_1}{\partial y_1} + X_2\frac{\partial x_2}{\partial y_1}, \quad Y_2 = X_1\frac{\partial x_1}{\partial y_2} + X_2\frac{\partial x_2}{\partial y_2}; \tag{1}$$

and if by the same change of variable we transform the differential system

$$\frac{dx_1}{X^{(1)}} = \frac{dx_2}{X^{(2)}} \quad \text{into} \quad \frac{dy_1}{Y^{(1)}} = \frac{dy_2}{Y^{(2)}},$$

---

[7] The lithographed *Lezioni* already cited is not obtainable either in new or second hand copies and is to be found in very few American libraries; it is to be had, however, at the Harvard library, the Boston Public library, and the library of Washington University (St. Louis). Ricci's first presentation of the essentials of the theory is scattered through a considerable number of papers in different Italian journals, particularly journals of the learned societies. See, e. g., *Rend. R. Acc. Lincei*, **5**, 112–118 (1889); *Studi off. d. Univ. Padovana a. Bolognese n. VIII centenario ecc.*, Vol. III (1888); Atti R. Ist. Veneto, (7) **4**, 1–29 (1897), *Ibid.*, **5**, 643–681 (1894), *Ibid.*, **6**, 445–488 (1895); *Rend. R. Acc. Lincei* (5) **4**, 232–237 (1895), *Ibid.*, **11**, 355–362 (1902). A general sketch of the method is found in *Bull. Sci. Math., Paris*, (2) **16**, 167–189 (1892) and a very elaborate outline not only of the foundations of the theory but of many of its applications is given by Ricci and Levi-Civita in *Math. Ann.* **54**, 125–201 (1900). More recently Grossmann, *Verallgem. Relativitätstheorie* (with Einstein), Teubner, 1913 (from *Zs. Math. Physik*, Vol. 62), mentions a few of the salient features of the method in a modified notation. It is however only in the *Lezioni* that the treatment of the elementary parts of the theory is given in comfortable detail. Moreover, the *Math. Encyc.* and the few authors who cite Ricci do so in a manner which suggests strongly that his method is practically unknown. These facts are offered in justification of our reproducing here material which has previously been published.

we find, on working out the details,

$$Y^{(1)} = X^{(1)} \frac{\partial y_1}{\partial x_1} + X^{(2)} \frac{\partial y_1}{\partial x_2}, \quad Y^{(2)} = X^{(1)} \frac{\partial y_2}{\partial x_1} + X^{(2)} \frac{\partial y_2}{\partial x_2}. \tag{2}$$

We can write the transformation from $X$ to $Y$ in the two cases in the forms

$$Y_r = \Sigma_s X_s \frac{\partial x_s}{\partial y_r}, \tag{1'}$$

$$Y^{(r)} = \Sigma_s X^{(s)} \frac{\partial y_r}{\partial x_s}. \tag{2'}$$

The system $X_1$, $X_2$ is said to be transformed *covariantly*. The system $Y^{(1)}$, $Y^{(2)}$ is said to be transformed *contravariantly*. Furthermore if we have any system of $X$'s which is so defined that the transformed system follows the rule (1′), it is called *covariant* and the members of the system are denoted by lower indices; whereas if the system follows the rule of transformation (2′), it is called *contravariant* and the members of the system are denoted by upper indices. The system of differentials $dx_1$, $dx_2$, by the formula for the total differential, is seen to follow rule (2′) and therefore the system of differentials of the independent variables are the members of a contravariant system; but we observe that the indices are lower, in conformity with ordinary practice, and not upper as the rule here would require.

If we were dealing with more than two variables $x_1$, $x_2$, we should still find that the transformation of the differential

$$X_1 dx_1 + X_2 dx_2 + \cdots\cdots + X_n dx_n$$

led to the rule (1′), where $s$ runs from 1 to $n$, for changing $X$'s into $Y$'s; and that the transformation of the system of equations

$$\frac{dx_1}{X^{(1)}} = \frac{dx_2}{X^{(2)}} = \cdots\cdots = \frac{dx_n}{X^{(n)}}$$

led to the set of equations (2′), where $s$ runs from 1 to $n$.

**5. Generalization of vector analysis.** If we could follow the ideas of Grassmann-Gibbs,[8] we should consider the sets of quantities

$$X_1, X_2, \cdot \cdot \cdot, X_n \quad \text{or } X^{(1)}, X^{(2)}, \cdot \cdot \cdot, X^{(n)}$$

as components of a vector along the directions $dx_i$ or upon the planes perpendicular to those directions. It proves, however, to be impossible to establish here more than an analogy; for it is actually untrue that these elements are such components.

That the $X$'s may not generally be interpreted as components of a vector is clear from the expression for the differential of work in terms of generalized coördinates, namely,

$$dW = Q_1dq_1 + Q_2dq_2 + \ldots + Q_ndq_n.$$

The set of generalized forces $Q_i$ is covariant under a transformation of the $q$'s, but the generalized forces are not the projections of the resultant force either upon the directions $dq_i$ or upon the planes perpendicular thereto in the $n$-dimensional representative space of the $q$'s; for instance, in polar coördinates in the plane, $dW = R\,dr + r\Theta d\theta$, where $R$ and $\Theta$ (not $r\Theta$) are the radial and tangential components of the force.

We have therefore to deal not with a generalized vector-analysis but with a generalization of vector analysis when we deal with systems $X_s$ or $X^{(s)}$. A method of converting such a system into one which represents the components of a vector will be mentioned later (note 17 to § 12).

So long as we remain in the vicinity of a particular point and deal only with differentials of the first order, the transformations (1′) and (2′) are linear with constant coefficients of the type

$$Y_r = \Sigma_s\, c_{sr}X_s, \qquad c_{sr} = \frac{\partial x_s}{\partial y_r},$$

$$Y^{(r)} = \Sigma_s\, \gamma_{sr}X^{(s)}, \qquad \gamma_{sr} = \frac{\partial y_r}{\partial x_s},$$

and the first section of our presentation of Ricci's method will therefore be strictly algebraic theory of the linear transformation. When, later, we come to differentiation we shall have to take into account the variability of the coefficients of the transformation.

---

[8] Grassmann, *Ausdehnungslehren von* 1844 *u.* 1862, also *Gesammelte Werke;* Gibbs, *Scientific Papers*, Vol. II; Gibbs-Wilson, *Vector Analysis*; Wilson, *Trans. Conn. Acad. Arts Sci.*, **14**, 1–57, (1908).

**6. Sets of elements.** We deal with sets of elements of different orders; thus our system is a generalization of matrical as well as of vectorial analysis. The fundamental elements are sets of quantities $X$ with $m$ indices, each of which may take all values from 1 to $n$. For example, if $n = 2$,

$m = 0$, $X$, no index;
$m = 1$, $X_1, \quad X_2$, one index;
$m = 2$, $X_{11}, X_{21}, X_{21}, X_{22}$, two indices;
$m = 3$, $X_{111}, X_{112}, X_{121}, X_{122}, X_{211}, X_{212}, X_{221}, X_{222}$, three indices.

In general there are $n^m$ quantities in the system of order $m$ with 1 to $n$ as the range for each index.

These systems of successive orders are analogous to the scalars, vectors, dyadics, triadics, . . . . of Gibbs, and to the scalars, vectors, open products with 1, 2, . . . openings of Grassmann; the matrical analogy would take us to matrices of higher dimensions than the usual two. The *addition* of two systems of the same order and the multiplication of a system by a constant are according to definitions obviously suggested by the analogies, i. e., the systems are *linear*.

Multiplication[9] of a system of order $m$ into a system of order $m'$ consists in multiplying each element of the first system into each element of the second and gives a system of order $m + m'$. For example,

$$(X_1, X_2)\,(X_{11}, X_{12}, X_{22}, X_{22}) = X_1X_{11}, X_1X_{12}, X_1X_{21}, X_1X_{22}, X_2X_{11}, X_2X_{12}, X_2X_{21}, X_2X_{22},$$

which is a system of order 3 and may be written $X_{ijk}$, $i, j, k = 1, 2$.

By following the method of Gibbs[10] we may construct an outer or "combinatory" product of two systems of order 1, as

$$(X_1, X_2, X_3) \times (Y_1, Y_2, Y_3) = X_2Y_3 - Y_3Y_2, \quad X_3Y_1 - X_1Y_3, \quad X_1Y_2 - X_2Y_1$$

---

[9] Grossmann calls the multiplication "outer" from analogy with Grassmann's outer product with which it has little in common; the real analogy is with Gibbs' indeterminate and Grassmann's open product.

[10] On Multiple Algebra, *Scientific Papers*, Vol. II, pp. 91–117.

for the case $n = 3$; and for the general case the elements of the product would be $X_{ij} = X_iY_j - X_jY_i$. This system of the second order is skew symmetric, that is, $X_{ij} = -X_{ji}$, $X_{jj} = 0$. We could likewise form an "algebraic" product $X_{ij} = X_iY_j + X_jY_i$, which is symmetric. And in general we could form the combinatory and algebraic products of $k$ systems.

If we wish actually to write the systems as hypercomplex numbers with "units" attached, we have

$$X_1e_1 + X_2e_2,$$
$$X_{11}e_1e_1 + X_{12}e_1e_2 + X_{21}e_2e_1 + X_{22}e_2e_2,$$

and so on. The product of these two systems would be similarly expressed with the units $e_1e_1e_1$, $e_1e_1e_2$, . . . exactly as the triadic which arises from the product of a vector and a dyadic.

If we wish to consider the units $e_1$, $e_2$, or $e_1e_1$, $e_1e_2$, $e_2e_1$, $e_2e_2$, etc., replaced by the set of independent variables, $x_1$, $x_2$, or $x_1y_1$, $x_1y_2$, $x_2y_1$, $x_2y_2$, etc., the expressions become

$$X_1x_1 + X_2x_2, \qquad X_{11}x_1y_1 + X_{12}x_1y_2 + X_{21}x_2y_1 + X_{22}x_2y_2,$$

and so on,— that is, they become linear, bilinear, trilinear, . . ., forms. Ricci's system of the $m$th order with range 1 to $n$ is therefore analogous to an $m$-linear form in $n$ variables.

**7. Transformations of sets of elements.** Consider next the linear transformation[11]

$$x_i = \Sigma_j c_{ij} y_j. \tag{3}$$

These equations may be solved by multiplying by the cofactors $C_{ik}$ each divided by the determinant $|c_{ij}|$, that is, by $\gamma_{ik} = C_{ik}/|c_{ij}|$, and summing with respect to $i$. Then

$$y_k = \Sigma_i \gamma_{ik} x_i \quad \text{or } y_i = \Sigma_j \gamma_{ji} x_j. \tag{3'}$$

If $u_i$, $v_i$ are variables contragredient to $x_i$ and $y_i$, the transformation upon the $u$'s and $v$'s is

$$v_i = \Sigma_j c_{ji} u_j \quad \text{or } u_i = \Sigma_j \gamma_{ij} v_j. \tag{4}$$

---

[11] We may refer to Bôcher's *Introduction to Higher Algebra* for the theory of linear transformations, linear dependence, cogredient and contragredient variables, bilinear forms, square matrices, etc.

If the transformation (3) be effected upon the variables of the linear, bilinear, . . . . . ., $m$-linear forms, there arise new forms in which the coefficients are $(X_i)$, $(X_{ij})$, . . . ., if we now use Ricci's notation,[12] in place of $Y_i$, $Y_{ij}$ used above. The law of transformation between the $X$'s and $(X)$'s is important and is obtained as follows:

$$\Sigma_i X_i x_i = \Sigma_i X_i \Sigma_j c_{ij} y_j = \Sigma_j (\Sigma_i c_{ij} X_i) y_j = \Sigma_j (X_j) y_j.$$

Hence

$$(X_j) = \Sigma_i c_{ij} X_i \text{ or } (X_i) = \Sigma_j c_{ji} X_j. \qquad (5)$$

If we solve, we have

$$X_i = \Sigma_j \gamma_{ij} (X_j). \qquad (5')$$

Similarly if we take a bilinear form, we find

$$\begin{aligned}\Sigma_{ij} X_{ij} x_i \xi_j &= \Sigma_{ij} X_{ij} \Sigma_k c_{ik} y_k \Sigma c_{jl} \eta_l \\ &= \Sigma_{kl} (\Sigma_{ij} c_{ik} c_{jl} X_{ij}) y_k \eta_l = \Sigma_{kl} (X_{kl}) y_k \eta_l.\end{aligned}$$

Hence changing subscripts we have

$$(X_{ij}) = \Sigma_{kl} c_{ki} c_{lj} X_{kl}, \qquad X_{ij} = \Sigma_{kl} \gamma_{ik} \gamma_{jl} (X_{kl}). \qquad (6)$$

In general for a system of order $m$, the transformation of the $m$-linear form shows that the transformation of the system follows the rule

$$(X_{i_1 i_2 \dots i_m}) = \Sigma_{j_1 j_2 \dots j_m} c_{j_1 i_1} c_{j_2 i_2} \dots c_{j_m i_m} X_{j_1 j_2 \dots j_m} \qquad (7)$$

or

$$X_{i_1 i_2 i_3 \dots i_m} = \Sigma_{j_1 j_2 \dots j_m} \gamma_{i_1 j_1} \gamma_{i_2 j_2} \dots \gamma_{i_m j_m} (X_{j_1 j_2 \dots j_m}). \qquad (7')$$

The results of this article may be written more compactly in matrical notation. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$, with a similar meaning for $\mathbf{y}$, be an extensive magnitude. Let $\mathbf{M}$ be the matrix

$$\mathbf{M} = \begin{Vmatrix} c_{11} & c_{12} \dots & c_{1n} \\ c_{21} & c_{22} \dots & c_{2n} \\ \dots & \dots & \dots \\ c_{n1} & c_{n2} \dots & c_{nn} \end{Vmatrix}$$

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[12] Ricci, *Lezioni*, p. 49. Although the use of ( ) for the transformed quantities appears awkward it is less so than any notation which has occurred to us.

of the coefficients of (3). Let $\mathbf{M}^{-1}$ and $\mathbf{M}_C$ be the reciprocal and conjugate. Then,

$$\mathbf{x} = \mathbf{M}\cdot\mathbf{y}, \quad \mathbf{y} = \mathbf{M}^{-1}\cdot\mathbf{x}, \quad \mathbf{u} = \mathbf{v}\cdot\mathbf{M}^{-1}, \quad \mathbf{v} = \mathbf{u}\cdot\mathbf{M},$$

provided the products are defined as usual and the dot is used in the sense of Gibbs. The transformations of $\mathbf{X}$ and $\mathbf{XY}$ into $(\mathbf{X})$ and $(\mathbf{XY})$ are

$$\mathbf{X}\cdot\mathbf{x} = \mathbf{X}\cdot\mathbf{M}\cdot\mathbf{y} = (\mathbf{X})\cdot\mathbf{y}, \quad (\mathbf{X}) = \mathbf{X}\cdot\mathbf{M}, \quad \mathbf{X} = (\mathbf{X})\cdot\mathbf{M}^{-1},$$
$$\mathbf{XY}:\mathbf{x}\boldsymbol{\xi} = \mathbf{XY}:[\mathbf{M}\cdot\mathbf{y}\ \mathbf{M}\cdot\boldsymbol{\eta}] = (\mathbf{XY}):\mathbf{y}\boldsymbol{\eta},$$
$$(\mathbf{XY}) = \mathbf{XY}:\mathbf{MM}, \quad \mathbf{XY} = (\mathbf{XY}):\mathbf{M}^{-1}\mathbf{M}^{-1}.$$

The double products (containing two dots) are to be interpreted as indicating the union of *corresponding* elements, that is,

$$\mathbf{XY}:\mathbf{MM} = [\mathbf{X}\cdot\mathbf{M}]\,[\mathbf{Y}\cdot\mathbf{M}] \qquad \text{and} \qquad \mathbf{XY}:\mathbf{x}\boldsymbol{\xi} = [\mathbf{X}\cdot\mathbf{x}]\,[\mathbf{Y}\cdot\boldsymbol{\xi}].$$

The expression $\mathbf{XY}:\mathbf{MM}$ may be written also as $\mathbf{M}_C\cdot\mathbf{XY}\cdot\mathbf{M}$. The use of a formal product of the dyad type $\mathbf{XY}$ for any system of the second order is legitimate because the systems are linear.

8. **An adjoined quadratic form.** Now if we have a given fundamental quadratic form

$$\Sigma_{ij}a_{ij}x_ix_j, \qquad a_{ij} = a_{ji}, \tag{8}$$

the transformation of the coefficients $a_{ij}$ will be the same as that of the elements $X_{ij}$ found above. It is for this reason that we say that the system $X_{ij}$ transforms *covariantly* with $a_{ij}$ or with the given form (8); and we shall further say that a system of $X$'s with any number of (lower) indices which transforms according to (7) or (7′) is a *covariant system*. The simplest case, given by (5) or (5′), shows that the covariant system of order 1 is transformed like the contragredient variables in (4).

If now we have a system, which we may denote by $X^{(i)}$, instead of by $X_i$, which transforms like the cogredient variables, or in an analogous manner, we call the system *contravariant*;[13] that is if,

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[13] Grossmann employs Greek letters with lower indices to designate a contravariant system instead of Ricci's letters with upper indices. A trial of this notation convinces us that however awkward Ricci's notation appears it is more convenient than that of Grossmann; especially in view of the principal of duality (§ 9), the lack of correspondence between Greek and Roman letters, and the undesirability of immobilizing alphabets in a definite sense.

$$(X^{(i)}) = \Sigma_j\gamma_{ji}X^{(j)}, \qquad X^{(i)} = \Sigma_j c_{ij}(X^{(j)}), \tag{9}$$

$$(X^{(ij)}) = \Sigma_{kl}\gamma_{ki}\gamma_{lj}X^{(kl)}, \qquad X^{(ij)} = \Sigma_{kl}c_{ik}c_{jl}(X^{(kl)}), \tag{10}$$

and

$$(X^{(i_1i_2\,\cdot\,\cdot\,i_m)}) = \Sigma_{j_1j_2\,.\,.\,j_m}\gamma_{j_1i_1}\gamma_{j_2i_2}\,.\,.\,.\,\gamma_{j_mi_m}X^{(j_1j_2\,\cdot\,\cdot\,j_m)}, \tag{11}$$

$$X^{(i_1i_2\,\cdot\,\cdot\,i_m)} = \Sigma_{j_1j_2\ldots j_m}c_{i_1j_1}c_{i_2j_2}\ldots c_{i_mj_m}(X^{(j_1j_2\,\cdot\,\cdot\,j_m)}), \tag{11$'$}$$

these systems $X$ with upper indices are contravariant of orders 1, 2, and $m$, respectively.

An important contravariant system is formed of the elements $\mathbf{a}_{ij}$ which are the cofactors of the determinant of the quadratic form (8) each divided by the determinant of the form. We may prove this as follows. Let $\epsilon_{ik}$ be 0 or 1 according as $j \neq k$ or $j = k$. Then,

$$\Sigma_i a_{ij}\mathbf{a}_{ik} = \epsilon_{jk}.$$

Substitute for $a_{ij}$ from (6). Then

$$\Sigma_i\Sigma_{pq}\gamma_{ip}\gamma_{jq}(a_{pq})\mathbf{a}_{ik} = \epsilon_{jk}.$$

Multiply by $c_{js}$ and sum over $j$; the expression reduces by virtue of the fact that $\Sigma_j c_{js}\gamma_{jq}$ is zero unless $s = q$ and unity if $s = q$. (We have therefore

$$\Sigma_{jq}\gamma_{jq}c_{js}(a_{pq}) = (a_{ps}) \tag{12}$$

which is a formula often used for reducing certain double sums to a single term.) Hence

$$\Sigma_i\Sigma_p\gamma_{ip}(a_{ps})\mathbf{a}_{ik} = \Sigma_j\epsilon_{jk}c_{js}.$$

Multiply by $(\mathbf{a}_{ts})$ and sum over $s$; then $p = t$ alone gives something. (We have then

$$\Sigma_{sp}\gamma_{ip}(a_{ps})\,(\mathbf{a}_{ts}) = \gamma_{it} \tag{13}$$

which like (12) reduces a double sum to a single term.) Hence

$$\Sigma_i\gamma_{it}\mathbf{a}_{ik} = \Sigma_{js}\epsilon_{jk}c_{js}(\mathbf{a}_{ts}).$$

Multiply by $c_{rt}$ and sum over $t$. The double sum $\Sigma_{it}$ on the left reduces to the single term $a_{rk}$ by (12), and we have

$$a_{rk} = \Sigma_{jst}\epsilon_{jk}c_{js}c_{rt}(a_{ts}).$$

Now $j = k$ alone contributes something. Hence, finally,

$$a_{rk} = \Sigma_{st}c_{ks}c_{rt}(a_{ts}).$$

If we compare this with (10), we see that the transformation of the $a$'s is contravariant, and the theorem is proved. We shall therefore in conformity with our general notation use upper indices and indeed write

$$a_{ij} = a^{(ij)}$$

for the cofactor of $a_{ij}$ in (8) divided by the determinant $|a_{ij}|$.

If $\mathbf{X}^\circ = X^{(1)}, X^{(2)}, \ldots, X^{(n)}$ be the notation for a contravariant system, the results of this article may be written

$$(\mathbf{X}^\circ) = \mathbf{M}^{-1}\cdot\mathbf{X}^\circ, \quad \mathbf{X}^\circ = \mathbf{M}\cdot(\mathbf{X}^\circ), \quad (\mathbf{X}^\circ\mathbf{Y}^\circ) = \mathbf{M}^{-1}\mathbf{M}^{-1}:\mathbf{X}^\circ\mathbf{Y}^\circ, \text{ etc.}$$

If $\mathbf{A}$ be the matrix $||a_{ji}||$, and I the idemfactor we have further

$$\mathbf{A}\cdot\mathbf{A}^{-1} = \mathrm{I}, \quad \mathbf{A} = (\mathbf{A}):\mathbf{M}^{-1}\mathbf{M}^{-1}, \quad [(\mathbf{A}):\mathbf{M}^{-1}\mathbf{M}^{-1}]\cdot\mathbf{A}^{-1} = \mathrm{I}.$$

Now if $[\mathbf{C}:\mathbf{MN}]\cdot\mathbf{D} = \mathrm{I}$, then $\mathbf{C}\cdot[\mathbf{NM}:\mathbf{D}] = \mathrm{I}$. For

$$[\mathbf{C}:\mathbf{MN}]\cdot\mathbf{D} = \mathbf{M}_{\mathrm{C}}\cdot\mathbf{C}\cdot\mathbf{N}\cdot\mathbf{D}, \qquad \mathbf{C}\cdot[\mathbf{NM}:\mathbf{D}] = \mathbf{C}\cdot\mathbf{N}\cdot\mathbf{D}\cdot\mathbf{M}_{\mathrm{C}},$$
$$\mathbf{M}_{\mathrm{C}}\cdot\mathbf{C}\cdot\mathbf{N}\cdot\mathbf{D} = \mathrm{I}, \qquad \mathbf{C}\cdot\mathbf{N}\cdot\mathbf{D} = \mathbf{M}_{\mathrm{C}}^{-1}, \qquad \mathbf{C}\cdot\mathbf{N}\cdot\mathbf{D}\cdot\mathbf{M}_{\mathrm{C}} = \mathrm{I}.$$

Hence $$[(\mathbf{A}):\mathbf{M}^{-1}\mathbf{M}^{-1}]\cdot\mathbf{A}^{-1} = (\mathbf{A})\cdot[\mathbf{M}^{-1}\mathbf{M}^{-1}:\mathbf{A}^{-1}] = \mathrm{I}$$

or $$(\mathbf{A}^{-1}) = \mathbf{M}^{-1}\mathbf{M}^{-1}:\mathbf{A}^{-1}, \qquad \mathbf{A}^{-1} = \mathbf{MM}:(\mathbf{A}^{-1}).$$

This analysis could, of course, be carried out without the conversion of the double products into simple products; the conversion has been used because it may seem simpler to those familiar with matrices (products with a single opening only).

**9. Dual systems.**[14] Consider next the system of the first order

$$X^{(i)} = \Sigma_j a^{(ij)} X_j, \tag{15}$$

formed of a system $X$ of the first order and the contravariant systems of the second order $a^{(ij)}$. We shall prove that this system $X^{(i)}$ is contravariant of the first order,— which will justify the use of upper indices. Carry out the transformation above; then, by (10) and (5),

$$(X^{(i)}) = \Sigma_j a^{(ij)}(X_j) = \Sigma_j \Sigma_{kl} \gamma_{ki} \gamma_{lj} a^{(kl)} \Sigma_p c_{pj} X_p.$$

The sum taken over $j$ requires $l = p$. Hence by (12),

$$(X^{(i)}) = \Sigma_{kl} \gamma_{ki} a^{(kl)} X_l = \Sigma_k \gamma_{ki} X^{(k)},$$

and the theorem is proved. We thus have, associated with every covariant system, a contravariant system relative to the quadratic form (8).

If we proceed in a similar manner for systems of the second order, we may construct,

$$X^{(ij)} = \Sigma_{kl} a^{(ik)} a^{(jl)} X_{kl}. \tag{15'}$$

This likewise is seen to be a contravariant system. In general if we have a covariant system of order $m$, we may define a contravariant system of equal order by the equation

$$X^{(i_1 i_2 \ldots l_m)} = \Sigma_{j_1 j_2 \ldots j_m} a^{(i_1 j_1)} a^{(i_2 j_2)} \ldots a^{(i_m j_m)} X_{j_1 j_1 \ldots j_m}. \tag{15''}$$

Moreover this relation is reciprocal; for we may pass back to the original system by the formula,

$$X_{k_1 k_2 \ldots k_m} = \Sigma_{l_1 l_2 \ldots l_m} a_{l_1 k_1} a_{l_2 k_2} \cdots a_{l_m k_m} X^{(l_1 l_2 \cdots l_m)}. \tag{16}$$

To prove this we have merely to substitute from (15″) in (16), taking $j = k$, $i = l$, and use the fact that $\Sigma_j a^{(ij)} a_{lj} X_l = X_i$. Thus to every covariant system of order $m$ corresponds a contravariant system of like order and conversely to every contravariant system corresponds a covariant system of like order, with the systems occurring in dual pairs. In particular the systems $a_{ij}$ and $a^{(ij)}$ are dual since,

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[14] Ricci uses the term *reciprocal* systems in place of *dual* systems and there are advantages in this use; but we have preferred to reserve the term reciprocal for *sets of systems*, thereby following the notation of Gibbs in his vector analysis. The term dual suggests itself strongly in connection with a quadratic form.

$$a^{(ij)} = \Sigma_{kl} a^{(ik)} a^{(jl)} a_{kl}, \qquad a_{ij} = \Sigma_{kl} a_{ki} a_{lj} a^{(kl)}.$$

The results of this section may again be put in matrical form and gain in brevity. We set $\mathbf{X}^\circ = \mathbf{X}\cdot\mathbf{A}^{-1}$ or $\mathbf{X}^\circ = \mathbf{A}^{-1}\cdot\mathbf{X}$, it matters not which, since $\mathbf{A}$ is self conjugate. Then,

$$\mathbf{X}^\circ = \mathbf{X}\cdot\mathbf{A}^{-1} = [(\mathbf{X})\cdot\mathbf{M}^{-1}]\cdot[\mathbf{MM}:(\mathbf{A}^{-1})] = (\mathbf{X})\cdot(\mathbf{A}^{-1})\cdot\mathbf{M}_C = \mathbf{M}\cdot(\mathbf{X})\cdot(\mathbf{A}^{-1}),$$

which shows that $\mathbf{X}^\circ$ transforms contravariantly. The terms $X^{(ij)}$ may be treated as a symbolic product $\mathbf{X}^\circ\mathbf{Y}^\circ$ and the result is that $\mathbf{XY}:\mathbf{A}^{-1}\mathbf{A}^{-1}$ is contravariant, etc. The dual is obtained by writing $\mathbf{X} = \mathbf{X}^\circ\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{X}^\circ$.

10. **Composition of systems.** If we have any two systems $X$, $Y$, of order $m$, one covariant, the other contravariant, we may form

$$I = \Sigma_{i_1 i_2 \cdots i_m} X_{i_1 i_2 \cdots i_m} Y^{(i_1 i_2 \cdots i_m)}. \tag{17}$$

This system $I$ contains only one element and is invariant. For

$$\Sigma_{i_1 i_2 \cdots i_m} X_{i_1 i_2 \cdots i_m} Y^{(i_1 i_2 \cdots i_m)} = \Sigma_{i_1 i_2 \cdots i_m} \Sigma_{j_1 j_2 \cdots j_m} \gamma_{i_1 j_1} \gamma_{i_2 j_2} \cdots \gamma_{i_m j_m} (X_{j_1 j_2 \cdots j_m}) \Sigma_{k_1 k_2 \cdots k_n} c_{i_1 k_1} c_{i_2 k_2} \cdots c_{i_m k_m} (X^{(k_1 k_2 \cdots k_m)}),$$

and this reduces to $\Sigma_{j_1 j_2 \cdots j_m}(X_{j_1 j_2 \cdots j_m})\,(Y^{(j_1 j_2 \cdots j_m)})$; because when summed on $i$ the right hand member gives something only when $k = j$, and then gives 1.

In like manner if we have a system $X$ of order $m + p$ and a contravariant system $Y$ of order $m$, we may get a system

$$Z_{i_1 i_2 \cdots i_p} = \Sigma_{j_1 j_2 \cdots j_m} X_{i_1 i_2 \cdots i_p j_1 j_2 \cdots j_m} Y^{(j_1 j_2 \cdots j_m)}, \tag{18}$$

of order $p$, which is covariant. By a similar definition,

$$Z^{(i_1 i_2 \cdots i_p)} = \Sigma_{j_1 j_2 \cdots j_m} X^{(i_1 i_2 \cdots i_p j_1 j_2 \cdots j_m)} Y_{j_1 j_2 \cdots j_m}. \tag{19}$$

We may combine a contravariant system of order $m + p$ with a covariant system of order $m$ to get a contravariant system of order $p$. The proofs of these facts are as above for the special case when the orders are equal.

The process by which covariant and contravariant systems are combined in (17), (18), (19) to obtain a system of order equal to the difference of the two orders is called *composition*.[15] (In the definition we have placed the common indices at the end. We may generalize the definition by distributing the indices in any way. Thus (15) and (16) may be considered as cases of composition of a system of order $2m$ with one of order $m$.)

Composition is very simple in matrical notation.

$$\mathbf{X}\cdot\mathbf{Y}^\circ = \mathbf{X}\cdot\mathbf{A}^{-1}\cdot\mathbf{Y} = \mathbf{X}\underset{\cdot}{\mathbf{Y}}:\mathbf{A}^{-1}$$

is clearly invariant. If proof were needed, we could write

$$\mathbf{X}\mathbf{Y}:\mathbf{A}^{-1} = [(\mathbf{X}\mathbf{Y}):\mathbf{M}^{-1}\mathbf{M}^{-1}]:[\mathbf{M}\mathbf{M}:(\mathbf{A}^{-1})] = (\mathbf{X}\mathbf{Y}):(\mathbf{A}^{-1}).$$

We have simply to take into account what elements the dots actually unite in the multiplications.

**11. Mutually reciprocal[16] n-tuples.** For any covariant system $\lambda_r$, consisting of $n$ functions of the variables $x_1, x_2, \ldots x_n$, and the dual system $\lambda^{(r)}$ we have found by (15), (16) the relations

$$\lambda^{(r)} = \Sigma_s a^{(rs)}\lambda_s, \qquad \lambda_r = \Sigma_s a_{rs}\lambda^{(s)}. \tag{20}$$

Suppose that we have $n$ systems ${}_1\lambda_r, {}_2\lambda_r, \ldots, {}_n\lambda_r$ and the corresponding dual systems ${}_1\lambda^{(r)}, {}_2\lambda^{(r)}, \ldots, {}_n\lambda^{(r)}$. The $n$ systems ${}_i\lambda_r$ will be called *independent* if the determinant $| {}_i\lambda_r |$ does not vanish. As $| a_{rs} | \neq 0$, it follows at once that the dual systems are also independent. We may define contravariant systems ${}_i\lambda'^{(s)}$ in terms of ${}_i\lambda_r$ by the equations

$$\Sigma_i \cdot {}_i\lambda'^{(s)} \cdot {}_i\lambda_r = \epsilon_{rs}, \qquad \epsilon_{rs} = \begin{cases} 0, r \neq s, \\ 1, r = s. \end{cases} \tag{21}$$

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[15] Composition is a sort of inverse of multiplication in that the result of composition is to subtract the orders of the factors, whereas multiplication adds the orders. Composition itself may be regarded as a species of multiplication in the general sense in which Gibbs used the term, and has close analogies with regressive multiplication or with the inner product as defined by G. N. Lewis, *Proc. Amer. Acad. Arts Sci.*, **46**, 165–181 (1910), also (with Wilson) *Ibid.*, **48**, 389–507 (1912), especially § 29.

[16] See note 14.

For if $s$ be held fixed and $r$ take the values 1, 2,......, $n$, there are $n$ equations (21) which are linear and non-homogeneous in the $n$ variables ${}_i\lambda'^{(s)}$, and the equations are consistent because the determinant $|\,{}_i\lambda_r\,|$ of the coefficients does not vanish. If we replace ${}_i\lambda'^{(s)}$ and ${}_i\lambda_r$ by their values from (20) we have

$$\Sigma_i\Sigma_t a^{(st)}.{}_i\lambda'_t\Sigma_u a_{ru}.{}_i\lambda^{(u)} = \epsilon_{rs}.$$

Then

$$\Sigma_{pq}a^{(rq)}a_{sp}\Sigma_{itu}a^{(st)}.{}_i\lambda'_t a_{ru}.{}_i\lambda^{(u)} = \Sigma_{pq}a^{(rq)}a_{sp}\epsilon_{rs}.$$

Now by the reduction formula (13) the left hand side may twice be simplified. On the right hand side $\epsilon_{rs}$ vanishes unless $r = s$ and the double sum reduces to $\epsilon_{pq}$. Hence

$$\Sigma_i.{}_i\lambda'_p.{}_i\lambda^{(q)} = \epsilon_{pq}. \tag{21'}$$

We see therefore that there is a reciprocal relation (21′) to (21) between the ${}_i\lambda_r$, ${}_i\lambda'_r$, ${}_i\lambda^{(r)}$, ${}_i\lambda'^{(r)}$.

The $n$ systems ${}_i\lambda_r$ may be called a covariant $n$-tuple; the systems ${}_i\lambda^{(r)}$ the contravariant $n$-tuple; these are mutually dual in pairs. The set of $n$ systems ${}_i\lambda'^{(r)}$ will be called *reciprocal* to the set ${}_i\lambda^{(r)}$ and the set ${}_i\lambda'_r$ reciprocal to the set ${}_i\lambda_r$. We may give a geometric analogy in support of this nomenclature. If we have a conic and three points $P$, $Q$, $R$, we may obtain the duals, the lines $p$, $q$, $r$. The points, however, determine three lines $QR$, $RP$, $PQ$ and of these the duals are the points $qr$, $rp$, $pq$. The sets $P$, $Q$, $R$ and $qr$, $rp$, $pq$ are reciprocal; and similarly $p$, $q$, $r$ and $QR$, $RP$, $PQ$. Another analogy would arise in spherical geometry where $ABC$ and $A'B'C'$ are polar triangles; the sets $A$, $B$, $C$ and $A'$, $B'$, $C'$, being reciprocal. The use of reciprocal systems in vector analysis is prominent in the system of Gibbs, particularly for the solution of equations. If the $n$ sets ${}_i\lambda_r$ form an *orthogonal* $n$-tuple, the reciprocal sets will be proportional to them — a *unit* orthogonal $n$-tuple is self-reciprocal (see infra, §13).

We may obtain in addition to the defining relations, the following between reciprocal $n$-tuples.

$$\Sigma_i.{}_i\lambda'_s.{}_i\lambda_t = a_{st}, \quad \Sigma_i.{}_i\lambda'^{(s)}.{}_i\lambda^{(t)} = a^{(st)}. \tag{22}$$

These are proved in the usual fashion. If we compare the relations (21) which define the elements ${}_i\lambda'^{(s)}$ in terms of the elements ${}_i\lambda_r$ with the relations $\Sigma_j c_{ir}\gamma_{is} = \epsilon_{rs}$ between the elements $c_{ir}$ of any non-vanishing determinant $|c_{ir}|$ and the elements $\gamma_{is}$ obtained by dividing the cofactor $C_{is}$ by the determinant, we see at once that *the elements* ${}_i\lambda'^{(s)}$ *are the cofactors of* ${}_i\lambda_s$ *divided by* $|{}_i\lambda_r|$, — and similarly from (21′) the elements ${}_i\lambda'_p$ are the cofactors of ${}_i\lambda^{(p)}$ divided by $|{}_i\lambda^{(p)}|$. These relations are also reciprocal, i. e., the elements ${}_i\lambda^{(p)}$ and ${}_i\lambda_r$ are respectively the cofactors of ${}_i\lambda'_p$ and ${}_i\lambda'^{(r)}$ divided by the determinants $|{}_i\lambda'_p|$ and $|{}_i\lambda'^{(r)}|$. Hence by summing the other way, namely upon the index $r$ we may get the relations

$$\Sigma_r \cdot {}_i\lambda_r \cdot {}_j\lambda'^{(r)} = \epsilon_{ij}, \quad \Sigma_r \cdot {}_i\lambda'_r \cdot {}_j\lambda^{(r)} = \epsilon_{ij}. \tag{22′}$$

**12. A standard form for systems.** If we have a contravariant $n$-tuple ${}_i\lambda^{(r)}$ and any covariant system $X_r$ we may form by composition the $n$ invariants

$$c_i = \Sigma_r X_r \cdot {}_i\lambda^{(r)}.$$

These equations may be solved with the aid of the reciprocal $n$-tuple. For, by (21′),

$$\Sigma_i c_i \cdot {}_i\lambda'_s = \Sigma_{ri} X_r \cdot {}_i\lambda'_s \cdot {}_i\lambda^{(r)} = \Sigma_r X_r \epsilon_{rs} = X_s.$$

Hence

$$X_s = \Sigma_i c_i \cdot {}_i\lambda'_s. \tag{23}$$

Any system $X_s$ is therefore representable as a linear function of ${}_i\lambda'_s$ with invariant coefficients. In like manner

$$X^{(s)} = \Sigma_i c_i \cdot {}_i\lambda'^{(s)}, \quad c_i = \Sigma_r X^{(r)}{}_i\lambda_r. \tag{23′}$$

In general for systems of any order we may write

$$\begin{aligned} X_{r_1 r_2 \cdots r_k} &= \Sigma_{i_1 i_2 \cdots i_k} c_{i_1 i_2 \cdots i_k} \cdot {}_{i_1}\lambda'_{r_1} \cdot {}_{i_2}\lambda'_{r_2} \cdots {}_{i_k}\lambda'_{r_k}, \\ c_{i_1 i_2 \cdots i_k} &= \Sigma_{r_1 r_2 \cdots r_k} X_{r_1 r_2 \cdots r_k} \cdot {}_{i_1}\lambda^{(r_1)} \cdot {}_{i_2}\lambda^{(r_2)} \cdots {}_{i_k}\lambda^{(r_k)}, \end{aligned} \tag{23″}$$

and

$$\begin{aligned} X^{(r_1 r_2 \cdots r_k)} &= \Sigma_{i_1 i_2 \cdots i_k} c_{i_1 i_2 \cdots i_k} \cdot {}_{i_2}\lambda'^{(r_1)} \cdot {}_{i_2}\lambda'^{(r_2)} \cdots {}_{i_k}\lambda'^{(r_k)}, \\ c_{i_1 i_2 \cdots i_k} &= \Sigma_{r_1 r_2 \cdots r_k} X^{(r_1 r_2 \cdots r_k)} \cdot {}_{i_1}\lambda_{r_2} \cdot {}_{i_2}\lambda_{r_2} \cdots {}_{i_k}\lambda_{r_k}. \end{aligned} \tag{23‴}$$

Any system of order $m$ is linearly dependent, with invariant coefficients, on the product system of the $m$th order made up of the λ″s.

As the λ's thus form *a basis* for the expression of systems in general we may set up readily the progressive product of Grassmann; for

$$\begin{vmatrix} X_r & X_s \\ Y_r & Y_s \end{vmatrix} = \Sigma_{ij} c_i c_j \begin{vmatrix} {}_i\lambda'_r & {}_j\lambda'_r \\ {}_i\lambda'_s & {}_j\lambda'_s \end{vmatrix}, \text{ etc.}$$

In the system of Grassmann the progressive product represents the space determined by the elements (a parallelogram in the case of two vectors); but the interpretation here is not so direct because the systems $X_r$ of the first order are not components of a vector,— they have to be multiplied by certain factors to obtain components of a vector.[17] In like manner the terms $X_r Y_s - X_s Y_r$ are not components of a plane but may be converted into such by proper factors.

**13. Orthogonal unit n-tuples.** We may define orthogonality relative to a given quadratic form as in non-euclidean geometry. We shall now however take the form as differential, namely, as

$$ds^2 = \Sigma_{rs} a_{rs} dx_r dx_s .$$

Since the elements $dx_r$ form a contravariant system (§4) a direction in space may be defined by any contravariant system $\lambda^{(r)}$ if we set up the simultaneous differential equations

$$\frac{dx_1}{\lambda^{(1)}} = \frac{dx_2}{\lambda^{(2)}} = \frac{dx_3}{\lambda^{(3)}} = = \ldots\ldots = \frac{dx_n}{\lambda^{(n)}} ; \tag{24}$$

and it is in this way that the contravariant systems used above, and previously defined as contravariant systems, are associated with special directions.

If we have two systems ${}_i\lambda^{(r)}$, ${}_j\lambda^{(s)}$ we define as is customary in differential or non-euclidean geometry, the angle between the directions by the formula

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[17] It is shown by Ricci and Levi-Civita (*Math. Ann.*, **60**) that if two dual systems of the first order $X_r$, $X^{(r)}$ are divided by $\sqrt{a_{rr}}$ and $\sqrt{a^{(rr)}}$, the resulting expressions $X_r/\sqrt{a_{rr}}$, $X^{(r)}/\sqrt{a^{(rr)}}$ may be regarded respectively as the orthogonal projections of one and the same vector upon the tangents to the coördinate lines $x_r$ and upon the normals to the coördinate surfaces; whereas the expressions $X^{(r)}\sqrt{a_{rr}}$ and $X_r\sqrt{a^{(rr)}}$ represent respectively the components of the same vector along the same lines and the same normals. This process of rendering a system vectorial might be called vectorization and could be extended to vectors of higher order (Stufe).

$$\cos\theta = \frac{\Sigma_{rs}a_{rs}\cdot{}_i\lambda^{(r)}\cdot{}_j\lambda^{(s)}}{\sqrt{\Sigma_{rs}a_{rs}\cdot{}_i\lambda^{(r)}\cdot{}_i\lambda^{(s)}}\ \sqrt{\Sigma_{rs}a_{rs}\cdot{}_j\lambda^{(r)}\cdot{}_j\lambda^{(s)}}}, \tag{25}$$

where the $\lambda$'s are proportional to the differentials by (24). The condition of orthogonality for the two directions ${}_i\lambda^{(r)}$, ${}_j\lambda^{(s)}$ is therefore

$$\Sigma_{rs}a_{rs}\cdot{}_i\lambda^{(r)}\cdot{}_j\lambda^{(s)} = 0.$$

This may be written $\Sigma_s\cdot{}_i\lambda_s\cdot{}_j\lambda^{(s)} = 0$, by using the covariant system.[18] Our results may be simplified by considering the systems ${}_i\lambda^{(r)}$, ${}_j\lambda^{(s)}$ in (24) as first multiplied by such a factor that the radicals in (25) reduce to unity, that is, so that $\Sigma_{rs}a_{rs}\cdot{}_i\lambda^{(r)}\cdot{}_i\lambda^{(s)} = 1$. Such a system may be called a *unit* system. The conditions for a unit orthogonal $n$-tuple are therefore,

$$\Sigma_s\cdot{}_i\lambda_s\cdot{}_j\lambda^{(s)} = \epsilon_{ij}. \tag{26}$$

Now if we multiply (26) by ${}_j\lambda'_r$, sum over $j$, and apply (21′) we have ${}_i\lambda_r = {}_i\lambda'_r$. and in like manner we should have ${}_i\lambda^{(r)} = {}_i\lambda'^{(r)}$. Hence for a unit orthogonal $n$-tuple the reciprocal and given sets of systems are identical. This gives from (21) the relation

$$\Sigma_i\cdot{}_i\lambda_r\cdot{}_i\lambda^{(s)} = \epsilon_{rs}, \tag{27}$$

in addition to (26) for unit orthogonal $n$-tuples. The relations (26) and (27) are like those connecting the directions cosines of an orthogonal set in ordinary space. We may get from (22) the relations

$$\Sigma_i\cdot{}_i\lambda_r\cdot{}_i\lambda_s = a_{rs}, \quad \Sigma_i\cdot{}_i\lambda^{(r)}{}_i\lambda^{(s)} = a^{(rs)} \tag{28}$$

**14. Transformations of variables.** Though the forms in which we are interested are differential and the transformations of variable arbitrary,

$$x_1 = x_1(y_1, y_2, \ldots, y_n), \ldots\ldots, \quad x_n = x_n(y_1, y_2, \ldots, y_n),$$

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[18] If we compare this condition of perpendicularity with (22′) we see that the direction ${}_i\lambda'^{(r)}$ is perpendicular to the direction ${}_j\lambda^{(r)}$ for all values of $i$ except $i = j$. If we consider all the directions linearly derived from ${}_i\lambda^{(r)}$, $i \neq j$, we find that they determine the $(n-1)$-space perpendicular to ${}_j\lambda'^{(r)}$.

the transformation of the differentials is linear;— thus

$$dx_i = \Sigma_j \frac{\partial x_i}{\partial y_j} dy_j = \Sigma_j c_{ij} dy_j, \qquad c_{ij} = \frac{\partial x_i}{\partial y_j},$$

with the difference over the algebraic theory that the coefficients $c_{ij}$ are variable. As the work done to this point does not involve derivatives of the $c$'s or in any way depend on their constancy, the whole work remains valid. As the particular relations

$$c_{ij} = \frac{\partial x_i}{\partial y_j}, \qquad \gamma_{ji} = \frac{\partial y_i}{\partial x_j}$$

now hold we may define covariant and contravariant systems of order $k$ as those for which

$$X_{i_1 i_2 \cdots i_k} = \Sigma_{j_1 j_2 \cdots j_k} (X_{j_1 j_2 \cdots j_k}) \frac{\partial y_{j_1}}{\partial x_{i_1}} \frac{\partial y_{j_2}}{\partial x_{i_2}} \cdots \frac{\partial y_{j_k}}{\partial x_{i_k}}, \tag{29}$$

$$(X_{i_1 i_2 \cdots i_k}) = \Sigma_{j_1 j_2 \cdots j_k} X_{j_1 j_2 \cdots j_k} \frac{\partial x_{j_1}}{\partial y_{i_1}} \frac{\partial x_{j_2}}{\partial y_{i_2}} \cdots \frac{\partial x_{j_k}}{\partial y_{i_k}}, \tag{29'}$$

$$X^{(i_1 i_2 \cdots i_k)} = \Sigma_{j_1 j_2 \cdots j_k} (X^{(j_1 j_2 \cdots j_k)}) \frac{\partial x_{i_1}}{\partial y_{j_1}} \frac{\partial x_{i_2}}{\partial y_{j_2}} \cdots \frac{\partial x_{i_k}}{\partial y_{j_k}}, \tag{30}$$

$$(X^{(i_1 i_2 \cdots i_k)}) = \Sigma_{j_1 j_2 \cdots j_k} X^{(j_1 j_2 \cdots j_k)} \frac{\partial y_{i_1}}{\partial x_{j_1}} \frac{\partial y_{i_2}}{\partial x_{j_2}} \cdots \frac{\partial y_{i_k}}{\partial x_{j_k}}. \tag{30'}$$

If we have a function of the variables, the derivatives $f_i = \partial f / \partial x_i$ form a system of the first order. We know that,

$$\frac{\partial f}{\partial x_i} = \Sigma_j \frac{\partial f}{\partial y_j} \frac{\partial y_j}{\partial x_i} = \Sigma_j \left(\frac{\partial f}{\partial x_j}\right) \frac{\partial y_j}{\partial x_i},$$

since $\partial f / \partial y_j = (\partial f / \partial x_j)$ by definition. Hence we see that the first derivatives of any function (system of order 0) form a covariant system of order 1.

If we have a general covariant system $X_i$ of the first order, the derivatives of the elements of the system with respect to the variables,

$X_{ij} = \partial X_j/\partial x_j$, would form a system of the second order. Let us consider the transformation of this system.

$$\begin{aligned} X_{ij} = \frac{\partial X_i}{\partial x_j} = \frac{\partial}{\partial x_j}\Sigma_k \gamma_{ik}(X_k) &= \Sigma_k \gamma_{ik} \frac{\partial(X_k)}{\partial x_j} + \Sigma_k (X_k)\frac{\partial \gamma_{ik}}{\partial x_j} \\ &= \Sigma_k \gamma_{ik} \Sigma_l \frac{\partial(X_k)}{\partial y_l}\frac{\partial y_l}{\partial x_j} + \Sigma_k (X_k)\frac{\partial \gamma_{ik}}{\partial x_j} \\ &= \Sigma \gamma_{ik}\gamma_{jl}(X_{kl}) + \Sigma_k (X_k)\frac{\partial^2 y_k}{\partial x_i \partial x_j}. \end{aligned} \tag{31}$$

If it were not for the second term, the transformation would be covariant, but the presence of this term shows that the derivatives of a covariant system of the first order do not form a covariant system of the second order.

The same is true for covariant systems of any order,— their derivatives do not form a covariant system. For instance in the case of a covariant system of the second order $X_{rs}$, by a similar transformation,

$$\frac{\partial X_{rs}}{\partial x_t} = \Sigma_{ijk}\gamma_{ri}\gamma_{sj}\gamma_{tk}\frac{\partial(X_{ij})}{\partial y_k} + \Sigma_{ij}(X_{ij})\left[\frac{\partial^2 y_i}{\partial x_r \partial x_t}\frac{\partial y_j}{\partial x_s} + \frac{\partial^2 y_j}{\partial x_s \partial x_t}\frac{\partial y_i}{\partial x_r}\right], \tag{31'}$$

where $\partial y_i/\partial x_s = \gamma_{si}$ and $\partial^2 y_i/\partial x_r \partial x_t = \partial \gamma_{ri}/\partial x_t$.

The fundamental relation $dx_i = \Sigma_i c_{ij} dy_j$ may be written in matrical notation as $d\mathbf{x} = d\mathbf{y}\cdot\nabla_y \mathbf{x}$. It follows that $\mathbf{M}_C = \nabla_y \mathbf{x}$. We may also write $d\mathbf{y} = d\mathbf{x}\cdot\nabla_x \mathbf{y}$. Hence $\nabla_x \mathbf{y}$ and $\nabla_y \mathbf{x}$ are reciprocals. The relations (29) and (30) may be written as

$$\begin{aligned} \mathbf{X} &= \nabla_x \mathbf{y}\cdot(\mathbf{X}), & \mathbf{XY} &= \nabla_x \mathbf{y}\nabla_x \mathbf{y}:(\mathbf{XY}), \\ (\mathbf{X}) &= \nabla_y \mathbf{x}\cdot\mathbf{X}, & (\mathbf{XY}) &= \nabla_y \mathbf{x}\nabla_y \mathbf{x}:\mathbf{XY}, \\ \mathbf{X}^\circ &= (\mathbf{X}^\circ)\cdot\nabla_y \mathbf{x}, & \mathbf{X}^\circ\mathbf{Y}^\circ &= (\mathbf{X}^\circ\mathbf{Y}^\circ):\nabla_y \mathbf{x}\nabla_y \mathbf{x}, \\ (\mathbf{X}^\circ) &= \mathbf{X}^\circ\cdot\nabla_x \mathbf{y}, & (\mathbf{X}^\circ\mathbf{Y}^\circ) &= \mathbf{X}^\circ\mathbf{Y}^\circ:\nabla_x \mathbf{y}\nabla_x \mathbf{y}, \end{aligned}$$

and so for systems of any order.

The differentiation of a system of the zeroth order $f$ is accomplished as:

$$df = d(f), \qquad d\mathbf{x}\cdot\nabla_x f = d\mathbf{y}\cdot\nabla_y(f),$$

$$d\mathbf{x}\cdot\nabla_x f = d\mathbf{x}\cdot\nabla_x \mathbf{y}\cdot\nabla_y(f), \qquad \nabla f = \nabla_x \mathbf{y}\cdot(\nabla f).$$

This shows that $\nabla f$ is covariant of order 1. To differentiate a system $\mathbf{X}$ of order 1 we have

$$\begin{aligned} d\mathbf{X} &= \nabla_x\mathbf{y}\cdot d(\mathbf{X}) + d\nabla_x\mathbf{y}\cdot(\mathbf{X}) \\ d\mathbf{x}\cdot\nabla\mathbf{X} &= \nabla_x\mathbf{y}\cdot[d\mathbf{x}\cdot\nabla_x\mathbf{y}\cdot\nabla_y(\mathbf{X})] + d\mathbf{x}\cdot\nabla_x\nabla_x\mathbf{y}\cdot(\mathbf{X}) \\ \nabla\mathbf{X} &= \nabla_x\mathbf{y}\nabla_x\mathbf{y}:(\nabla\mathbf{X}) + \nabla_x\nabla_x\mathbf{y}\cdot(\mathbf{X}). \end{aligned}$$

**15. Solution for the second derivatives.**[19] As we are working with a fundamental adjoined quadratic form $\Sigma a_{rs}dx_r dx_s$, we regard the $a_{rs}$ and their derivatives as known. We may then write

$$\frac{\partial a_{rs}}{\partial x_t} = \Sigma_{ijk}\gamma_{ri}\gamma_{sj}\gamma_{tk}\frac{\partial(a_{ij})}{\partial y_k} + \Sigma_{ij}(a_{ij})\left[\frac{\partial^2 y_i}{\partial x_r\partial x_t}\gamma_{sj} + \frac{\partial^2 y_j}{\partial x_s\partial x_t}\gamma_{ri}\right]$$

and solve the six equations obtained by permuting $r$, $s$, $t$ for the six derivatives $\partial^2 y_i/\partial x_r\partial x_t$ as unknowns. We have

$$\begin{aligned} \frac{\partial a_{rt}}{\partial x_s} &= \Sigma_{ij}\,\gamma_{ri}\gamma_{tj}\gamma_{sk}\frac{\partial(a_{ij})}{\partial y_k} + \Sigma_{ij}(a_{ij})\left[\frac{\partial^2 y_i}{\partial x_r\partial x_s}\gamma_{tj} + \frac{\partial^2 y_j}{\partial x_s\partial x_t}\gamma_{ri}\right], \\ \frac{\partial a_{ts}}{\partial x_r} &= \Sigma_{ijk}\,\gamma_{ti}\gamma_{sj}\gamma_{rk}\frac{\partial(a_{ij})}{\partial y_k} + \Sigma_{ij}\,(a_{ij})\left[\frac{\partial^2 y_i}{\partial x_t\partial x_r}\gamma_{sj} + \frac{\partial^2 y_j}{\partial x_s\partial x_r}\gamma_{ti}\right]. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial a_{rt}}{\partial x_s} + \frac{\partial a_{ts}}{\partial x_r} - \frac{\partial a_{rs}}{\partial x_t} = \Sigma_{ijk}&\left[\gamma_{ri}\gamma_{tj}\gamma_{sk} + \gamma_{ti}\gamma_{sj}\gamma_{rk} - \gamma_{ri}\gamma_{sj}\gamma_{tk}\right]\frac{\partial(a_{ij})}{\partial y_k} \\ &+ \Sigma_{ij}(a_{ij})\left[\frac{\partial^2 y_i}{\partial x_r\partial x_s}\gamma_{tj} + \frac{\partial^2 y_j}{\partial x_s\partial x_r}\gamma_{ti}\right]. \end{aligned}$$

But as $(a_{ij}) = (a_{ji})$ we have,

$$\Sigma_{ij}(a_{ij})\,\frac{\partial^2 y_i}{\partial x_s\partial x_r}\gamma_{tj} = \Sigma_{ij}(a_{ij})\,\frac{\partial^2 y_j}{\partial x_s\partial x_r}\gamma_{ti}$$

---

[19] The solution for the second derivatives, though cumbersome, is exceedingly important for it is through this substitution that the Christoffel symbols actually arise (see Christoffel, *Gesammelte Werke*, or *Crelle J. Math.*, **70**, 46.) The method followed in so many books, viz., to write down the Christoffel symbols without any preliminaries seems decidedly artificial. We may point out that *when the analysis is carried on in matrical notation, as below, the elimination suggests itself much more readily than when we have so many subscripts and summation signs to manipulate as in the ordinary derivation.*

and the last bracket becomes a single term repeated. Moreover the first bracket may be changed by interchanging the indices $i, j, k$. For,

$$\Sigma_{ijk}\gamma_{ri}\gamma_{tj}\gamma_{sk}\frac{\partial(a_{ij})}{\partial y_k} = \Sigma_{ijk}\gamma_{ri}\gamma_{sj}\gamma_{tk}\frac{\partial(a_{ik})}{\partial y_j}$$

since in either case the summation is over all values of $j$ and $k$. Hence,

$$\frac{1}{2}\left[\frac{\partial a_{rt}}{\partial x_s} + \frac{\partial a_{ts}}{\partial x_r} - \frac{\partial a_{rs}}{\partial x_t}\right] = \tfrac{1}{2}\,\Sigma_{ijk}\gamma_{ri}\gamma_{sj}\gamma_{tk}\left[\frac{\partial(a_{ik})}{\partial y_j} + \frac{\partial(a_{kj})}{\partial y_i} - \frac{\partial(a_{ij})}{\partial y_k}\right]$$
$$+ \Sigma_{ij}(a_{ij})\frac{\partial^2 y_i}{\partial x_s \partial x_r}\gamma_{tj}.$$

This somewhat cumbrous form may be simplified by introducing the notation of the Christoffel symbols,

$$\begin{bmatrix} r\ s \\ t \end{bmatrix} = \frac{1}{2}\left[\frac{\partial a_{rt}}{\partial x_s} + \frac{\partial a_{ts}}{\partial x_r} - \frac{\partial a_{rs}}{\partial x_t}\right]. \tag{32}$$

The above expression then becomes

$$\begin{bmatrix} s\ r \\ t \end{bmatrix} = \Sigma_{ijk}\gamma_{ri}\gamma_{sj}\gamma_{tk}\left(\begin{bmatrix} i\ j \\ k \end{bmatrix}\right) + \Sigma_{ij}(a_{ij})\frac{\partial^2 y_i}{\partial x_s \partial x_r}\gamma_{tj}.$$

To complete the solution for the second derivatives, multiply by $c_{tl}$ and sum over $t$. Then

$$\Sigma_t c_{tl}\begin{bmatrix} r\ s \\ t \end{bmatrix} = \Sigma_{ij}\gamma_{ri}\gamma_{sj}\left(\begin{bmatrix} i\ j \\ l \end{bmatrix}\right) + \Sigma_i(a_{il})\ \frac{\partial^2 y_i}{\partial x_s \partial x_r}.$$

Next multiply by $(a^{(ml)})$ and sum over $l$. Then

$$\Sigma_{tl}c_{tl}(a^{(ml)})\begin{bmatrix} r\ s \\ t \end{bmatrix} = \Sigma_{ijl}\gamma_{ri}\gamma_{sj}(a^{(ml)})\left[\begin{pmatrix} i\ j \\ l \end{pmatrix}\right] + \frac{\partial^2 y_m}{\partial x_r \partial x_s},$$

and hence finally we have the expression

$$\frac{\partial^2 y_m}{\partial x_r \partial x_s} = \Sigma_{tl}c_{tl}(a^{(ml)})\begin{bmatrix} r\ s \\ t \end{bmatrix} - \Sigma_{ijl}\gamma_{ri}\gamma_{sj}(a^{(ml)})\left(\begin{bmatrix} i\ j \\ l \end{bmatrix}\right). \tag{33}$$

To differentiate the matrix **A** of the coefficients $a_{ij}$ and obtain an expression for the second derivative we proceed as follows.[20] As **A** is self conjugate we may write **A** symbolically for the analytic work as **XX**. Then as

$$\mathbf{A} = \nabla\mathbf{y}\nabla\mathbf{y}:(\mathbf{A}), \quad \mathbf{XX} = \nabla\mathbf{y}\nabla\mathbf{y}:(\mathbf{XX}).$$

If we use subscripts to indicate the variables to which the differentiations apply, we have

$$\mathbf{XX} = \nabla_1\nabla_2\mathbf{y}_1\cdot(\mathbf{X})\mathbf{y}_2\cdot(\mathbf{X}).$$

The symbols $\mathbf{y}_1\cdot(\mathbf{X})$ and $\mathbf{y}_2\cdot(\mathbf{X})$ are scalar, $\nabla_1$ and $\nabla_2$ extensive magnitudes. Now

$$\nabla\mathbf{XX} = [\nabla_1\nabla_1\nabla_2 + \nabla_2\nabla_1\nabla_2 + \nabla\,\nabla_1\nabla_2]\,\mathbf{y}_1\cdot(\mathbf{X})\mathbf{y}_2\cdot(\mathbf{X}),$$

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[20] Knoblauch in the preface of his *Grundlagen der Differential Geometrie* (1913) lays stress on the necessity of some operation such as his geometric differentiation to illuminate the formulas of differential geometry and while acknowledging the importance of Ricci's work, especially the *Lezioni*, complains that instead of using geometric derivatives he for the most part uses their "Coeffizienten System." A part of this difficulty is obviated by the use of the notations of multiple algebra as here employed by us and more of it by the large use of vectors that we make later in the work. By the combination of these two elements the analysis can be kept measurably simple and interpretable.

When discussing methods in differential geometry we must not omit that of Maschke; of which an account may be found in the following articles: Maschke, *Trans. Amer. Math. Soc.*, **1**, 197–204, *Ibid.*, **4**, 445–469, *Ibid.*, **7**, 69–80, 81–93; A. W. Smith, *Ibid.*, **7**, 33–60; Ingold, *Ibid.*, **11**, 449–474. For the actual use of the method in the theory of surfaces Smith's article is by far the most important of these references. One may say somewhat epigrammatically that Maschke's method contrasts with Ricci's in much the same way that the Clebsch-Aronhold method contrasts with Grassmann's. The fundamental element in Maschke's work is a *symbolic* treatment of the quadratic differential form. The reason that we have not used this method is because we have a natural preference for the non-symbolic method which is not overborne, for the simple work that we have in hand, by the gain in simplicity of operation of the symbolic method. In particular in regard to the present question of the solution for the second derivatives and the introduction of the Christoffel symbols we may observe that for Maschke's interpretation of $f_if_{kl}$ as a Christoffel symbol it is necessary to assume that the symbols $f_{kl}$ and $f_{lk}$ are equal. Under this assumption $f_if_{kl}$ appears as a Christoffel symbol and its appearance in this form may be taken as a justification for considering the symbols $f_{lk}$ and $f_{kl}$ as equal (for two of the indices in the Christoffel symbol are commutative). A very natural way to arrive at the Christoffel symbols is by Shaw's method (loc. cit., note 1) in which the symbols all have a geometric meaning; but unfortunately in order to follow this method we have to regard the surface as immersed in a space so that $ds^2 = d\mathbf{r}\cdot d\mathbf{r}$, and for theoretical purposes it is preferable at this stage to remain entirely upon the surface.

where the symbol $\nabla$ applies to the variables $\mathbf{X}$ and $(\mathbf{X})$. Next,

$$\mathbf{X}\nabla\mathbf{X} = [\nabla_1\nabla_1\nabla_2 + \nabla_1\nabla_2\nabla_2 + \nabla_1\nabla\,\nabla_2]\,\mathbf{y}_1\cdot(\mathbf{X})\mathbf{y}_2\cdot(\mathbf{X})$$

may be obtained by interchanging the first and second extensive magnitudes. And

$$\mathbf{X}\mathbf{X}\nabla = [\nabla_1\nabla_2\nabla_1 + \nabla_1\nabla_2\nabla_2 + \nabla_1\nabla_2\nabla]\,\mathbf{y}_1\cdot(\mathbf{X})\mathbf{y}_2\cdot(\mathbf{X})$$

follows from another interchange. Now

$$\begin{aligned}\nabla\mathbf{X}\mathbf{X} + \mathbf{X}\nabla\mathbf{X} - \mathbf{X}\mathbf{X}\nabla = {} & 2\nabla_1\nabla_1\nabla_2\mathbf{y}_1\cdot(\mathbf{X})\mathbf{y}_2\cdot(\mathbf{X}) \\ & + [\nabla\,\nabla_1\nabla_2 + \nabla_1\nabla\,\nabla_2 - \nabla_1\nabla_2\nabla]\mathbf{y}_1\cdot(\mathbf{X})\mathbf{y}_2\cdot(\mathbf{X}),\end{aligned}$$

because $\nabla_2\nabla_1\nabla_2\mathbf{y}_1\cdot(\mathbf{X})\mathbf{y}_2\cdot(\mathbf{X})$ and $\nabla_1\nabla_2\nabla_1\mathbf{y}_1\cdot(\mathbf{X})\mathbf{y}_2\cdot(\mathbf{X})$ are the same. Thus far $\nabla$ has denoted differentiation by $\mathbf{x}$. But $\nabla_x = \nabla_x\mathbf{y}\cdot\nabla_y = \nabla\mathbf{y}\cdot(\nabla)$. The terms in the bracket on the right may therefore be written

$$\nabla\,\nabla_1\nabla_2 = \nabla\mathbf{y}\cdot(\nabla)\nabla_1\nabla_2\mathbf{y}_1\cdot(\mathbf{X})\mathbf{y}_2\cdot(\mathbf{X}) = \nabla\mathbf{y}\nabla\mathbf{y}\nabla\mathbf{y}\,\vdots\,(\nabla\mathbf{X}\mathbf{X}),$$

and so on; hence

$$\begin{aligned}\nabla\mathbf{X}\mathbf{X} + \mathbf{X}\nabla\mathbf{X} - \mathbf{X}\mathbf{X}\nabla = 2\nabla\nabla\mathbf{y}\cdot(\mathbf{X}\mathbf{X})\cdot\nabla\mathbf{y}_C + \nabla\mathbf{y}\nabla\mathbf{y}: [(\nabla\mathbf{X}\mathbf{X} + \mathbf{X}\nabla\mathbf{X} \\ - \mathbf{X}\mathbf{X}\nabla)]\cdot\nabla\mathbf{y}_C\end{aligned}$$

or

$$\begin{aligned}2\nabla\nabla\mathbf{y} = [\nabla\mathbf{X}\mathbf{X} + \mathbf{X}\nabla\mathbf{X} - \mathbf{X}\mathbf{X}\nabla]\cdot\nabla_y\mathbf{x}_C\cdot(\mathbf{A}^{-1}) + \nabla\mathbf{y}\nabla\mathbf{y}: \\ [(\nabla\mathbf{X}\mathbf{X} + \mathbf{X}\nabla\mathbf{X} - \mathbf{X}\mathbf{X}\nabla)]\cdot\mathbf{A}^{-1}).\end{aligned}$$

The elements of this triadic are (compare 33)

$$\begin{aligned}2\frac{\partial^2 y_t}{\partial x_r\partial x_s} = {} & \Sigma_{pq}\left[\frac{\partial a_{rq}}{\partial x_r} + \frac{\partial a_{rq}}{\partial x_s} - \frac{\partial a_{rs}}{\partial x_q}\right] c_{qp}(a^{(pt)}) \\ & + \Sigma_{pqn}\left[\frac{\partial(a_{qn})}{\partial y_p} + \frac{\partial(a_{pn})}{\partial y_q} - \frac{\partial(a_{pq})}{\partial y_n}\right]\gamma_{rp}\gamma_{sq}(a^{(nt)}) \\ = {} & 2\,\Sigma_{pq}\gamma_{pt}a^{(qp)}\begin{bmatrix} r & s \\ & q \end{bmatrix} - 2\,\Sigma_{pqn}\gamma_{rp}\gamma_{sq}(a^{(nt)})\left(\begin{bmatrix} p & q \\ & n \end{bmatrix}\right).\end{aligned}$$

**16. Covariant differentiation of a simple system.** Let us now substitute from (33) for the second derivatives in (31). Then

$$\frac{\partial X_i}{\partial x_j} \underline{= \Sigma_{kl}\gamma_{ik}\gamma_{jl}} \frac{\partial (X_k)}{\partial y_l} + \Sigma_k(X_k)\Sigma_{tl}c_{tl}(a^{(kl)})\begin{bmatrix} i\,j \\ t \end{bmatrix}$$
$$- \Sigma_k(X_k)\Sigma_{rsl}\gamma_{ir}\gamma_{js}(a^{(kl)}) \begin{bmatrix} r\,j \\ l \end{bmatrix}$$

or

$$\frac{\partial X_i}{\partial x_j} \underline{- \Sigma_k(X_k)\Sigma_{tl}c_{tl}(a^{(kl)})} \begin{bmatrix} i\,j \\ t \end{bmatrix} = \Sigma_{kl}\gamma_{ik}\gamma_{jl} \frac{\partial (X_k)}{\partial y_l}$$
$$- \Sigma_t(X_t)\Sigma_{klp}\gamma_{ik}\gamma_{jl}(a^{(tp)}) \left(\begin{bmatrix} k\,l \\ p \end{bmatrix}\right).$$

Now the presence of the multipliers $\gamma_{ik}$, $\gamma_{jl}$ on the right makes it look as though the left might be a covariant of the second order and if we replace $(X_k)$ by its value $\Sigma_m X_m c_{mk}$ and $(a^{(kl)})$ by its value $\Sigma_{pq} a^{(pq)} \gamma_{pk}\gamma_{ql}$, we find that

$$\Sigma_k(X_k)\Sigma_{tl}c_{tl}(a^{(kl)}) \begin{bmatrix} i\,j \\ t \end{bmatrix} = \Sigma_{ktlmpq} X_m c_{mk} c_{tl} \gamma_{pk}\gamma_{ql} a^{(pq)} \begin{bmatrix} i\,j \\ t \end{bmatrix}$$
$$= \Sigma_{pq} X_p a^{(pq)} \begin{bmatrix} i\,j \\ q \end{bmatrix}.$$

Hence we have

$$\frac{\partial X_i}{\partial x_j} - \Sigma_{pq} X_p a^{(pq)} \begin{bmatrix} i\,j \\ q \end{bmatrix} = \sum_{kl}\gamma_{ik}\gamma_{jl} \left\{ \frac{\partial (X_k)}{\partial y_l} - \Sigma_{tp}(X_t)\,(a^{(tp)})\left(\begin{bmatrix} k\,l \\ p \end{bmatrix}\right)\right\}.$$

We therefore write, as the *covariant derivative* of the system $X_i$ of order 1, the covariant system of order 2,

$$X_{ij} = \frac{\partial X_i}{\partial x_j} - \Sigma_{pq} X_p a^{(pq)} \begin{bmatrix} i\,j \\ q \end{bmatrix}. \tag{34}$$

This system may be written a little more simply by introducing the Christoffel symbols of the second kind,

$$\Sigma_q a^{(pq)} \begin{bmatrix} i\,j \\ q \end{bmatrix} = \left\{ \begin{matrix} i\,j \\ p \end{matrix} \right\}. \tag{35}$$

This is a sort of partial dual of the symbol of the first kind. Then

$$X_{ij} = \frac{\partial X_i}{\partial x_j} - \Sigma_p X_p \left\{\begin{matrix} i\ j \\ p \end{matrix}\right\}. \tag{34$'$}$$

The partial derivatives $\partial X_i/\partial x_j$ are expressible in terms of the derived system $X_{ij}$ as

$$\frac{\partial X_i}{\partial x_j} = X_{ij} + \Sigma_p X_p \left\{\begin{matrix} i\ j \\ p \end{matrix}\right\}. \tag{34$''$}$$

If now we take the expression $\nabla\mathbf{X} = \nabla\mathbf{y}\nabla\mathbf{y}\mathbf{:}(\nabla\mathbf{X}) + \nabla\nabla\mathbf{y}\cdot(\mathbf{X})$, and substitute for $\nabla\nabla\mathbf{y}$, we have

$$\nabla\mathbf{X} = \nabla\mathbf{y}\nabla\mathbf{y}\mathbf{:}(\nabla\mathbf{X}) + \tfrac{1}{2}\begin{bmatrix} \mathbf{X}\ \mathbf{X} \\ \nabla \end{bmatrix}(\mathbf{A}^{-1})\cdot\nabla\mathbf{y}\cdot(\mathbf{X}) - \tfrac{1}{2}\nabla\mathbf{y}\nabla\mathbf{y}\mathbf{:}$$
$$\left(\begin{bmatrix} \mathbf{X}\ \mathbf{X} \\ \nabla \end{bmatrix}\right)\cdot(\mathbf{A}^{-1})\cdot(\mathbf{X})$$

or

$$\nabla\mathbf{X} - \tfrac{1}{2}\begin{bmatrix} \mathbf{X}\ \mathbf{X} \\ \nabla \end{bmatrix}\cdot\mathbf{A}^{-1}\cdot\mathbf{X} = \nabla\mathbf{y}\nabla\mathbf{y}\mathbf{:}\left\{(\nabla\mathbf{X}) - \tfrac{1}{2}\left(\begin{bmatrix} \mathbf{X}\ \mathbf{X} \\ \nabla \end{bmatrix}\right)\cdot\right.$$
$$\left.(\mathbf{A}^{-1})\cdot(\mathbf{X})\right\}.$$

If this be expanded we have, as before,

$$\frac{\partial X_i}{\partial x_j} - \Sigma_{pq}\begin{bmatrix} i\ j \\ q \end{bmatrix} a^{(qp)}X_p = \sum{}_{mn}\gamma_{im}\gamma_{in}\left\{\frac{\partial X_n}{\partial x_m} - \Sigma_{pq}\left(\begin{bmatrix} n\ m \\ q \end{bmatrix}\right)\right.$$
$$\left.(a^{(qp)})X_p\right\}.$$

**17. Covariant differentiation of systems of higher order.** To find the covariant derivative of a system of the second order we must substitute from (33) for the second derivatives in (31′) and reduce. There are two terms containing second derivatives. We have

$$\Sigma_{ij}(X_{ij})\frac{\partial^2 y_i}{\partial x_r \partial x_t}\gamma_{sj} = \Sigma_{ij}(X_{ij})\gamma_{sj}\Sigma_{pl}c_{pl}(a^{(il)})\begin{bmatrix} r\ t \\ p \end{bmatrix}$$
$$- \Sigma_{ij}(X_{ij})\gamma_{sj}\Sigma_{pql}\gamma_{rp}\gamma_{tq}(a^{(il)})\left(\begin{bmatrix} p\ q \\ l \end{bmatrix}\right).$$

The second term here is

$$\Sigma_{jpq}\gamma_{sj}\gamma_{rp}\gamma_{tq}\Sigma_{il}(X_{ij})(a^{(il)})\left(\begin{bmatrix} p\ q \\ l \end{bmatrix}\right) = \Sigma_{ijpq}\gamma_{sj}\gamma_{rp}\gamma_{tq}(X_{ij})\left(\begin{Bmatrix} p\ q \\ i \end{Bmatrix}\right).$$

The first term may be written as

$$\Sigma_{ijuvmnpql}X_{uv}c_{ui}c_{vj}c_{pl}\gamma_{sj}a^{(mn)}\gamma_{mi}\gamma_{nl}\begin{bmatrix} r\ t \\ p \end{bmatrix} =$$

$$\Sigma_{mn}X_{ms}a^{(mn)}\begin{bmatrix} r\ t \\ n \end{bmatrix} = \Sigma_m X_{ms}\begin{Bmatrix} r\ t \\ m \end{Bmatrix}.$$

Hence

$$\Sigma_{ij}(X_{ij})\frac{\partial^2 y_i}{\partial x_r \partial x_t}\gamma_{sj} = \Sigma_m\left[X_{ms}\begin{Bmatrix} r\ t \\ m \end{Bmatrix} - \Sigma_{jpq}\gamma_{sj}\gamma_{rp}\gamma_{tq}(X_{mj})\left(\begin{Bmatrix} p\ q \\ m \end{Bmatrix}\right)\right],$$

and in like manner,

$$\Sigma_{ij}(X_{ij})\frac{\partial^2 y_j}{\partial x_s \partial x_t}\gamma_{ri} = \Sigma_m\left[X_{rm}\begin{Bmatrix} s\ t \\ m \end{Bmatrix} - \Sigma_{jpq}\gamma_{rj}\gamma_{sp}\gamma_{tq}(X_{jm})\left(\begin{Bmatrix} p\ q \\ m \end{Bmatrix}\right)\right].$$

Hence

$$X_{rst} = \frac{\partial X_{rs}}{\partial x_t} - \Sigma_m\left[X_{ms}\begin{Bmatrix} r\ t \\ m \end{Bmatrix} + X_{rm}\begin{Bmatrix} s\ t \\ m \end{Bmatrix}\right] \tag{36}$$

transforms covariantly as of order three.

We may generalize to the next higher order as

$$X_{rstu} = \frac{\partial X_{rst}}{\partial x_u} - \Sigma_m\left[X_{mst}\begin{Bmatrix} r\ u \\ m \end{Bmatrix} + X_{rmt}\begin{Bmatrix} s\ u \\ m \end{Bmatrix} + X_{rsm}\begin{Bmatrix} t\ u \\ m \end{Bmatrix}\right]$$

and so on. These derivatives of higher order may also be written neatly by using matrical notation, but we shall carry that method no further.

A particular case of interest is the successive covariant derivatives of a function $F$. The first is merely the set $X_r = \partial F/\partial x_r$ as shown above (§ 14); the second is

$$X_{rs} = \frac{\partial^2 F}{\partial x_r \partial x_s} - \Sigma_m \frac{\partial F}{\partial x_m}\begin{Bmatrix} r\ s \\ m \end{Bmatrix}.$$

In this particular case the system is symmetric, $X_{rs} = X_{sr}$, because the Christoffel symbols are, as is known, symmetric. Moreover[21] if the covariant derivatives $X_{rs}$ of a system $X_r$ form a symmetric system $X_{rs} = X_{sr}$, then the elements $X_r$ of the system, must be the partial derivatives of the same function $F$.

**18. Contravariant differentiation.** If $X^{(r)}$ is a contravariant system of order 1, we should call a contravariant set $X^{(rs)}$ of order 2. which contains the derivatives $\partial X^{(r)}/\partial x_s$ and the coefficients $a_{rs}$ and their derivatives, the first contravariant derived set. We may obtain this set by considering the dual $X^{(rs)}$ of the first covariant derivative $X_{rs}$ of the set $X_r$, dual to the given set $X^{(r)}$. Thus,

$$\begin{aligned} X^{(uv)} &= \Sigma_{rs} a^{(ru)} a^{(sv)} X_{rs} = \Sigma_{rs} a^{(ru)} a^{(sv)} \left[ \frac{\partial X_r}{\partial x_s} - \Sigma_m X_m \left\{ \begin{matrix} r & s \\ & m & \end{matrix} \right\} \right] \\ &= \Sigma_{sr} a^{(sv)} \left[ \frac{\partial a^{(ru)} X_r}{\partial x_s} - X_r \frac{\partial a^{(ru)}}{\partial x_s} - a^{(ru)} \Sigma_m X_m \left\{ \begin{matrix} r & s \\ & m & \end{matrix} \right\} \right] \\ &= \Sigma_s a^{(sv)} \left[ \frac{\partial X^{(u)}}{\partial x_s} - \Sigma_{rt} X^{(t)} a_{rt} \frac{\partial a^{(ru)}}{\partial x_s} - \Sigma_{rmt} a^{(ru)} X^{(t)} a_{mt} \left\{ \begin{matrix} r & s \\ & m & \end{matrix} \right\} \right] \end{aligned}$$

Now as $\Sigma_r a_{rt} a^{(ru)} = \epsilon_{ut}$, we have

$$\Sigma_r a_{rt} \frac{\partial a^{(ru)}}{\partial x_s} = - \Sigma_r a^{(ru)} \frac{\partial a_{rt}}{\partial x_s}.$$

Hence

$$X^{(uv)} = \Sigma_s a^{(sv)} \left[ \frac{\partial X^{(u)}}{\partial x_s} + \Sigma_{rt} X^{(t)} a^{(ru)} \frac{\partial a_{rt}}{\partial x_s} - \Sigma_{rt} a^{(ru)} X^{(t)} \left[ \begin{matrix} r & s \\ & t & \end{matrix} \right] \right].$$

But, by (32),

$$\frac{\partial a_{rt}}{\partial x_s} - \left[ \begin{matrix} r & s \\ & t & \end{matrix} \right] = \left[ \begin{matrix} t & s \\ & r & \end{matrix} \right]. \tag{32'}$$

Hence

$$X^{(uv)} = \Sigma_s a^{(sv)} \left[ \frac{\partial X^{(u)}}{\partial x_s} + \Sigma_t X^{(t)} \left\{ \begin{matrix} t & s \\ & u & \end{matrix} \right\} \right]. \tag{37}$$

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[21] See Ricci, *Lezioni*, p. 70.

For a contravariant system of higher order the process is similar and the result is as follows:

$$X^{(uvw)} = \Sigma_s a^{(sw)} \left[ \frac{\partial X^{(uv)}}{\partial x_s} + \Sigma_t \left( X^{(ut)} \left\{ {t\,s \atop v} \right\} + X^{(tv)} \left\{ {t\,s \atop u} \right\} \right) \right], \qquad (37')$$

and similarly in general.

The partial derivatives of a contravariant set may then be obtained by solution. For,

$$\Sigma_v X^{(uv)} a_{rv} = \frac{\partial X^{(u)}}{\partial x_r} + \Sigma_t X^{(t)} \left\{ {t\,r \atop v} \right\}, \qquad (38)$$

$$\Sigma_w X^{(uvw)} a_{rw} = \frac{\partial X^{(uv)}}{\partial x_r} + \Sigma_t \left( X^{(ut)} \left\{ {t\,r \atop v} \right\} + X^{(tv)} \left\{ {t\,r \atop u} \right\} \right). \qquad (38')$$

**19. Properties of covariant differentiation.** If we apply (36) to the set $a_{rs}$ of the coefficients of the quadratic differential form, we find

$$\begin{aligned} a_{rst} &= \frac{\partial a_{rs}}{\partial x_t} - \Sigma_m \left[ a_{ms} \left\{ {r\,t \atop m} \right\} + a_{rm} \left\{ {s\,t \atop m} \right\} \right] \\ &= \frac{\partial a_{rs}}{\partial x_t} - \left[ {r\,t \atop s} \right] - \left[ {s\,t \atop r} \right] = 0, \end{aligned}$$

as follows from (32) and (32′). Hence the first covariant derived set of $a_{rs}$ vanishes identically. The same may be proved of the first contravariant derived set of $a^{(rs)}$; but as the set $a^{(rs)}$ is the dual of the set $a_{rs}$, no formal proof is necessary.

The covariant derivatives of a product of covariant factors follows the rule of ordinary differentiation. For example,

$$(X_r X_s)_t = X_{rt} X_s + X_r X_{st}, \qquad (39)$$

since

$$\begin{aligned} (X_r X_s)_t &= \frac{\partial X_r X_s}{\partial x_t} - \Sigma_m \left[ X_m X_s \left\{ {r\,t \atop m} \right\} + X_r X_m \left\{ {s\,t \atop m} \right\} \right] \\ &= \left[ \frac{\partial X_r}{\partial x_t} - \Sigma_m X_m \left\{ {r\,t \atop m} \right\} \right] X_s + \left[ \frac{\partial X_s}{\partial x} - \Sigma_m X_m \left\{ {s\,t \atop m} \right\} \right] X_r . \end{aligned}$$

The covariant derivative of the covariant system formed by the composition of a contravariant system of order $m$ and a covariant system of order $m + p$ may be written

$$Z_{i_1 i_2 \cdots i_p t} = \Sigma_{j_1 j_2 \cdots j_m} X_{i_1 i_2 \cdots i_p j_1 j_2 \cdots j_p t} Y^{(j_1 j_2 \cdots j_m)} \\ + \Sigma_{j_1 j_2 \cdots j_m s} X_{i_1 i_2 \cdots i_p j_1 j_2 \cdots j_m} Y^{(j_1 j_2 \cdots j_m s)} a_{st} \qquad (40)$$

There is a dual proposition for the contravariant derivative of a contravariant system formed by composition of a covariant system of order $p$ and a contravariant system of order $m + p$.

A special case of importance is the differentiation of the invariant which arises from the composition when the orders of the covariant and contravariant systems are equal. We have, from (40),

$$I_t = \Sigma_{j_1 j_2 \cdots j_m} X_{j_1 j_2 \cdots j_m t} Y^{(j_1 j_2 \cdots j_m)} + \\ \Sigma_{j_1 j_2 \cdots j_m s} X_{j_1 j_2 \cdots j_m} Y^{(j_1 j_2 \cdots j_m s)} a_{st} .$$

If we write for $Y^{(j_1 j_2 \cdots j_m s)}$ its value

$$Y^{(j_1 j_2 \cdots j_m s)} = \Sigma_{i_1 i_2 \cdots i_m r} a^{(i_1 j_1)} a^{(i_2 j_2)} \cdots a^{(i_m j_m)} a^{(rs)} Y_{i_1 i_2 \cdots i_m r} ,$$

we may sum over the $i$'s combining the $a$'s with the $X$'s; then, with proper change of indices,

$$I_t = \Sigma_{j_1 j_2 \cdots j_m} [X_{j_1 j_2 \cdots j_m t} Y^{(j_1 j_2 \cdots j_m)} + X^{(j_1 j_2 \cdots j_m)} Y_{j_1 j_2 \cdots j_m t}] . \quad (41)$$

**20. Relative covariant differentiation.**— Covariant differentiation is a process which derives from a covariant set of order $m$ another covariant set of order $m + 1$ containing the derivatives of the elements of the first set and certain derivatives of the coefficients of the quadratic form, namely the Christoffel symbols. We may obtain a covariant set of order $m + 1$ from one of order $m$ in other ways, without the use of Christoffel symbols but with the aid of the functions which define an $n$-tuple and its reciprocal.

Let us express $X_r$ in terms of the λ's as a basis (§ 12).

$$X_r = \Sigma_i c_{i \cdot i} \lambda'_r , \qquad c_i = \Sigma_{t \cdot i} \lambda^{(t)} X_t .$$

Now differentiate with respect to $x_s$. Then.

$$\frac{\partial X_r}{\partial x_s} = \Sigma_i c_i \frac{\partial_i\lambda'_r}{\partial x_s} + \Sigma_i \frac{\partial c_i}{\partial x_s}\,{}_i\lambda'_r\,. \qquad (42)$$

We next observe that ${}_i\lambda'_r$ is covariant and that

$$\frac{\partial c_i}{\partial x_s} = \Sigma_t \frac{\partial c_i}{\partial y_t}\frac{\partial y_t}{\partial x_s}.$$

As $c_i$ is an invariant, $\partial c_i/\partial y_t$ is the expression in the new variables corresponding to $\partial c_i/\partial x_s$. If we introduce the new $\lambda'$'s, we have

$$\Sigma_i \frac{\partial c_i}{\partial x_s}\,{}_i\lambda'_r = \Sigma_{itu}\left(\frac{\partial c_i}{\partial y_t}\right)({}_i\lambda'_u)\,\frac{\partial y_t}{\partial x_s}\frac{\partial y_u}{\partial x_r}.$$

Hence the set of terms $\Sigma_{i\cdot}{}_i\lambda'_r\partial c_i/\partial x_s$ is covariant of order 2. Now, replacing in (42) the invariant $c_i$ by $\Sigma_{t\cdot}{}_i\lambda^{(t)}X_t$ and transposing, we have as a covariant set of order 2,

$$X_{rs} = \frac{\partial X_r}{\partial x_s} - \Sigma_t X_t \Sigma_{i\cdot}{}_i\lambda^{(t)}\frac{\partial_i\lambda'_r}{\partial x_s}. \qquad (43)$$

(We may verify directly that $X_{rs}$ is a covariant set of order 2 by transforming it.)

If we had a set of order 2 expressed in terms of the basis, we find

$$X_{rs} = \Sigma_{ij} c_{ij}\cdot{}_i\lambda'_r\cdot{}_j\lambda'_s\,, \qquad \text{with } c_{ij} = \Sigma_{pq} X_{pq}\cdot{}_i\lambda^{(p)}{}_j\lambda^{(q)}.$$

Differentiate and transpose,

$$\frac{\partial X_{rs}}{\partial x_t} - \Sigma_{ij} c_{ij}\frac{\partial_i\lambda'_r}{\partial x_t}\,{}_j\lambda'_s - \Sigma_{ij} c_{ij}\cdot{}_i\lambda'_r\frac{\partial_j\lambda'_s}{\partial x_t} = \Sigma_{ij}\frac{\partial c_{ij}}{\partial x_t}\,{}_i\lambda'_r\cdot{}_j\lambda'_s\,.$$

The right hand member forms (for all different values of $r$, $s$, $t$,) a covariant system of order 3; so also must the left hand member. If now we replace $c_{ij}$ by its value and if we note that $\Sigma_{j\cdot}{}_j\lambda^{(q)}{}_j\lambda'_s = \epsilon_{sq}$ by (21′), we see that

$$X_{rst} = \frac{\partial X_{rs}}{\partial x_t} - \Sigma_p X_{ps}\Sigma_{i\cdot}{}_i\lambda^{(p)}\frac{\partial({}_i\lambda'_r)}{\partial x_t} - \Sigma_p X_{rp}\Sigma_{i\cdot}{}_i\lambda^{(p)}\frac{\partial({}_i\lambda'_s)}{\partial x_t} \qquad (43')$$

is a covariant system of order 3. And in like manner we could form from a system of order $m$ a covariant system of order $m+1$.

# CHAPTER II. THE GENERAL THEORY OF SURFACES.

**21. Normalization of element of arc.** In ordinary surface theory the second fundamental form may be derived[22] by considering a change of variable from the given or first fundamental form,

$$\varphi = \Sigma a_{rs} dx_r dx_s \text{ to } (\varphi) = \Sigma dy_k^2,$$

where $r$, $s$ have the range 1, 2, and $k$ the range 1, 2, 3. We shall refer to Ricci[23] for this development and proceed to the case in which we are interested, namely, in which the surface lies not in 3 but in $n > 3$ dimensions. The proof for this case is similar to Ricci's. We shall treat first the simplest assumption, namely, that $n = 4$, and shall mention the generalization to $n > 4$ for the most part without proof.

To simplify notations we shall use a small amount of vector analysis. A set of values of the variables $y_i$ may be written simply as $\mathbf{y}$. A sum of the form $\Sigma_k y_k z_k$ is then the *scalar product* $\mathbf{y} \cdot \mathbf{z}$. The use of vector analysis is possible and entirely appropriate when operating as now in a Euclidean space of $n$ dimensions. If any question as to the legitimacy of the application of Ricci's rules for the absolute calculus arises we may revert at once to the ordinary form of analysis without vectors by taking components (supposed to be along fixed orthogonal directions) of vector equations and by replacing scalar products by sums.

We have, then,

$$\Sigma_k dy_k^2 = d\mathbf{y} \cdot d\mathbf{y} = \Sigma a_{rs} dx_r dx_s, \qquad k = 1, \ldots, 4;\ r, s = 1, 2,$$

by virtue of some transformation

$$y_k = y_k(x_1, x_2) \qquad \text{or} \qquad \mathbf{y} = \mathbf{y}(x_1, x_2).$$

Now if $\mathbf{y}_r$ denote the partial derivative of $\mathbf{y}$ by $x_r$,

$$d\mathbf{y} \cdot d\mathbf{y} = \Sigma_{rs} \mathbf{y}_r \cdot \mathbf{y}_s dx_r dx_s, \tag{44}$$

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[22] It is not ordinarily derived in this way.
[23] *Lezioni*, Part II, Chap. 1.

and hence

$$a_{rs} = \mathbf{y}_r \cdot \mathbf{y}_s . \qquad (44')$$

Differentiate covariantly by the rule for a composed system (§ 19)

$$a_{rst} = 0 = \mathbf{y}_{rt} \cdot \mathbf{y}_s + \mathbf{y}_r \cdot \mathbf{y}_{st} .$$

As this relation holds for any $r$, $s$, $t$, we have also

$$0 = \mathbf{y}_{tr} \cdot \mathbf{y}_s + \mathbf{y}_t \cdot \mathbf{y}_{rs} , \qquad 0 = \mathbf{y}_{rs} \cdot \mathbf{y}_t + \mathbf{y}_r \cdot \mathbf{y}_{ts} .$$

As $\mathbf{y}$ is a function of $x_1$, $x_2$, the second covariant derivative is commutative like an ordinary derivative (§ 19), and by addition and subtraction among the three equations we have

$$\mathbf{y}_t \cdot \mathbf{y}_{rs} = 0 \qquad \text{or} \qquad \mathbf{y}_r \cdot \mathbf{y}_{st} = 0, \qquad (45)$$

for all values of $r$, $s$, $t$.

**22. Normal vectors.** Equations (45) mean that the second covariant derivatives $\mathbf{y}_{st}$ are perpendicular to the first derivatives $\mathbf{y}_r$. As $\mathbf{y}_r$ lies in the tangent plane and as $\mathbf{y}_{st}$ is perpendicular to $\mathbf{y}_r$ for $r = 1, 2$, we infer that the vectors $\mathbf{y}_{st}$ lie in the normal plane[24] to the surface.[25] If $\mathbf{z}$ and $\mathbf{w}$ are any two unit vectors in the normal plane we may write

$$\mathbf{y}_{rs} = b_{rs}\mathbf{z} + c_{rs}\mathbf{w} \qquad (46)$$

with

$$\mathbf{z} \cdot \mathbf{z} = \mathbf{w} \cdot \mathbf{w} = 1, \qquad \mathbf{z} \cdot \mathbf{w} = 0, \qquad (47)$$

$$\mathbf{z} \cdot \mathbf{y}_r = \mathbf{w} \cdot \mathbf{y}_r = 0, \qquad (47')$$

$$b_{rs} = \mathbf{z} \cdot \mathbf{y}_{rs} , \qquad c_{rs} = \mathbf{w} \cdot \mathbf{y}_{rs} . \qquad (47'')$$

Here $\mathbf{z}$, $\mathbf{w}$ are particular unit vectors in the normal plane and consequently are invariant of the coordinate system, $x_1$, $x_2$; they are, however, functions of $x_1$, $x_2$, namely, invariant functions. The set of quantities $b_{rs}$, $c_{rs}$ are therefore covariant. As $\mathbf{y}_{rs} = \mathbf{y}_{sr}$ we see that $b_{rs}$ and $c_{rs}$ are also symmetrical sets.

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[24] By the normal plane we mean the plane which is completely perpendicular to the tangent plane, that is, such that any line in one is perpendicular to every line in the other. These planes intersect in only one point.

[25] One great advantage of the covariant derivative is therefore brought to light; for the ordinary second derivative of $\mathbf{y}$ would not lie in the normal plane.

We may differentiate (47′) to find the derivatives of $\mathbf{z}$ and $\mathbf{w}$. Then

$$\mathbf{y}_{rs}\cdot\mathbf{z} + \mathbf{y}_r\cdot\mathbf{z}_s = 0, \qquad \mathbf{y}_{rs}\cdot\mathbf{w} + \mathbf{y}_r\cdot\mathbf{w}_s = 0.$$

Hence

$$\mathbf{y}_r\cdot\mathbf{z}_s = -\,b_{rs}, \qquad \mathbf{y}_r\cdot\mathbf{w}_s = -\,c_{rs}.$$

Also, from (47),

$$\mathbf{z}\cdot\mathbf{z}_s = 0, \qquad \mathbf{w}\cdot\mathbf{w}_s = 0, \qquad \mathbf{z}\cdot\mathbf{w}_s + \mathbf{w}\cdot\mathbf{z}_s = 0.$$

Let

$$\mathbf{z}\cdot\mathbf{w}_s = +\,\nu_s, \qquad \mathbf{w}\cdot\mathbf{z}_s = -\,\nu_s. \tag{48}$$

We have then four equations (since $r = 1, 2$) to solve for $\mathbf{z}_s$; one of the equations shows that $\mathbf{z}_s$ is perpendicular to $\mathbf{z}$ and the other three give the components of $\mathbf{z}_s$ along the tangent plane and along $\mathbf{w}$. Now

$$\mathbf{y}^{(p)}\cdot\mathbf{y}_r = \Sigma_h y_h^{(p)} y_{h|r} = \Sigma_{ht} y_{h|t} a^{(pt)} y_{h|r} = \Sigma_t a^{(pt)} a_{rt} = \epsilon_{rp}.$$

The solution for $\mathbf{z}_s$ may then be written by inspection as

$$\mathbf{z}_s = -\,\Sigma_p b_{ps}\mathbf{y}^{(p)} - \nu_s\mathbf{w}, \tag{49}$$

and checked; in like manner,

$$\mathbf{w}_s = -\,\Sigma_p c_{ps}\mathbf{y}^{(p)} + \nu_s\mathbf{z}. \tag{49′}$$

**23. Gauss-Codazzi relations.** The third derivatives of $\mathbf{y}$ may next be found by differentiating (covariantly) the expressions (46).

$$\mathbf{y}_{rst} = b_{rst}\mathbf{z} + c_{rst}\mathbf{w} + b_{rs}\mathbf{z}_t + c_{rs}\mathbf{w}_t,$$

or

$$\mathbf{y}_{rst} = \mathbf{z}[b_{rst} + c_{rs}\nu_t] + \mathbf{w}[c_{rst} - b_{rs}\nu_t] - \Sigma_p[b_{pt}b_{rs} + c_{pt}c_{rs}]\mathbf{y}^{(p)}. \tag{50}$$

Now by (36) the general form for a third derivative is

$$\begin{aligned} X_{rst} &= \frac{\partial X_{rs}}{\partial x_t} - \Sigma_m \left[ X_{ms} \begin{Bmatrix} r\ \ t \\ m \end{Bmatrix} + X_{rm} \begin{Bmatrix} s\ \ t \\ m \end{Bmatrix} \right] \\ &= \frac{\partial^2 X_r}{\partial x_s \partial x_t} - \Sigma_m \left[ X_m \frac{\partial}{\partial x_t} \begin{Bmatrix} r\ \ s \\ m \end{Bmatrix} + \begin{Bmatrix} r\ \ s \\ m \end{Bmatrix} \frac{\partial X_m}{\partial x_t} \right] - \Sigma_m \left[ X_{ms} \begin{Bmatrix} r\ \ t \\ m \end{Bmatrix} \right. \\ &\qquad \left. + X_{rm} \begin{Bmatrix} s\ \ t \\ m \end{Bmatrix} \right] \end{aligned}$$

and

$$\frac{\partial X_m}{\partial x_t} = X_{mt} + \Sigma_p X_p \begin{Bmatrix} m\ \ t \\ p \end{Bmatrix}.$$

Hence

$$\begin{aligned} X_{rst} = {} & \frac{\partial^2 X_r}{\partial x_s \partial x_t} - \Sigma_m X_m \frac{\partial}{\partial x_t} \begin{Bmatrix} r\ \ s \\ m \end{Bmatrix} - \Sigma_m \begin{Bmatrix} r\ \ s \\ m \end{Bmatrix} X_{mt} \\ & - \Sigma_{mp} X_p \begin{Bmatrix} r\ \ s \\ m \end{Bmatrix} \begin{Bmatrix} m\ \ t \\ p \end{Bmatrix} - \Sigma_m \left[ X_{ms} \begin{Bmatrix} r\ \ t \\ m \end{Bmatrix} + X_{rm} \begin{Bmatrix} s\ \ t \\ m \end{Bmatrix} \right]. \end{aligned}$$

and

$$\begin{aligned} X_{rst} - X_{rts} = {} & - \Sigma_m X_m \left[ \frac{\partial}{\partial x_t} \begin{Bmatrix} r\ \ s \\ m \end{Bmatrix} - \frac{\partial}{\partial x_s} \begin{Bmatrix} r\ \ t \\ m \end{Bmatrix} + \Sigma_p \left( \begin{Bmatrix} r\ \ s \\ p \end{Bmatrix} \begin{Bmatrix} p\ \ t \\ m \end{Bmatrix} \right. \right. \\ & \qquad \left. \left. - \begin{Bmatrix} r\ \ t \\ p \end{Bmatrix} \begin{Bmatrix} p\ \ s \\ m \end{Bmatrix} \right) \right] \\ & - \Sigma_m \left[ \begin{Bmatrix} r\ \ s \\ m \end{Bmatrix} X_{mt} - \begin{Bmatrix} r\ \ t \\ m \end{Bmatrix} X_{ms} + \begin{Bmatrix} r\ \ t \\ m \end{Bmatrix} X_{ms} \right. \\ & \qquad - \begin{Bmatrix} r\ \ s \\ m \end{Bmatrix} X_{mt} \\ & \left. + X_{rm} \begin{Bmatrix} s\ \ t \\ m \end{Bmatrix} - X_{rm} \begin{Bmatrix} s\ \ t \\ m \end{Bmatrix} \right] \\ = {} & - \Sigma_m X_m \left[ \frac{\partial}{\partial x_t} \begin{Bmatrix} r\ \ s \\ m \end{Bmatrix} - \frac{\partial}{\partial x_s} \begin{Bmatrix} r\ \ t \\ m \end{Bmatrix} + \Sigma_p \left( \begin{Bmatrix} r\ \ s \\ p \end{Bmatrix} \begin{Bmatrix} p\ \ t \\ m \end{Bmatrix} \right. \right. \\ & \qquad \left. \left. - \begin{Bmatrix} r\ \ t \\ p \end{Bmatrix} \begin{Bmatrix} p\ \ s \\ m \end{Bmatrix} \right) \right]. \end{aligned}$$

Thus the difference of the two third derivatives of a function is expressible in terms of the first derivatives $X_m$ and a combination of the derivatives of the Christoffel symbols with the symbols themselves. This combination is the Riemann symbol[26] $\{rm, st\}$ of the second kind and hence

$$X_{rst} - X_{rts} = -\Sigma_m X_m \{rm, st\} = -\Sigma_u X^{(u)}(ru, st), \qquad (51)$$

where

$$(ru, st) = \Sigma_m a_{mu} \{rm, st\} \qquad (51')$$

is a Riemann symbol of the first kind. As $(ru, st)$ and $(ur, st)$ differ only in sign, we have

$$X_{rst} - X_{rts} = \Sigma_u X^{(u)}(ur, st). \qquad (51'')$$

From (50) we may obtain $\mathbf{y}_{rst} - \mathbf{y}_{rts}$ and identify with

$$\mathbf{y}_{rst} - \mathbf{y}_{rts} = \Sigma_u \mathbf{y}^{(u)}(ur, st). \qquad (52)$$

As the vectors $\mathbf{y}^{(u)}$ are tangential, the components of $\mathbf{z}$ and $\mathbf{w}$ vanish in this direction. Hence we obtain the equations,

$$b_{rst} - b_{rts} = c_{rt}\nu_s - c_{rs}\nu_t. \qquad (53)$$

$$c_{rst} - c_{rts} = -b_{rs}\nu_t - b_{rt}\nu_s. \qquad (53')$$

$$(pr, st) = [(b_{ps}b_{rt} - b_{pt}b_{rs}) + (c_{ps}c_{rt} - c_{pt}c_{rs})]. \qquad (53'')$$

**24. Extension to n > 4.** Thus far the four dimensional case has been treated. The generalization is simple. Instead of two independent normals $\mathbf{z}$, $\mathbf{w}$, we have $n - 2$ normals $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{n-2}$ and may write

$$\mathbf{y}_{rs} = {}_1b_{rs}\mathbf{z}_1 + {}_2b_{rs}\mathbf{z}_2 + \ldots + {}_{n-2}b_{rs}\mathbf{z}_{n-2}, \qquad (54)$$

$$\mathbf{z}_i \cdot \mathbf{z}_j = \epsilon_{ij}, \quad \mathbf{z}_i \cdot \mathbf{y}_s = 0, \quad i = 1, 2, \ldots, n - 2. \qquad (55)$$

If we differentiate, we have

$$\mathbf{z}_{i|s} \cdot \mathbf{z}_j + \mathbf{z}_i \cdot \mathbf{z}_{j|s} = 0, \quad \mathbf{z}_{i|r} \cdot \mathbf{y}_s + \mathbf{z}_i \cdot \mathbf{y}_{rs} = 0,$$

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[26] Pascal, *Repertorio* (Italian), Vol. II, p. 850, except for a typographical error.

If $i \neq j$, we set

$$\mathbf{z}_i \cdot \mathbf{z}_{j|s} = \nu_{ij|s}, \quad \mathbf{z}_j \cdot \mathbf{z}_{i|s} = -\nu_{ij|s} = \nu_{ji|s}, \tag{56}$$

and

$$\mathbf{z}_{i|r} \cdot \mathbf{y}_s = -{}_ib_{rs}. \tag{56'}$$

We can then obtain by the same process as before,

$${}_ib_{rst} - {}_ib_{rts} = \Sigma_{j=1}^{n-2} ({}_jb_{rs}\nu_{ji|t} - {}_jb_{rt}\nu_{ji|s}), \tag{57}$$

$$(pr, st) = \Sigma_{i=1}^{n-2} ({}_ib_{ps} \cdot {}_ib_{rt} - {}_ib_{pt} \cdot {}_ib_{rs}). \tag{57'}$$

Moreover we may obtain by a somewhat detailed analysis in the case $n = 4, 5, \ldots$ a relation involving the second derivatives of $\nu$ as

$$\nu_{rs} - \nu_{sr} = \Sigma_{pq} (b_{pr}c_{qs} - b_{ps}c_{qr}) \, a^{(pq)}, \quad n = 4, \tag{58}$$

$$\nu_{ji|rs} - \nu_{ji|sr} + \Sigma_{l=1}^{n-2} (\nu_{lj|r} \nu_{li|s} - \nu_{lj|s} \nu_{li|r})$$

$$= \Sigma_{pq} a^{(pq)} ({}_ib_{pr} \cdot {}_jb_{qs} - {}_ib_{ps} \cdot {}_jb_{qr})]. \tag{58'}$$

In the case of a binary (first) fundamental form $\varphi = \Sigma a_{rs} dx_r dx_s$, the Riemann symbol $(pr, st)$ reduces to a single one, namely $(12, 12)$, and we may write

$$(12, 12) = aG, \tag{59}$$

where $G$ is an invariant, $(G) = G$, called the Gaussian invariant or *Gaussian curvature.* If $n = 4$ equation (53″) may be written

$$| b | + | c | = aG, \tag{59'}$$

and in higher dimensions we have, from (57′),

$$\Sigma_i | {}_ib | = aG, \tag{59''}$$

where $| b |$, $| c |$, $| {}_ib |$ are the determinants formed of the terms $b_{rs}$, $c_{rs}$, ${}_ib_{rs}$. In case $n = 3$ we have simply $| b | = aG$.

**25. The Vector Second Form.** In three dimensions we construct a form,

$$\psi = \Sigma b_{rs} dx_r dx_s,$$

from the symmetric system $b_{rs}$ and call it the second fundamental form of the surface, defined by the first form $\varphi$ as one of a class of applicables, and thus we have the surface defined by $\varphi$ and $\psi$ as a rigid surface. In higher dimensions we construct $n-2$ forms $\psi_1, \psi_2, \ldots, \psi_{n-2}$ (two, when $n = 4$) from the $n-2$ symmetric systems ${}_ib_{rs}$ and this set of $n-2$ forms are *the second fundamental forms.* The different forms are not, however, entirely determined because with a different choice of the unit vectors $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{n-2}$ in the normal $(n-2)$-space, there is a change in the quantities ${}_ib_{rs}$. The set of forms $\psi_i$ taken with $\varphi$ and the generalized Gauss-Codazzi relations (57), (57′), (58), (58′) will determine the surface as a rigid surface in $n$-dimensions.[27] We shall not, however, enter into a proof of this proposition which is adequately treated by Ricci and not important for our work.

It has been stated that the systems ${}_ib_{rs}$ are not entirely determined. The relations between different systems may be illustrated in the case $n = 4$. We had $\mathbf{y}_{rs} = b_{rs}\mathbf{z} + c_{rs}\mathbf{w}$, that is, $b_{rs}$ and $c_{rs}$ are respectively the components of $\mathbf{y}_{rs}$ along $\mathbf{z}$ and along $\mathbf{w}$. If a new choice $\mathbf{z}'$, $\mathbf{w}'$ were made, the quantities $b'_{rs}$, $c'_{rs}$ would be the components of $\mathbf{y}_{rs}$ along $\mathbf{z}'$, $\mathbf{w}'$. Hence the relations $b'_{rs}$, $c'_{rs}$ and $b_{rs}$, $c_{rs}$ are those which express a rotation, namely,

$$b_{rs} = b'_{rs}\cos\theta - c'_{rs}\sin\theta, \quad c_{rs} = b'_{rs}\sin\theta + c'_{rs}\cos\theta.$$

In general if $n > 4$, the relations between ${}_ib_{rs}$ and ${}_ib'_{rs}$ must be those which determine an orthogonal transformation in the normal $(n-2)$-space, since ${}_ib_{rs}$ and ${}_ib'_{rs}$ are merely the components of $\mathbf{y}_{rs}$ along two different systems of orthogonal lines in that space. This amount and only this amount of indetermination is involved in our set of second fundamental forms $\psi_i$.

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[27] The generalization of the Gauss-Codazzi equations to hypersurfaces (for which the element of arc is a quadratic form of class 1) has been obtained by a number of authors, including Ricci, and do not contain the $\nu$'s which by Ricci's development (*Lezioni*, Introduction, Chap. 4) are necessary in case the class of the surface is greater than one. Levi (loc. cit., note 2) develops the theory of surfaces in a very different way. For him the element of arc is apparently not a particularly fundamental form but merely one of a set of fundamental forms. That is to say where we, following Ricci, have a first fundamental form (which is scalar) and a second fundamental form (60) which is vectorial, both quadratic, Levi has an infinite set of $(\mu + \nu)$-linear forms $F_{\mu\nu}$ $(\mu, \nu = 1, 2, \ldots)$ of which the first, $F_{11}$, is $ds^2$. He shows that the problem of finding the absolute invariants reduces to that of finding the simultaneous invariants of the forms $F_{\mu\nu}$ and he finds five special invariants $\Delta_i$ $(i = 1, \ldots, 5)$ which form a complete system of independent invariants. Our analysis leads us very naturally to five invariants which are equivalent to Levi's (see note 39).

Instead of carrying $n-2$ second fundamental forms $\psi_i$ we shall combine them into *a single vector second fundamental form*

$$\Psi = \mathbf{z}_1\psi_1 + \mathbf{z}_2\psi_2 + \ldots + \mathbf{z}_{n-2}\psi_{n-2} = \Sigma \mathbf{y}_{rs}dx_r dx_s \qquad (60)$$

*in the normal* $(n-2)$*-space.* If the vector form is regarded as given, the surface may be regarded as not fixed relative to arbitrary axes in space; only the shape of the surface is determined.

**26. Canonical orthogonal curve systems.**[28] We have defined a set of curves on a surface by the differential equations obtained by equating the ratios $dx_r : \lambda^{(r)}$(§13; here $r = 1, 2$). The quantities $\lambda^{(r)}$ are the contravariant system defining the curves; the dual system $\lambda_r$ is a covariant system which may also be regarded as defining the curves. We have defined perpendicularity and hence orthogonal systems of curves. If we give the definition

$$\lambda^{(r)} = \frac{dx_r}{ds} \qquad (61)$$

we have a special system $\lambda^{(r)}$ which satisfies the relation

$$\Sigma_r \lambda_r \lambda^{(r)} = 1, \qquad (61')$$

and we shall here assume this system. The orthogonal curves defined by $\bar{\lambda}^{(r)}$ or $\bar{\lambda}_r$ will satisfy the relation

$$\Sigma_r \lambda^{(r)} \bar{\lambda}_r = \Sigma_r \lambda_r \bar{\lambda}^{(r)} = 0. \qquad (62)$$

If we impose the further condition

$$\Sigma_r \bar{\lambda}_r \bar{\lambda}^{(r)} = 1, \qquad (62')$$

we have a set of relations which will determine $\bar{\lambda}^{(r)}$ or $\bar{\lambda}_r$ except for sign (the arbitrariness of sign corresponds to the two opposite directions along the curve). For from (62) $\bar{\lambda}^{(r)} = (-1)^{r+1}\rho\lambda_{r+1}$, it being understood that all even values of the index are equivalent and all odd values also equivalent. Then from (62′),

$$\Sigma_{rs} \bar{\lambda}^{(r)} \bar{\lambda}^{(s)} a_{rs} = 1 = \Sigma_{rs} \rho^2 (-1)^{r+s} \lambda_{r+1} \lambda_{s+1} a_{rs}.$$

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[28] Ricci, *Lezioni*, p. 106, and *Atti. R. Ist. Veneto*, (7) **4**, 1–29 (1893).

Now $$a^{(r+1,\,s+1)} = (-1)^{r+s}a_{rs}/a.$$

Hence $$1/\rho^2 = a\Sigma_{rs}\lambda_{r+1}\lambda_{s+1}a^{(r+1,\,s+1)} = a\Sigma_r\lambda_{r+1}\lambda^{(r+1)} = a,$$

and $$\rho = 1/\sqrt{a}.$$

Hence the system $\bar{\lambda}^{(r)}$ is

$$\bar{\lambda}^{(r)} = (-1)^{r+1}\lambda_{r+1}/\sqrt{a}.$$

Further we see easily that

$$\bar{\lambda}_r = (-1)^{r+1}\sqrt{a}\lambda^{(r+1)}. \qquad (63)$$

The system $\bar{\lambda}^{(r)}$ or $\bar{\lambda}_r$ is called the canonical orthogonal system for $\lambda^{(r)}$ or $\lambda_r$. The repetition of the process of forming the canonical system leads to the negative of the original system (not to the system itself). For

$$\bar{\lambda}_r = (-1)^{r+1}\sqrt{a}\,\bar{\lambda}^{(r+1)} = (-1)^3\sqrt{a}\lambda_r/\sqrt{a} = -\lambda_r.$$

If we have a given system $\lambda^{(r)}$ and let $\varphi_s$ be the covariant system obtained by the composition

$$\varphi_s = \Sigma_r\bar{\lambda}^{(r)}\lambda_{rs}, \qquad (64)$$

we have by solution, as may easily be verified,

$$\lambda_{rs} = \bar{\lambda}_r\varphi_s. \qquad (64')$$

Also $$\varphi_s = -\Sigma_r\lambda^{(r)}\bar{\lambda}_{rs}, \quad \bar{\lambda}_{rs} = -\lambda_r\varphi_s. \qquad (64'')$$

Thus by the introduction of $\varphi_s$ the system $\lambda_{rs}$ of order two is written as the product $\bar{\lambda}_r\varphi_s$ of two systems of the first order and at the same time $\bar{\lambda}_{rs}$ appears as the product $-\lambda_r\varphi_s$. The system $\varphi_s$ is called the derived system from the $\lambda$'s.

**27. Expressions of the second forms.** If we consider a covariant system $b_{rs}$ we may form the three invariants,

$$\begin{aligned} \alpha &= \Sigma_{rs}\lambda^{(r)}\lambda^{(s)}b_{rs}. \\ \beta &= \Sigma_{rs}\bar{\lambda}^{(r)}\bar{\lambda}^{(s)}b_{rs}. \\ \mu &= \Sigma_{rs}\bar{\lambda}^{(r)}\lambda^{(s)}b_{rs} = \Sigma_{rs}\lambda^{(r)}\bar{\lambda}^{(s)}b_{rs}. \end{aligned} \qquad (65)$$

The solution for the $b$'s gives at once,

$$b_{rs} = a\lambda_r\lambda_s + \mu(\bar{\lambda}_r\lambda_s + \lambda_r\bar{\lambda}_s) + \beta\bar{\lambda}_r\bar{\lambda}_s. \tag{65'}$$

The determinant of the $b$'s is then

$$|\,b\,| = a\Sigma_r b_{rs} b^{(rs)} = a(a\beta - \mu^2). \tag{65''}$$

If we are working with several systems ${}_ib_{rs}$ we have for each a set of invariants $a_i, \beta_i, \mu_i$ formed from (65). The second fundamental forms are therefore

$$\psi_i = \Sigma_{rs}[a_i\lambda_r\lambda_s + \mu_i(\lambda_r\bar{\lambda}_s + \bar{\lambda}_r\lambda_s) + \beta_i\bar{\lambda}_r\bar{\lambda}_s]dx_r dx_s. \tag{66}$$

The vector fundamental form is

$$\Psi = \Sigma_{rs}[\boldsymbol{\alpha}\lambda_r\lambda_s + \boldsymbol{\mu}(\lambda_r\bar{\lambda}_s + \bar{\lambda}_r\lambda_s) + \boldsymbol{\beta}\bar{\lambda}_r\bar{\lambda}_s]\, dx_r dx_s \tag{67}$$

where $$\boldsymbol{\alpha} = \Sigma a_i\mathbf{z}_i, \quad \boldsymbol{\mu} = \Sigma\mu_i\mathbf{z}_i, \quad \boldsymbol{\beta} = \Sigma\beta_i\mathbf{z}_i, \tag{67'}$$

$i$ running from 1 to $n-2$. The vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\mu}$ are invariant vectors in the normal $(n-2)$-space. From (65′), (67′) we have immediately,

$$\mathbf{y}_{rs} = \boldsymbol{\alpha}\lambda_r\lambda_s + \boldsymbol{\mu}(\lambda_r\bar{\lambda}_s + \bar{\lambda}_r\lambda_s) + \boldsymbol{\beta}\bar{\lambda}_r\bar{\lambda}_s. \tag{68}$$

Then from (65″) and (59″) we have,

$$G = \Sigma_i(a_i\beta_i - \mu_i^2) = \boldsymbol{\alpha}\cdot\boldsymbol{\beta} - \boldsymbol{\mu}^2. \tag{69}$$

Hence the result: *The Gaussian invariant $G$ is the scalar product of the vector invariants $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ diminished by the square of the vector invariant $\boldsymbol{\mu}$.*

**28. Moving rectangular axes.** The elements $\mathbf{y}_r$ or $y_{h|r}$, $h = 1, 2, \ldots, n$, are tangent to the surface. If we form

$$\boldsymbol{\xi} = \Sigma_r\lambda^{(r)}\mathbf{y}_r = \Sigma_r\lambda_r\mathbf{y}^{(r)}, \qquad \boldsymbol{\eta} = \Sigma_r\bar{\lambda}^{(r)}\mathbf{y}_r = \Sigma_r\bar{\lambda}_r\mathbf{y}^{(r)}, \tag{70}$$

we have two vectors tangent to the surface. Moreover these are: 1°, unit vectors; 2°, mutually perpendicular; 3°, tangent respectively to the curves $\lambda^{(r)}$ and to their orthogonal trajectories $\bar{\lambda}^{(r)}$. To prove 1° and 2° we note that

$$\boldsymbol{\xi}\cdot\boldsymbol{\xi} = \Sigma_{rs}\lambda^{(r)}\lambda^{(s)}\mathbf{y}_r\cdot\mathbf{y}_s = \Sigma_{rs}\lambda^{(r)}\lambda^{(s)}a_{rs} = \Sigma_r\lambda^{(r)}\lambda_r = 1,$$

and similar equations hold. For 3° observe that

$$\boldsymbol{\xi} = \Sigma_t \frac{\partial x_t}{\partial s}\frac{\partial \mathbf{y}_r}{\partial x_t} = \frac{\partial \mathbf{y}_r}{\partial s},$$

$d\mathbf{y}_r$ being the differential along the curves $\lambda^{(r)}$.

In case of four dimensions we shall use $\boldsymbol{\zeta}$, $\boldsymbol{\omega}$ (to correspond with $\boldsymbol{\xi}$, $\boldsymbol{\eta}$) in place of $\mathbf{z}$, $\mathbf{w}$ as the unit normal vectors — in higher dimensions $\mathbf{z}_1, \mathbf{z}_2, \ldots \mathbf{z}_{n-2}$. We have therefore such relations as (47) or (55). The systems $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$, $\boldsymbol{\omega}$ or $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\mathbf{z}_i$, $i = 1, 2, \ldots, n-2$, are therefore systems of moving axes in which $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ move along definite orthogonal trajectories upon the surface.

The rate of change of the unit vectors $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ are, by covariant differentiation of (70),

$$\boldsymbol{\xi}_r = \Sigma_s\lambda^{(s)}\mathbf{y}_{rs} + \Sigma_s\lambda_{rs}\mathbf{y}^{(s)},$$

From (68), (61′), (62),

$$\begin{aligned}\Sigma_s\lambda^{(s)}\mathbf{y}_{rs}, &= \Sigma_s[\boldsymbol{\alpha}\lambda_r\lambda_s\lambda^{(s)} + \boldsymbol{\mu}(\lambda_r\bar{\lambda}_s + \bar{\lambda}_r\lambda_s)\lambda^{(s)} + \boldsymbol{\beta}\bar{\lambda}_r\bar{\lambda}_s\lambda^{(s)}]\\ &= \boldsymbol{\alpha}\lambda_r + \boldsymbol{\mu}\bar{\lambda}_r = \Sigma_i(a_i\mathbf{z}_i\lambda_r + \mu_i\mathbf{z}_i\bar{\lambda}_r),\\ \Sigma_s\lambda_{rs}\mathbf{y}^{(s)} &= \Sigma_s\lambda_s\varphi_r\mathbf{y}^{(s)} = \varphi_r\boldsymbol{\eta}.\end{aligned}$$

Hence

$$\begin{aligned}\boldsymbol{\xi}_r &= \boldsymbol{\alpha}\lambda_r + \boldsymbol{\mu}\bar{\lambda}_r + \boldsymbol{\eta}\varphi_r,\\ \boldsymbol{\eta}_r &= \boldsymbol{\mu}\lambda_r + \boldsymbol{\beta}\bar{\lambda}_r - \boldsymbol{\xi}\varphi_r.\end{aligned} \tag{71}$$

The rates of change of the normals are found from the relations (55), (56), (56′).

$$\begin{gathered}\mathbf{z}_i\cdot\mathbf{y}_r = 0, \qquad \mathbf{z}_{is}\cdot\mathbf{y}_r + \mathbf{z}_i\cdot\mathbf{y}_{rs} = 0,\\ \mathbf{z}_{is}\cdot\mathbf{y}_r = -\,{}_ib_{rs} \quad\text{and}\quad \mathbf{z}_{is}\cdot\mathbf{z}_j = \nu_{ji|s}.\end{gathered}$$

These equations give the components of $\mathbf{z}_{i|s}$ along the surface and along the normals. Hence,

$$\begin{aligned}\mathbf{z}_{i|r} &= -\,\Sigma_s b_{rs}\mathbf{y}^{(s)} + \Sigma_j\nu_{ji|r}\,\mathbf{z}_j\\ &= -\,\Sigma_s[a_i\lambda_r\lambda_s + \mu_i(\bar{\lambda}_r\lambda_s + \lambda_r\bar{\lambda}_s) + \beta_i\bar{\lambda}_r\bar{\lambda}_s]\mathbf{y}^{(s)} + \Sigma_j\nu_{ji|r}\,\mathbf{z},\end{aligned}$$

or

$$\mathbf{z}_{i|r} = -\,\boldsymbol{\xi}(a_i\lambda_r + \mu_i\bar{\lambda}_r) - \boldsymbol{\eta}(\mu_i\lambda_r + \beta_i\bar{\lambda}_r) + \Sigma_j\nu_{ji|r}\,\mathbf{z}_j. \tag{72}$$

If in four dimensions we use $\boldsymbol{\zeta}$, $\boldsymbol{\omega}$, $b_{rs}$, $c_{rs}$ to avoid subscripts, we have

$$\begin{aligned}\boldsymbol{\zeta}_r &= -\Sigma_s b_{rs}\mathbf{y}^{(s)} + \nu_r\boldsymbol{\omega} \\ &= -\boldsymbol{\xi}(a_1\lambda_r + \mu_1\overline{\lambda}_r) + \boldsymbol{\eta}(\mu_1\lambda_r + \beta_1\overline{\lambda}_r) + \nu_r\boldsymbol{\omega}, \qquad (72') \\ \boldsymbol{\omega}_r &= -\Sigma_s c_{rs}\mathbf{y}^{(s)} - \nu_r\boldsymbol{\zeta} \\ &= -\boldsymbol{\xi}(a_2\lambda_r + \mu_2\overline{\lambda}_r) + \boldsymbol{\eta}(\mu_2\lambda_r + \beta_2\overline{\lambda}_r) - \nu_r\boldsymbol{\zeta}.\end{aligned}$$

It is important to observe that the theory of the 2-surface in four or more dimensions is not the same as the theory of the moving axes: for the 2, or $n-2$ normals, to the surface are to a large extent indeterminate so far as the surface itself is concerned. It is the set of quantities $\nu$ which render the normal system definite and upon which the rate of change of the normal vectors depends as in the above equations. The theory of the set of moving axes is a step further than the theory of the surface and as far as the surface alone is concerned we may disregard the $\nu$'s so long as we do not need to differentiate the normals. In this respect there is the same difference between surface theory and the theory of moving axes (of which two are tangent to the surface) as between the theory of a twisted curve in three dimensions and the theory of moving axes of which only one is tangent to the curve. If the differential theory of a curve is treated from the point of view of the quadratic form (in one variable), the $\nu$ which must be introduced in the case of a twisted curve in three dimensions is related to the radius of torsion. In the curve theory the set of axes is rendered definite by assuming that the normal axes are along the principal normal and binormal and if we desire to keep moving axes in our theory of surfaces it will be desirable to specialize the normal axes in some such way as in the case of curves in three dimensions.

**29. Tangent plane and normal space.** Two elements which have strictly to do with the surface alone are the tangent plane and the normal plane or $(n-2)$-space. Following the notation of Gibbs (for the outer product) we may write the unit tangent plane and its differential as

$$\mathbf{M} = \boldsymbol{\xi}\times\boldsymbol{\eta}, \qquad d\mathbf{M} = d\boldsymbol{\xi}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times d\boldsymbol{\eta}. \qquad (73)$$

The unit normal space is,

$$\begin{aligned}\mathbf{N} &= \boldsymbol{\zeta}\times\boldsymbol{\omega} \quad \text{or} \quad \mathbf{N} = \mathbf{z}_1\times\mathbf{z}_2\times\ldots\times\mathbf{z}_{n-2}, \qquad (73') \\ d\mathbf{N} &= d\boldsymbol{\zeta}\times\boldsymbol{\omega} + \boldsymbol{\zeta}\times d\boldsymbol{\omega} \quad \text{or} \quad d\mathbf{N} = d\mathbf{z}_1\times\mathbf{z}_2\times\mathbf{z}_3\times\ldots\times\mathbf{z}_{n-2} + \text{etc.}\end{aligned}$$

In terms of the notation introduced above we have, from (71),

$$d\boldsymbol{\xi} = \Sigma_r \boldsymbol{\xi}_r dx_r = \boldsymbol{\alpha}\Sigma_r \lambda_r dx_r + \boldsymbol{\mu}\Sigma_r \overline{\lambda}_r dx_r + \boldsymbol{\eta}\Sigma_r \varphi_r dx_r\,,$$

$$d\boldsymbol{\eta} = \Sigma_r \boldsymbol{\eta}_r dx_r = \boldsymbol{\mu}\Sigma_r \lambda_r dx_r + \boldsymbol{\beta}\Sigma_r \overline{\lambda}_r dx_r - \boldsymbol{\xi}\Sigma_r \varphi_r dx_r\,.$$

Hence

$$d\mathbf{M} = \boldsymbol{\alpha}\times\boldsymbol{\eta}\Sigma\lambda_r dx_r + \boldsymbol{\mu}\times\boldsymbol{\eta}\Sigma\overline{\lambda}_r dx_r - \boldsymbol{\mu}\times\boldsymbol{\xi}\Sigma\lambda_r dx_r - \boldsymbol{\beta}\times\boldsymbol{\xi}\Sigma\overline{\lambda}_r dx_r\,. \qquad (74)$$

Now $\quad d\mathbf{y} = \Sigma \mathbf{y}_r dx_r\,, \qquad \boldsymbol{\xi} = \Sigma\lambda^{(r)}\mathbf{y}_r\,, \qquad \boldsymbol{\eta} = \Sigma\overline{\lambda}^{(r)}\mathbf{y}_r\,.$

The last two equations may be solved by inspection as

$$\mathbf{y}_r = \boldsymbol{\xi}\lambda_r + \boldsymbol{\eta}\overline{\lambda}_r\,,$$

$$d\mathbf{y} = \boldsymbol{\xi}\Sigma\lambda_r dx_r + \boldsymbol{\eta}\Sigma\overline{\lambda}_r dx_r\,. \qquad (74')$$

Hence

$$\begin{aligned} d\mathbf{y}\times d\mathbf{M} &= \boldsymbol{\xi}\times\boldsymbol{\alpha}\times\boldsymbol{\eta}\Sigma\lambda_r dx_r \Sigma\lambda_r dx_r + \boldsymbol{\xi}\times\boldsymbol{\mu}\times\boldsymbol{\eta}\Sigma\lambda_r dx_r\Sigma\overline{\lambda}_r dx_r \\ &\quad - \boldsymbol{\eta}\times\boldsymbol{\mu}\times\boldsymbol{\xi}\Sigma\overline{\lambda}_r dx_r\Sigma\lambda_r dx_r - \boldsymbol{\eta}\times\boldsymbol{\beta}\times\boldsymbol{\xi}\Sigma\overline{\lambda}_r dx_r\Sigma\overline{\lambda}_r dx_r \\ &= -\,\boldsymbol{\xi}\times\boldsymbol{\eta}\times\{\boldsymbol{\alpha}\Sigma_{rs}\lambda_r\lambda_s dx_r dx_s + \boldsymbol{\beta}\Sigma_{rs}\overline{\lambda}_r\overline{\lambda}_s dx_r dx_s \\ &\qquad + \boldsymbol{\mu}\Sigma_{rs}(\lambda_r\overline{\lambda}_s + \overline{\lambda}_r\lambda_s)dx_r dx_s\} \end{aligned}$$

or

$$d\mathbf{y}\times d\mathbf{M} = -\,\mathbf{M}\times\Sigma_{rs}\mathbf{y}_{rs}dx_r dx_s = -\,\mathbf{M}\times\Psi. \qquad (75)$$

This expression may be solved for $\Psi$ by multiplying by **M**. Thus,[29]

$$\mathbf{M}\cdot(d\mathbf{y}\times d\mathbf{M}) = -\,\mathbf{M}\cdot(\mathbf{M}\times\Psi). \qquad (75')$$

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[29] We shall use as a definition of the inner product that due to G. N. Lewis (loc. cit., note 15) which has the advantage over the inner product of Grassmann that it is commutative. The interpretation of the inner product of a $p$-dimensional parallelepiped and a $q$-dimensional parallelepiped where $q > p$ is a $(q - p)$-dimensional parallelepiped in the $q$-space perpendicular to the $p$-space. The rules of operation with inner products have been developed for a non-Euclidean case by Wilson and Lewis (loc. cit., note 15) and the rules for the Euclidean case are not different except for an occasional change of sign. As the product is distributive the rules may all be verified on or derived from products of unit vectors. (For the transformation used in the text at this point see Wilson and Lewis, p. 439). One of the most important rules is that represented by such expansions as,

$$(\mathbf{m}\times\mathbf{n})\cdot(\mathbf{p}\times\mathbf{q}\times\mathbf{r}) = (\mathbf{p}\times\mathbf{q}\times\mathbf{r})\cdot(\mathbf{m}\times\mathbf{n}) = (\mathbf{m}\times\mathbf{n})\cdot(\mathbf{q}\times\mathbf{r})\mathbf{p} + (\mathbf{m}\times\mathbf{n})\cdot(\mathbf{r}\times\mathbf{p})\mathbf{q} + (\mathbf{m}\times\mathbf{n})\cdot(\mathbf{p}\times\mathbf{q})\mathbf{r}.$$

The general rule is to take from the larger factor as many of its factors as there are factors in the smaller factor to form with them a scalar product, taking all

But $$\mathbf{C}\cdot(\mathbf{b}\times\mathbf{A}) = (\mathbf{C}\cdot\mathbf{A})\mathbf{b} + (\mathbf{b}\cdot\mathbf{C})\cdot\mathbf{A},$$

Here, $\mathbf{M}\cdot d\mathbf{M} = 0$ and $\mathbf{M}\cdot\Psi = 0$. Hence

$$(d\mathbf{y}\cdot\mathbf{M})\cdot d\mathbf{M} = -\Psi. \tag{76}$$

The vector $d\mathbf{y}\cdot\mathbf{M}$ is a 1-vector in $\mathbf{M}$ perpendicular to $d\mathbf{y}$ and $(d\mathbf{y}\cdot\mathbf{M})\cdot d\mathbf{M}$ is a 1-vector in $d\mathbf{M}$ perpendicular to $d\mathbf{y}\cdot\mathbf{M}$.

The expressions (75′) or (76) hold of course in three dimensions as the work by which they were obtained is independent of the number of dimensions, greater than two. In ordinary surface theory we have

$$\Psi = \Sigma_{rs} b_{rs} dx_r dx_s = -\Sigma_h dy_h d\zeta_h = -d\mathbf{y}\cdot d\boldsymbol{\zeta},$$

where $\zeta_h$ are the direction cosines of the normal $\boldsymbol{\zeta}$. If we multiply this by $\boldsymbol{\zeta}$ to make a vector form we have

$$\Psi = \boldsymbol{\zeta}\Sigma_{rs} b_{rs} dx_r dx_s = -\boldsymbol{\zeta}(d\mathbf{y}\cdot d\boldsymbol{\zeta}) = -d\boldsymbol{\zeta}\cdot(\boldsymbol{\zeta}\times d\mathbf{y}).$$

The form is expressed in terms of the normal and its differential instead of in terms of the tangent plane and its differential. We may make the change by taking complements,[30]

$$[d\boldsymbol{\zeta}\cdot(\boldsymbol{\zeta}\times d\mathbf{y})]^{**} = -[d\boldsymbol{\zeta}\times(\boldsymbol{\zeta}\times d\mathbf{y})^{*}]^{*} = -[d\boldsymbol{\zeta}\times(d\mathbf{y}\cdot\mathbf{M})]^{*} = (d\mathbf{y}\cdot\mathbf{M})\cdot d\mathbf{M}.$$

We have therefore arrived at *a formula* $\Psi = -(d\mathbf{y}\cdot\mathbf{M})\cdot d\mathbf{M}$ *for the (vector) second fundamental form which is the immediate generalization of the formula in three dimensions.*

If we desire to express the second fundamental form in terms of the normal $(n-2)$-space $\mathbf{N}$ instead of in terms of $\mathbf{M}$ we can do so.

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possible combinations and adding with due regard to sign. For the case in which the two factors are of equal order we have

$$(\mathbf{m}\times\mathbf{n})\cdot(\mathbf{p}\times\mathbf{q}) = \begin{vmatrix} \mathbf{m}\cdot\mathbf{p} & \mathbf{n}\cdot\mathbf{p} \\ \mathbf{m}\cdot\mathbf{q} & \mathbf{n}\cdot\mathbf{q} \end{vmatrix}.$$

These rules for obvious reasons are similar to those for regressive or mixed products and the rule quoted at this point in the text is like Müller's theorems (see Whitehead, *Universal Algebra*, p. 192). The complement, denoted by *, which is used below is similar to Grassmann's supplement, except possibly for sign.

[30] See Wilson and Lewis (loc. cit., note 15), p. 435.

**30. Square of element of surface.** Consider $d\mathbf{M}\cdot d\mathbf{M}$ which is numerically equal to $d\mathbf{N}\cdot d\mathbf{N}$.

$$d\mathbf{M}\cdot d\mathbf{M} = [\boldsymbol{\alpha}\times\boldsymbol{\eta}\Sigma\lambda_r dx_r + \boldsymbol{\mu}\times\boldsymbol{\eta}\Sigma\bar{\lambda}_r dx_r - \boldsymbol{\mu}\times\boldsymbol{\xi}\Sigma\lambda_r dx_r - \boldsymbol{\beta}\times\boldsymbol{\xi}\Sigma\bar{\lambda}_r dx_r]^2.$$

Now $$(\boldsymbol{\alpha}\times\boldsymbol{\eta})\cdot(\boldsymbol{\alpha}\times\boldsymbol{\eta}) = \boldsymbol{\alpha}\cdot\boldsymbol{\alpha}\boldsymbol{\eta}\cdot\boldsymbol{\eta} - (\boldsymbol{\alpha}\cdot\boldsymbol{\eta})^2 = \boldsymbol{\alpha}^2,$$

$$(\boldsymbol{\alpha}\times\boldsymbol{\eta})\cdot(\boldsymbol{\mu}\times\boldsymbol{\eta}) = \boldsymbol{\alpha}\cdot\boldsymbol{\mu}, \qquad (\boldsymbol{\alpha}\times\boldsymbol{\eta})\cdot(\boldsymbol{\mu}\times\boldsymbol{\xi}) = 0, \text{ etc.}$$

$$\begin{aligned} d\mathbf{M}\cdot d\mathbf{M} = {} & \boldsymbol{\alpha}^2\Sigma\lambda_r\lambda_s dx_r dx_s + \boldsymbol{\mu}^2\Sigma\bar{\lambda}_r\bar{\lambda}_s dx_r dx_s + \boldsymbol{\mu}^2\Sigma\lambda_r\lambda_s dx_r dx_s \\ & + \boldsymbol{\beta}^2\Sigma\bar{\lambda}_r\bar{\lambda}_s dx_r dx_s + 2\boldsymbol{\alpha}\cdot\boldsymbol{\mu}\Sigma\lambda_r\bar{\lambda}_s dx_r dx_s + 2\boldsymbol{\beta}\cdot\boldsymbol{\mu}\Sigma\bar{\lambda}_r\lambda_s dx_r dx_s. \end{aligned}$$

By (69) we have $\boldsymbol{\mu}^2 = \boldsymbol{\alpha}\cdot\boldsymbol{\beta} - G$. Hence

$$\begin{aligned} d\mathbf{M}\cdot d\mathbf{M} = {} & - G[\Sigma\lambda_r\lambda_s dx_r dx_s + \bar{\lambda}_r\bar{\lambda}_s dx_r dx_s] + \boldsymbol{\alpha}\cdot\boldsymbol{\beta}[\Sigma\lambda_r\lambda_s dx_r dx_s \\ & + \bar{\lambda}_r\bar{\lambda}_s dx_r dx_s] \\ & + \boldsymbol{\alpha}\cdot\boldsymbol{\alpha}\Sigma\lambda_r\lambda_s dx_r dx_s + \boldsymbol{\beta}\cdot\boldsymbol{\beta}\Sigma\bar{\lambda}_r\bar{\lambda}_s dx_r dx_s \\ & + (\boldsymbol{\alpha}\cdot\boldsymbol{\mu} + \boldsymbol{\beta}\cdot\boldsymbol{\mu})[\Sigma\lambda_r\bar{\lambda}_s dx_r dx_s + \Sigma\bar{\lambda}_r\lambda_s dx_r dx_s]. \end{aligned}$$

Now $a_{rs}$ may be expressed in terms of the $\lambda$'s as $b_{rs}$ was expressed in (65′). Then,

$$a_{rs} = c_1\lambda_r\lambda_s + c_2(\lambda_r\bar{\lambda}_s + \bar{\lambda}_r\lambda_s) + c_3\bar{\lambda}_r\bar{\lambda}_s.$$

When the invariants $c_1$, $c_2$, $c_3$ are determined by means of (65) we find $c_1 = 1$, $c_2 = 0$, $c_3 = 1$. Hence

$$a_{rs} = \lambda_r\lambda_s + \bar{\lambda}_r\bar{\lambda}_s. \tag{77}$$

$$\begin{aligned} d\mathbf{M}\cdot d\mathbf{M} = {} & - G\Sigma a_{rs} dx_r dx_s + (\boldsymbol{\alpha} + \boldsymbol{\beta})\cdot[\boldsymbol{\alpha}\Sigma\lambda_r\lambda_s dx_r dx_s + \boldsymbol{\mu}\Sigma(\bar{\lambda}_r\bar{\lambda}_s \\ & + \bar{\lambda}_r\lambda_s) dx_r dx_s + \boldsymbol{\beta}\Sigma\bar{\lambda}_r\bar{\lambda}_s dx_r dx_s], \end{aligned}$$

or

$$d\mathbf{M}\cdot d\mathbf{M} = - G\varphi + (\boldsymbol{\alpha} + \boldsymbol{\beta})\cdot\Psi. \tag{78}$$

Hence: *The square of the differential of the tangent plane is equal to the scalar product of the vector invariant* $\boldsymbol{\alpha} + \boldsymbol{\beta}$ *and the vector second fundamental form* $\Psi$ *less the product of the Gaussian invariant* $G$ *and the first fundamental form* $\varphi$.

This relation holds also in three dimensions: but in this case $\boldsymbol{\alpha} + \boldsymbol{\beta}$ and $\Psi$ are generally regarded as scalar quantities, $d\mathbf{M}\cdot d\mathbf{M}$ is replaced by the square of the differential of the normal,— and, furthermore, this quantity is interpreted as the differential of arc of the Gaussian spherical representation of the surface. No spherical representation of the same simple sort as obtained in three dimensions exists for higher dimensions, though (78) is common to all dimensions.[31]

**31. Geodesics.**[32] The shortest lines on a surface are determined by means of the first fundamental form alone and might properly have been treated before. We shall however take them up at this point. To minimize

$$s = \int [\Sigma_{rs} a_{rs} dx_r dx_s]^{\frac{1}{2}}$$

we follow the ordinary procedure of variation:

$$\delta s = \frac{1}{ds} \int \left[ \Sigma_{rs} \delta a_{rs} dx_r dx_s + 2\Sigma_{rs} a_{rs} \delta dx_r dx_s \right]$$

$$= \int \left[ \frac{1}{ds} \Sigma_{rst} \frac{\partial a_{rs}}{\partial x_t} dx_r dx_s \delta x_t - 2\Sigma_{rs} d\left( a_{rs} \frac{dx_s}{ds} \right) \delta x_r \right].$$

Now by (61) and $\lambda_r = \Sigma_s a_{rs} \lambda^s$,

$$\delta s = \int ds \left[ \Sigma_{rst} \frac{\partial a_{rs}}{\partial x_t} \lambda^{(r)} \lambda^{(s)} \delta x_t - 2\Sigma_r \frac{d\lambda_r}{ds} \delta x_r \right].$$

By (34)

$$\frac{d\lambda_r}{ds} = \Sigma_s \frac{d\lambda_r}{\partial x_s} \frac{dx_s}{ds} = \Sigma_s \lambda_{rs} \frac{dx_s}{ds} + \Sigma_{sqp} \begin{bmatrix} r & s \\ & q \end{bmatrix} \lambda_p a^{(pq)} \frac{dx_s}{ds}$$

$$= \Sigma_s \lambda_{rs} \lambda^{(s)} + \Sigma_{sq} \begin{bmatrix} r & s \\ & q \end{bmatrix} \lambda^{(q)} \lambda^{(s)}.$$

Hence the condition $\delta s = 0$ gives, when we set $t$ for $r$ in the second term and $r$ for $q$ in the third sum,

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[31] To have a spherical representation which will generalize we should mark on the unit sphere the great circle which is the trace upon the sphere of the diametral plane parallel to the tangent plane of the surface instead of the point which is the trace of the normal. This representation would therefore be the polar of the ordinary spherical representation.

[32] In this section we merely follow Ricci's *Lezioni*.

$$\Sigma_{rs}\frac{\partial a_{rs}}{\partial x_t}\lambda^{(r)}\lambda^{(s)}-2\Sigma_s\lambda_{ts}\lambda^{(s)}-2\Sigma_{sr}\begin{bmatrix} t\ s \\ r \end{bmatrix}\lambda^{(r)}\lambda^{(s)}=0.$$

The first and last terms cancel and hence the condition for a geodesic in the notation of the covariant derivatives is

$$\Sigma_s\lambda_{ts}\lambda^{(s)}=0. \tag{79}$$

In terms of the system $\varphi_s$ derived from the λ's the condition is, by (64′),

$$\Sigma_s\overline{\lambda}_t\varphi_s\lambda^{(s)}=0 \quad \text{or} \quad \Sigma_s\varphi_s\lambda^{(s)}=0. \tag{79′}$$

The quantity $\Sigma\varphi_s\lambda^{(s)}$ is an invariant which vanishes when λ is a system of geodesics.

**32. Curvature; Interpretation of $\mathbf{a}$ and $\gamma$.** The moving axis $\boldsymbol{\xi}$ is tangent to the curves λ. The curvature of these curves is $d\boldsymbol{\xi}/ds$ and from (71) takes the form

$$\frac{d\boldsymbol{\xi}}{ds}=\Sigma_r\boldsymbol{\xi}_r\frac{dx_r}{ds}=\Sigma_r\boldsymbol{\xi}_r\lambda^{(r)}=\mathbf{a}\Sigma\lambda_r\lambda^{(r)}+\boldsymbol{\mu}\Sigma\overline{\lambda}_r\lambda^{(r)}+\boldsymbol{\eta}\Sigma\varphi_r\lambda^{(r)}.$$

Hence

$$\mathbf{c}=\frac{d\boldsymbol{\xi}}{ds}=\mathbf{a}+\gamma\boldsymbol{\eta}, \tag{80}$$

if

$$\gamma=\Sigma\varphi_r\lambda^{(r)}, \tag{81}$$

where $\gamma$ is the invariant which vanishes (as has been seen) for geodesics. *The curvature of a surface curve therefore has two components one normal to the surface and equal to the vector invariant* $\mathbf{a}$, *one in the surface perpendicular to* λ *and of magnitude* $\gamma$. We have therefore an interpretation of the vector invariant $\mathbf{a}$, namely, the component of the curvature perpendicular to the surface. We have also an interpretation of $\gamma$ as the tangential component of the curvature. A geodesic being a curve which has no tangential component of curvature, the curvature of a geodesic is wholly normal to the surface, i. e., *the osculating plane of the geodesic is normal to the surface, no matter what the number of dimensions in which the surface lies.* We may consequently say that: *the vector* $\mathbf{a}$ *is the curvature of the geodesic which is* *tangent to the curve* $\lambda$, since $\boldsymbol{\alpha}$ depends only on the direction of the tangent $\lambda^{(r)}$ as shown by (65).

*If a curve is projected on a plane (or any plane space) passing through a tangent line to the curve, the curvature of the projection at the point of tangency is equal to the projection of the curvature of the given curve at that point.* To see this note first that the elements of arc on the given curve ($ds$) and the projected curve ($ds'$) differ at the point of contact by infinitesimals higher than the second because their ratios involve the cosine of a small angle. The elements $ds$ and $ds'$ are therefore equivalent for first and second derivatives. The projection of a vector $\mathbf{r}$ on a space $S_k$ represented by a unit vector $\mathbf{S}_k$ is

$$\mathbf{r}' = (-1)^{k-1}(\mathbf{r}\cdot\mathbf{S}_k)\cdot\mathbf{S}_k .$$

Then,

$$\mathbf{c}' = \frac{d^2\mathbf{r}'}{ds'^2} = (-1)^{k-1}\left(\frac{d^2\mathbf{r}}{ds^2}\cdot\mathbf{S}_k\right)\cdot\mathbf{S}_k = (-1)^{k-1}(\mathbf{c}\cdot\mathbf{S}_k)\cdot\mathbf{S}_k .$$

We could in like manner show that if we project a curve on a plane space through the osculating plane of the curve, the torsion of the projection is equal to projection of the torsion at that point: and so on.

We have $\mathbf{c} = \boldsymbol{\alpha} + \gamma\boldsymbol{\eta}$. If we project the curve $\lambda$ on the tangent plane to the surface, we have for the curvature of the projection,

$$\begin{aligned}\mathbf{c}' &= -(\mathbf{c}\cdot\mathbf{M})\cdot\mathbf{M} = -[(\boldsymbol{\alpha}+\gamma\boldsymbol{\eta})\cdot(\boldsymbol{\xi}\times\boldsymbol{\eta})]\cdot(\boldsymbol{\xi}\times\boldsymbol{\eta})\\ &= -\gamma\boldsymbol{\xi}\cdot(\boldsymbol{\xi}\times\boldsymbol{\eta}) = \gamma\boldsymbol{\eta} .\end{aligned}$$

Hence the curvature of the projection upon the tangent plane is $\gamma$ in magnitude. The invariant $\gamma$ is therefore the curvature of the projection of the curve upon the tangent plane,— this is called the geodesic curvature (which must be clearly distinguished from the curvature of the geodesic tangent to the curve).

If we project on a normal plane determined by $\boldsymbol{\xi}$ and any normal $\mathbf{n}$ we have

$$\mathbf{c}' = -[(\boldsymbol{\alpha}+\gamma\boldsymbol{\eta})\cdot(\boldsymbol{\xi}\times\mathbf{n})]\cdot(\boldsymbol{\xi}\times\mathbf{n}) = (\boldsymbol{\alpha}\cdot\mathbf{n})\cdot\mathbf{n} .$$

Hence the curvature of the projection is the component of $\boldsymbol{\alpha}$ along $\mathbf{n}$. If $\mathbf{n}$ had coincided with $\boldsymbol{\alpha}$ in direction, the curvature of the projection would have been $\boldsymbol{\alpha}$.

Consider now a section of the surface by a normal space $S_{n-1}$ of $n-1$ dimensions containing the tangent line $\boldsymbol{\xi}$ and the normal space

$\mathbf{N}_{n-2}$. The geodesic tangent to $\lambda$ has for curvature $\boldsymbol{\alpha}$, as has been seen, and hence its osculating plane $\boldsymbol{\xi}\times\boldsymbol{\alpha}$ lies in $S_{n-1}$. The geodesic has therefore three consecutive points in $S_{n-1}$, i. e., to infinitesimals of the third order the geodesic coincides with the normal section and hence the curvature of the geodesic and of the normal section are equal. Consequently: *we may interpret* $\boldsymbol{\alpha}$ *as the curvature of the normal section of the surface.* So far as curvature is concerned we may replace the normal section by the geodesic.

Now $\mathbf{c} = \boldsymbol{\alpha} + \gamma\boldsymbol{\eta}$ is the curvature of any section (for the curve on the surface and the section of the surface by a space $S_{n-1}$ containing the osculating plane of the curve are exchangeable as far as curvature is concerned) and the projection of $\mathbf{c} = \boldsymbol{\alpha} + \gamma\boldsymbol{\eta}$ on the normal is $\boldsymbol{\alpha}$ itself. Hence we have *Meusnier's theorem* that: *The projection of the curvature of any section on the normal section is the curvature of the normal section.* (Meusnier's theorem may be found in various degrees of generalization in the literature, e. g., in Levi's long article cited in note 2).

As $\mathbf{c} = \boldsymbol{\alpha} + \gamma\boldsymbol{\eta}$, $\mathbf{c}^2 = \boldsymbol{\alpha}^2 + \gamma^2$ and hence: *The magnitude of the curvature of a section is the square root of the sum of the squares of the normal and geodesic curvatures.*

### 33. Interpretation of $\boldsymbol{\beta}$ and $\boldsymbol{\mu}$.

If we treat $d\boldsymbol{\eta}$ as we treated $d\boldsymbol{\xi}$ we find

$$\frac{d\boldsymbol{\eta}}{ds} = \boldsymbol{\mu} - \gamma\boldsymbol{\xi}. \tag{82}$$

Now $\boldsymbol{\eta}$ is a normal to the curve $\lambda$ lying in the surface and $d\boldsymbol{\eta}/ds$ is the rate of change of this surface normal. If we consider the geodesic tangent to $\lambda$ we have $\gamma = 0$, and hence: *The vector* $\boldsymbol{\mu}$ (*which is perpendicular to the surface*) *may be interpreted as the rate of change of the surface-normal to a geodesic.* In three dimensions the surface normal is the binormal of the geodesic (with the proper convention as to sign) and hence in three dimensions $\mu$ is the torsion of the geodesic tangent to $\lambda$. In higher dimensions this interpretation is no longer valid because the osculating three space of the geodesic need not contain the tangent plane $\mathbf{M}$.

We may next form

$$\frac{d\boldsymbol{\xi}}{ds} = \Sigma_r\boldsymbol{\xi}_r\frac{\partial x_r}{ds} = \Sigma_r\boldsymbol{\xi}_r\bar{\lambda}^{(r)} = \boldsymbol{\mu} + \boldsymbol{\eta}\Sigma_r\varphi_r\bar{\lambda}^{(r)},$$

$$\frac{d\boldsymbol{\eta}}{ds} = \Sigma_r\boldsymbol{\eta}_r\frac{dx_r}{ds} = \Sigma_r\boldsymbol{\eta}_r\bar{\lambda}^{(r)} = \boldsymbol{\beta} - \boldsymbol{\xi}\Sigma_r\varphi_r\bar{\lambda}^{(r)},$$

by which we denote the rate of change of $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ with respect to the arc upon the orthogonal trajectories $\bar{\lambda}_r$ of $\lambda_r$.

Now the derived system $\varphi_r$ for $\lambda_r$ is related to the derived system $\bar{\varphi}_r$ for $\bar{\lambda}_r$ by the relation

$$\bar{\varphi}_r = -\varphi_r,$$

as may be seen from (63) and (64). Let

$$\bar{\gamma} = \Sigma_r \bar{\varphi}_r \bar{\lambda}^{(r)} = -\Sigma_r \varphi_r \bar{\lambda}^{(r)}. \tag{83}$$

Then $\bar{\gamma}$ is the geodesic curvature of the normal trajectories $\bar{\lambda}$.

$$\frac{d\boldsymbol{\xi}}{d\bar{s}} = \boldsymbol{\mu} - \bar{\gamma}\boldsymbol{\eta}, \quad \frac{d\boldsymbol{\eta}}{d\bar{s}} = \boldsymbol{\beta} + \bar{\gamma}\boldsymbol{\xi}. \tag{84}$$

By reasoning like that previously used we note that: *The vector* $\boldsymbol{\beta}$ *is the normal curvature of the orthogonal trajectories of* $\lambda$. Moreover, as the relation of $\boldsymbol{\eta}$ to $\boldsymbol{\xi}$ is the same as that of $-\boldsymbol{\xi}$ to $\boldsymbol{\eta}$ we may interpret $\boldsymbol{\mu}$ as the rate of change of the surface-normal to the geodesic tangent to $\bar{\lambda}$ changed in sign, that is, *the rate of change of the surface-normals tangent to normal geodesics are equal and opposite (vectors).* This corresponds to the theorem in three dimensions that the geodesic torsions in perpendicular directions are equal and opposite. In three dimensions, where $\mu$ is scalar the inference is immediate that there are a pair of orthogonal directions for which the geodesic torsion is zero — the lines of curvature. But in the general case $\boldsymbol{\mu}$ is a vector and may change sign without passing through zero, and we cannot affirm the existence of directions for which the rate of change of the surface-normal vanishes.

### 34. The mean curvature.

From (68) we get,

$$\Sigma_{rs} a^{(rs)} \mathbf{y}_{rs} = \boldsymbol{\alpha} \Sigma_r \lambda_r \lambda^{(r)} + \boldsymbol{\mu} \Sigma_r (\bar{\lambda}_r \lambda^{(r)} + \lambda_r \bar{\lambda}^{(r)}) + \boldsymbol{\beta} \Sigma_r \bar{\lambda}_r \bar{\lambda}^{(r)},$$

or

$$\Sigma_{rs} a^{(rs)} \mathbf{y}_{rs} = \boldsymbol{\alpha} + \boldsymbol{\beta}. \tag{85}$$

This equation from its form on the right appears to depend on $\lambda$, but from the form on the left is seen to be independent of $\lambda$. Hence: *The vector* $\boldsymbol{\alpha} + \boldsymbol{\beta}$ *is an invariant normal vector associated with a point of the surface* — it is a special and particularly important normal selected from all possible normals. As $\boldsymbol{\alpha}$ is the curvature of one section and $\boldsymbol{\beta}$ of the orthogonal section, we have the result that: *The sum of the normal curvatures in two orthogonal directions is independent of the directions. The sum* $\boldsymbol{\alpha} + \boldsymbol{\beta}$ *will be written as* $2\mathbf{h}$, *where* $\mathbf{h}$ *is called the mean (vector) curvature.*[33]

Since $\boldsymbol{\alpha} + \boldsymbol{\beta}$ is constant and the vector $\boldsymbol{\alpha} - \boldsymbol{\beta}$ is the other diagonal of the parallelogram on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, the vector $\boldsymbol{\alpha} - \boldsymbol{\beta}$ must pass through a fixed point on the mean curvature vector (namely, the extremity of that vector) and the termini of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ must describe a central curve about that point.

If we introduce a new pair of orthogonal directions $\lambda'$ making an angle $\theta$ with $\lambda_r$ we have

$$\lambda'_r = \lambda_r\cos\theta + \bar{\lambda}_r\sin\theta, \qquad \bar{\lambda}'_r = \bar{\lambda}_r\cos\theta - \lambda_r\sin\theta,$$

whence
$$\cos\theta = \Sigma\lambda^{(r)}\lambda'_r, \qquad \sin\theta = \Sigma\bar{\lambda}^{(r)}\lambda'_r,$$

$$\begin{aligned} \lambda_r &= \lambda'_r\cos\theta - \bar{\lambda}'_r\sin\theta, & \bar{\lambda}_r &= \lambda'_r\sin\theta + \bar{\lambda}'_r\cos\theta, \\ \lambda^{(r)} &= \lambda'^{(r)}\cos\theta - \bar{\lambda}'^{(r)}\sin\theta, & \bar{\lambda}^{(r)} &= \lambda'^{(r)}\sin\theta + \bar{\lambda}'^{(r)}\cos\theta. \end{aligned} \tag{86}$$

Now from (65) we have, in vector form,

$$\begin{gathered} \boldsymbol{\alpha} = \Sigma_{rs}\lambda^{(r)}\lambda^{(s)}\mathbf{y}_{rs}, \qquad \boldsymbol{\beta} = \Sigma_{rs}\bar{\lambda}^{(r)}\bar{\lambda}^{(s)}\mathbf{y}_{rs}, \\ \boldsymbol{\mu} = \Sigma_{rs}\bar{\lambda}^{(r)}\lambda^{(s)}\mathbf{y}_{rs} = \Sigma_{rs}\lambda^{(r)}\bar{\lambda}^{(s)}\mathbf{y}_{rs}. \end{gathered} \tag{87}$$

If we substitute for the $\lambda$'s in terms of the $\lambda'$'s we get the relations between $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\mu}$ and $\boldsymbol{\alpha}'$, $\boldsymbol{\beta}'$, $\boldsymbol{\mu}'$ for different directions in the surface. Thus

$$\begin{aligned} \boldsymbol{\alpha} &= \boldsymbol{\alpha}'\cos^2\theta - 2\boldsymbol{\mu}'\sin\theta\cos\theta + \boldsymbol{\beta}'\sin^2\theta \\ \boldsymbol{\beta} &= \boldsymbol{\beta}'\cos^2\theta + 2\boldsymbol{\mu}'\sin\theta\cos\theta + \boldsymbol{\alpha}'\sin^2\theta \\ \boldsymbol{\mu} &= \boldsymbol{\mu}'(\cos^2\theta - \sin^2\theta) + (\boldsymbol{\alpha}' + \boldsymbol{\beta}')\sin\theta\cos\theta, \end{aligned} \tag{88}$$

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[33] By mean curvature we designate the half sum of the curvatures $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. This is a true mean. In three dimensional surface theory the mean curvature often if not generally stands for the sum of the curvatures (See Eisenhart, *Differential Geometry*, page 123; E. E. Levi, loc. cit., page 69). We may quote as Levi does a theorem of Killing: the sum of the squares of the mean curvatures of the $n - 2$ three dimensional surfaces obtained by projecting an $n$-dimensional surface on $n - 2$ mutually perpendicular three spaces passing through the tangent plane, is constant. That is, is independent of the $n - 2$ normals selected to determine the three-spaces. The value of this invariant is $(2\mathbf{h})^2$. The theorem is of course merely the scalar form of our relation (67').

or

$$\begin{aligned} \boldsymbol{\alpha}' &= \boldsymbol{\alpha}\cos^2\theta + 2\boldsymbol{\mu}\sin\theta\cos\theta + \boldsymbol{\beta}\sin^2\theta, \\ \boldsymbol{\beta}' &= \boldsymbol{\beta}\cos^2\theta - 2\boldsymbol{\mu}\sin\theta\cos\theta + \boldsymbol{\alpha}\sin^2\theta, \\ \boldsymbol{\mu}' &= \boldsymbol{\mu}(\cos^2\theta - \sin^2\theta) - (\boldsymbol{\alpha} - \boldsymbol{\beta})\sin\theta\cos\theta. \end{aligned} \tag{88'}$$

Hence if we write

$$\begin{aligned} \mathbf{h} &= \tfrac{1}{2}(\boldsymbol{\alpha} + \boldsymbol{\beta}) \text{ and } \boldsymbol{\delta} = \tfrac{1}{2}(\boldsymbol{\alpha} - \boldsymbol{\beta}), \\ \boldsymbol{\alpha}' &= \mathbf{h} + \boldsymbol{\mu}\sin 2\theta + \boldsymbol{\delta}\cos 2\theta, \\ \boldsymbol{\beta}' &= \mathbf{h} - \boldsymbol{\mu}\sin 2\theta - \boldsymbol{\delta}\cos 2\theta, \\ \boldsymbol{\mu}' &= \boldsymbol{\mu}\cos 2\theta - \boldsymbol{\delta}\sin 2\theta, \\ \boldsymbol{\delta}' &= \boldsymbol{\delta}\cos 2\theta + \boldsymbol{\mu}\sin 2\theta. \end{aligned} \tag{89}$$

35. **The indicatrix.** From equations (89) we infer that: *As* $\theta$ *changes, the extremity of* $\boldsymbol{\alpha}'$ *describes an ellipse of which* $\boldsymbol{\mu}$ *and* $\boldsymbol{\delta}$ *are conjugate radii and of which the center is given by* $\mathbf{h}$; *the extremity of* $\boldsymbol{\beta}$ *describes the same ellipse at the opposite end of the diameter from* $\boldsymbol{\alpha}$; *and* $\boldsymbol{\mu}'$, *laid off from the center of the ellipse, describes the same ellipse, each position of* $\boldsymbol{\mu}'$ *being conjugate to the line joining* $\boldsymbol{\alpha}'$ *and* $\boldsymbol{\beta}'$ *and advanced by the excentric angle* $\pi/2$ *from* $\boldsymbol{\alpha}'$ *toward* $\boldsymbol{\beta}'$. (The $\boldsymbol{\mu}$ that goes with the orthogonal trajectories is clearly $-\boldsymbol{\mu}$ as previously proved).

*The conic, which we thus get, lying in the normal space, may be called the* INDICATRIX. In four dimensions the whole figure including $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ lies in a plane, namely the normal plane; in higher dimensions the figure will not generally lie in a plane, the ellipse with the lines $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}'$ forming a conical surface lying in a normal three space. *No matter how many dimensions a surface may lie in, the properties of normal curvature at any particular point may be described in a 3-space; for such properties surfaces in more than five dimensions need not be discussed.*

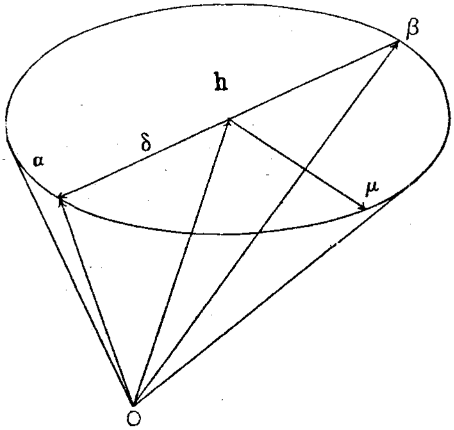

FIGURE 1.

The relation (69), that is, $\boldsymbol{\alpha}\cdot\boldsymbol{\beta} - \boldsymbol{\mu}^2 = G$, may be interpreted on our indicatrix. For

$$\boldsymbol{\alpha}\cdot\boldsymbol{\beta} = \mathbf{h}^2 - \boldsymbol{\delta}^2, \quad \mathbf{h}^2 - (\boldsymbol{\delta}^2 + \boldsymbol{\mu}^2) = G. \tag{90}$$

Now the sum $\boldsymbol{\delta}^2 + \boldsymbol{\mu}^2$ of the squares of two conjugate radii of an ellipse is constant and equal to $a^2 + b^2$, the sum of the squares of the semi-axes. Hence: *The Gaussian invariant G is the difference of the square of the mean curvature and the sum of the squares of the semi-axes of the indicatrix.*[34]

**36. Minimal surfaces.**[35] The vector element of area of a surface may be written as

$$\mathbf{P}dx_1dx_2 = \frac{\partial\mathbf{y}}{\partial x_1} \times \frac{\partial\mathbf{y}}{\partial x_2}\, dx_1dx_2.$$

To find the condition for a minimal surface we write

$$0 = \delta\int\int(\mathbf{P}\cdot\mathbf{P})^{\frac{1}{2}}dx_1dx_2 = \int\int\frac{\delta\mathbf{P}\cdot\mathbf{P}}{(\mathbf{P}\cdot\mathbf{P})^{\frac{1}{2}}}\, dx_1dx_2.$$

If **M** is the unit tangent plane as heretofore, the condition becomes

$$0 = \int\int\delta\mathbf{P}\cdot\mathbf{M}dx_1dx_2,$$

$$\delta\mathbf{P} = \frac{\partial\delta\mathbf{y}}{\partial x_1} \times \frac{\partial\mathbf{y}}{\partial x_2} - \frac{\partial\delta\mathbf{y}}{\partial x_2} \times \frac{\partial\mathbf{y}}{\partial x_1}.$$

We have to integrate two terms by parts, one of which is

$$\int\int\frac{\partial\delta\mathbf{y}}{\partial x_1} \times \frac{\partial\mathbf{y}}{\partial x_2}\cdot\mathbf{M}dx_1dx_2 = -\int\int\delta\mathbf{y} \times \frac{\partial}{\partial x_1}\left(\frac{\partial\mathbf{y}}{\partial x_2}\cdot\mathbf{M}\right)dx_1dx_2,$$

omitting the integrated term which vanishes at the limits; we have then

$$\int\int\delta\mathbf{y} \times \left[\frac{\partial}{\partial x_1}\left(\frac{\partial\mathbf{y}}{\partial x_2}\cdot\mathbf{M}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial\mathbf{y}}{\partial x_1}\cdot\mathbf{M}\right)\right]dx_1dx_2 = 0.$$

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[34] This result is stated by Levi, loc. cit., p. 71.

[35] For special developments on minimum surfaces see Levi, loc. cit., p. 90. Eisenhart, *Amer. J. Math.*, **34**, 215–236 (1912), where references to earlier work will be found.

As $\delta\mathbf{y}$ is arbitrary we infer that the condition for a minimal surface is

$$\frac{\partial}{\partial x_1}\left(\frac{\partial \mathbf{y}}{\partial x_2}\cdot\mathbf{M}\right)-\frac{\partial}{\partial x_2}\left(\frac{\partial \mathbf{y}}{\partial x_1}\cdot\mathbf{M}\right)=0.$$

The equation further simplifies to

$$\frac{\partial \mathbf{y}}{\partial x_2}\cdot\frac{\partial \mathbf{M}}{\partial x_1}-\frac{\partial \mathbf{y}}{\partial x_1}\cdot\frac{\partial \mathbf{M}}{\partial x_2}=0.$$

We may use (74) and (74′) to modify the results to

$$\begin{gathered}(\boldsymbol{\xi}\lambda_2+\boldsymbol{\eta}\bar{\lambda}_2)\cdot(\boldsymbol{\alpha}\times\boldsymbol{\eta}\lambda_1+\boldsymbol{\mu}\times\boldsymbol{\eta}\bar{\lambda}_1-\boldsymbol{\mu}\times\boldsymbol{\xi}\lambda_1-\boldsymbol{\beta}\times\boldsymbol{\xi}\bar{\lambda}_1)\\ -(\boldsymbol{\xi}\lambda_1+\boldsymbol{\eta}\bar{\lambda}_1)\cdot(\boldsymbol{\alpha}\times\boldsymbol{\eta}\lambda_2+\boldsymbol{\mu}\times\boldsymbol{\eta}\bar{\lambda}_2-\boldsymbol{\mu}\times\boldsymbol{\xi}\lambda_2-\boldsymbol{\beta}\times\boldsymbol{\xi}\bar{\lambda}_2)=0.\end{gathered}$$

When we multiply the equation out we find

$$(\boldsymbol{\alpha}+\boldsymbol{\beta})\,(\bar{\lambda}_1\lambda_2-\lambda_1\bar{\lambda}_2)=0.$$

The term $\bar{\lambda}_1\lambda_2-\lambda_1\bar{\lambda}_2$ cannot vanish because it is equal to $-\sqrt{a}$ as may readily be shown from the defining relations of $\lambda$ and $\bar{\lambda}$;

$$\lambda_1\bar{\lambda}_2-\bar{\lambda}_1\lambda_2=\sqrt{a}. \tag{91}$$

Hence the condition for a minimal surface is $\boldsymbol{\alpha}+\boldsymbol{\beta}=0$. Thus: *In any number of dimensions the condition for a minimal surface is that the mean curvature shall vanish at each point of the surface.*[36] This is the immediate generalization of the condition in three dimensions.

By reference to (78) we see that for a minimal surface, $d\mathbf{M}\cdot d\mathbf{M} = -G ds^2$. This relation in three dimensions is interpreted as showing that the spherical representation of a minimal surface is conformal: for $d\mathbf{M}\cdot d\mathbf{M}=d\mathbf{n}\cdot d\mathbf{n}$, $\mathbf{n}$ being a unit normal, and $d\mathbf{n}\cdot d\mathbf{n}$ is the differential of arc in the spherical representation. In higher dimensions we can merely say that: *The magnitude of $d\mathbf{M}/ds$ is the same for all directions through a point on a minimal surface.*

**37. The intersection of consecutive normals.** Let $\mathbf{N}$ be the unit normal space of $n-2$ dimensions at any point of the surface, and $\mathbf{r}$ a vector from that point. The equation of the normal space is

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[36] This result has been stated by Levi, loc. cit.

$\mathbf{r}\times\mathbf{N} = 0$. If $d\mathbf{r} = \boldsymbol{\xi}ds$ is an infinitesimal displacement along the surface and $\mathbf{N} + d\mathbf{N}$ the normal at its extremity, the equation of the adjacent normal space becomes

$$(\mathbf{r} - d\mathbf{r})\times(\mathbf{N} + d\mathbf{N}) = 0 \quad\text{or}\quad \mathbf{r}\times d\mathbf{N} - d\mathbf{r}\times\mathbf{N} = 0.$$

The intersection of the two normal spaces is determined by the simultaneous equations

$$\mathbf{r}\times\mathbf{N} = 0, \quad \mathbf{r}\times\frac{d\mathbf{N}}{ds} - \boldsymbol{\xi}\times\mathbf{N} = 0.$$

If we take complements we may write these equations

$$\mathbf{r}\cdot\mathbf{M} = 0, \quad \mathbf{r}\cdot\frac{d\mathbf{M}}{ds} - \boldsymbol{\xi}\cdot\mathbf{M} = 0. \tag{92}$$

The first equation merely states that $\mathbf{r}$ is perpendicular to $\mathbf{M}$ and we shall therefore consider only such values of $\mathbf{r}$ in the second equation. From (73), (80), (82), we have,

$$\mathbf{r}\cdot(\boldsymbol{\alpha} + \gamma\boldsymbol{\eta})\times\boldsymbol{\eta} + \boldsymbol{\xi}\times(\boldsymbol{\mu} - \gamma\boldsymbol{\xi}) - \boldsymbol{\xi}\cdot(\boldsymbol{\xi}\times\boldsymbol{\eta}) = 0,$$

or

$$\mathbf{r}\cdot(\boldsymbol{\alpha}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\mu}) + \boldsymbol{\eta} = 0,$$

$$-\,\mathbf{r}\cdot\boldsymbol{\alpha}\boldsymbol{\eta} + \mathbf{r}\cdot\boldsymbol{\mu}\boldsymbol{\xi} = -\,\boldsymbol{\eta}.$$

Hence

$$\mathbf{r}\cdot\boldsymbol{\alpha} = 1, \quad \mathbf{r}\cdot\boldsymbol{\mu} = 0. \tag{92$'$}$$

*Special Cases.* Consider first the case $n = 4$. Here the indicatrix is a conic in a plane through the surface-point $O$. The vector $\boldsymbol{\alpha}$ runs from $O$ to a point of this conic. If we lay off from $O$ the radius of curvature instead of the curvature itself, we get a point $Q$ which is the inverse of $P$ with respect to $O$. The locus of $Q$ is therefore a bicircular quartic. If we draw through $Q$ a line perpendicular to $\boldsymbol{\alpha}$, we have a line for which $\mathbf{r}\cdot\boldsymbol{\alpha} = 1$; and the point where this line cuts the perpendicular from $O$ upon $\boldsymbol{\mu}$, or upon the tangent to the indicatrix at $P$, is a point $P'$ which is the common solution of (92′) and which therefore is the point of intersection of the normal plane $\mathbf{N}$ with the adjacent normal plane in the direction $\boldsymbol{\xi}$.

If we consider the triangles $OPM$ and $OQP'$ we see that $OM\cdot OP' = OQ\cdot OP = 1$. Hence $P'$ and $M$ are inverse points. But the locus of $M$ is the pedal of the indicatrix and hence we have the theorem: *The inverse of the pedal of the indicatrix is the locus of points where consecutive normal planes about a point intersect the normal plane at the point.*[37]

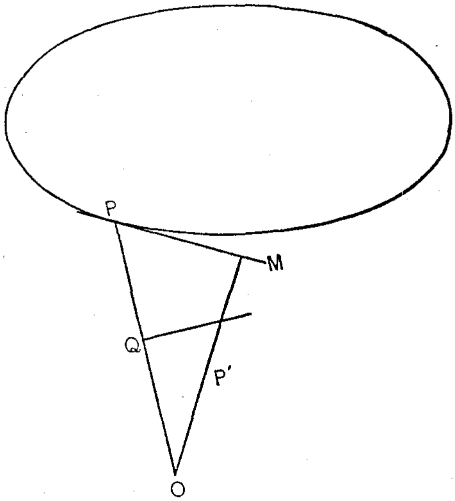

FIGURE 2.

Consider next the case $n = 5$. Here the indicatrix is a conic which may or may not lie in a plane through $O$. In the latter special case the reasoning before holds except for the fact that the solution for $\mathbf{r}$ in (92′) is no longer a point, but a line through that point perpendicular to the plane of the conic. The locus of intersection of consecutive normal spaces is therefore a right cylinder of which the directrix is the conic which is the inverse of the pedal of the indicatrix. This is merely a direct extension of the case previously treated.

*The general case.* If the indicatrix does not lie in a plane with $O$, and if we lay off along $\boldsymbol{\alpha}$ the distance equal to the radius of curvature, instead of equal to the curvature, we get a point $Q$ which lies both on the cone determined by $O$ as vertex and the indicatrix as directrix and on the sphere through $O$ which is the inverse of the plane of the indicatrix. The locus of $Q$ is therefore a sphero-conic. The plane $\mathbf{r}\cdot\boldsymbol{\alpha} = 1$ passes through the point $Q$ and is perpendicular to $\boldsymbol{\alpha}$; it therefore passes through the point $O'$ of the sphere diametrically opposite to $O$, this point $O'$ being also the inverse of the foot $F$ of the perpendicular $OF$ from $O$ upon the plane of the indicatrix.

Now $\mathbf{r}\cdot\boldsymbol{\mu} = 0$ is the plane through $O$ perpendicular to $\boldsymbol{\mu}$, and hence perpendicular to the plane of the indicatrix, and hence finally $\mathbf{r}\cdot\boldsymbol{\mu} = 0$ is a plane through the line $OF$. The intersection of $\mathbf{r}\cdot\boldsymbol{\mu} = 0$ and $\mathbf{r}\cdot\boldsymbol{\alpha} = 1$ is therefore a line through $O'$ perpendicular alike to $\boldsymbol{\mu}$ and $\boldsymbol{\alpha}$, and consequently perpendicular to the plane tangent to the cone (described by $\boldsymbol{\alpha}$) through the element $\boldsymbol{\alpha}$ (since $\boldsymbol{\mu}$ is parallel to the tangent to the indicatrix at the extremity of $\boldsymbol{\alpha}$).

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[37] Kommerell, loc. cit.

*Cones I and II.* We may now state that: *The locus of intersection of consecutive normal spaces* $N_3$ *generate a cone* (*thus all the consecutive normal spaces pass through a point*). *This cone is a quadric cone.* For if $\mathbf{r}\cdot\Phi^{-1}\cdot\mathbf{r} = 0$, where $\Phi^{-1}$ is a self conjugate dyadic, be the equation of the cone described by $\boldsymbol{\alpha}$, which we shall call Cone I, the normal to the tangent plane is determinable from the equation $\mathbf{r}\cdot\Phi^{-1}\cdot d\mathbf{r} = 0$ and $\mathbf{r}\cdot\Phi^{-1} = \mathbf{n}$ or $\mathbf{r} = \Phi\cdot\mathbf{n}$. Hence the locus of $\mathbf{n}$ is $\mathbf{n}\cdot\Phi\cdot\mathbf{n} = 0$, the reciprocal cone to Cone I. This reciprocal cone we shall call Cone II; its vertex is at $O'$, not at $O$. *As the Cones* I *and* II *are reciprocal, we can infer not only that the normals to the tangent planes to* I *generate* II *but that reciprocally the normals to the tangent planes to* II *generate* I. The special case previously treated where $O$ lies in the plane of the indicatrix falls under the general case because $O'$ has retreated to infinity and consequently Cone II becomes a cylinder.

In case $n > 5$ we may, if we desire, restrict ourselves to the normal space $N_3$ in which the indicatrix lies. We shall then have precisely the relations just proved for the case $n = 5$. But when $n > 5$ the equations (92′) have additional solutions in the rest of the total normal space $N_{n-2}$ external to the particular $N_3$. *The adjacent normal spaces* $N_{n-2}$ *intersect in a space* $N_{n-4}$ *which is perpendicular to* $N_3$ *and contains in* $N_3$ *an element in Cone* II.

**38. The fundamental dyadic $\Phi$.** The forms of $\Phi$ and $\Phi^{-1}$ which determine Cones II and I may be found in the general case as follows. Let $\mathbf{h}'$, $\boldsymbol{\mu}'$, $\boldsymbol{\delta}'$ be the reciprocal set to $\mathbf{h}$, $\boldsymbol{\mu}$, and $\boldsymbol{\delta}$, and consider

$$\Phi = c(\mathbf{hh} - \boldsymbol{\mu\mu} - \boldsymbol{\delta\delta}), \quad \Phi^{-1} = c^{-1}(\mathbf{h}'\mathbf{h}' - \boldsymbol{\mu}'\boldsymbol{\mu}' - \boldsymbol{\delta}'\boldsymbol{\delta}'),$$

where $c$ is any constant. The vectors $\mathbf{h} + \boldsymbol{\delta}$ lie on $\mathbf{r}\cdot\Phi^{-1}\cdot\mathbf{r} = 0$. But

$$(\mathbf{h} + \boldsymbol{\delta})\cdot(\mathbf{h}'\mathbf{h}' - \boldsymbol{\mu}'\boldsymbol{\mu}' - \boldsymbol{\delta}'\boldsymbol{\delta}')\cdot(\mathbf{h} + \boldsymbol{\delta}) = 0.$$

Moreover the expression $\boldsymbol{\mu\mu} + \boldsymbol{\delta\delta}$ is invariant of the system $\lambda$ as may be seen from (89) where the accents denote new values of $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$ not the reciprocal set as here. As $\mathbf{h}$ is independent of $\lambda$, the diadics $\Phi$ and $\Phi^{-1}$ are independent of $\lambda$ and any value of $\mathbf{h} + \boldsymbol{\delta}$ will satisfy $\mathbf{r}\cdot\Phi^{-1}\cdot\mathbf{r} = 0$. The expression written down for $\Phi$ is therefore correct.

The dyadic $\Phi$ may be expressed directly and simply in terms of the vector coefficients of the form $\Psi$. Consider the dyadic $\Omega$ which is the discriminant matrix of $\Psi$, namely,

$$\Omega = \left\| \begin{matrix} \mathbf{y}_{11} & \mathbf{y}_{12} \\ \mathbf{y}_{21} & \mathbf{y}_{22} \end{matrix} \right\| = \mathbf{y}_{11}\mathbf{y}_{22} - \mathbf{y}_{12}\mathbf{y}_{21}. \tag{93}$$

As $\mathbf{y}_{11}\mathbf{y}_{22} \neq \mathbf{y}_{22}\mathbf{y}_{11}$ this dyadic is not self conjugate.

$$\begin{aligned}\Omega &= [\boldsymbol{\alpha}\lambda_1^2 + 2\boldsymbol{\mu}\lambda_1\bar{\lambda}_1 + \boldsymbol{\beta}\bar{\lambda}_1^2][\boldsymbol{\alpha}\lambda_2^2 + 2\boldsymbol{\mu}\lambda_2\bar{\lambda}_2 + \boldsymbol{\beta}\bar{\lambda}_2^2] \\ &\quad - [\boldsymbol{\alpha}\lambda_1\lambda_2 + \boldsymbol{\mu}(\lambda_1\bar{\lambda}_2 + \lambda_2\bar{\lambda}_1) + \boldsymbol{\beta}\bar{\lambda}_1\bar{\lambda}_2]^2 \\ &= [-\boldsymbol{\mu\mu}(\lambda_1\bar{\lambda}_2 - \lambda_2\bar{\lambda}_1) + \boldsymbol{\alpha\beta}\lambda_1\bar{\lambda}_2 - \boldsymbol{\beta\alpha}\bar{\lambda}_1\lambda_2 \\ &\quad + (\boldsymbol{\alpha\mu} - \boldsymbol{\mu\alpha})\lambda_1\lambda_2 + (\boldsymbol{\mu\beta} - \boldsymbol{\beta\mu})\bar{\lambda}_1\bar{\lambda}_2](\lambda_1\bar{\lambda}_2 - \lambda_2\bar{\lambda}_1).\end{aligned}$$

The first term is self-conjugate and the last two are anti-self-conjugate.

$$\begin{aligned}\tfrac{1}{2}(\Omega + \Omega_C) &= [\tfrac{1}{2}(\boldsymbol{\alpha\beta} + \boldsymbol{\beta\alpha}) - \boldsymbol{\mu\mu}](\lambda_1\bar{\lambda}_2 - \lambda_2\bar{\lambda}_1)^2 \\ &= [\mathbf{hh} - \boldsymbol{\mu\mu} - \boldsymbol{\delta\delta}]a,\end{aligned}$$

by (91). We find therefore that: *The selfconjugate part of the vector matrix $\Omega$ is the dyadic $\Phi$ which defines Cone* II, with the multiplier $c = a$. We shall use for $\Phi$ the value

$$\Phi = [\mathbf{hh} - \boldsymbol{\mu\mu} - \boldsymbol{\delta\delta}]a, \quad \text{Cone II}, \tag{94}$$

including the multiplier $a$; and for $\Phi^{-1}$,

$$\Phi^{-1} = [\mathbf{h}'\mathbf{h}' - \boldsymbol{\mu}'\boldsymbol{\mu}' - \boldsymbol{\delta}'\boldsymbol{\delta}']a^{-1}, \quad \text{Cone I}. \tag{94'}$$

The value of the scalar invariant $\Omega_S$ of the dyadic $\Omega$ and of the self-conjugate part of $\Omega$ are the same. Hence,

$$\Omega_S = \Phi_S = \mathbf{y}_{11}\cdot\mathbf{y}_{22} - \mathbf{y}_{12}^2 = (\mathbf{h}^2 - \boldsymbol{\mu}^2 - \boldsymbol{\delta}^2)a = Ga. \tag{95}$$

We have therefore the result that: *The Gaussian curvature $G$ is*

$$G = \frac{\Omega_S}{a} = \frac{\mathbf{y}_{11}\cdot\mathbf{y}_{22} - \mathbf{y}_{12}^2}{a}, \tag{95'}$$

the quotient of the scalar of the matrix of the second fundamental form by the discriminant of the first fundamental form, in complete analogy with the result in three dimensions which expresses $G$ as the quotient of the determinants of the two fundamental forms. It has already been seen that the mean curvature $\mathbf{h}$ is expressed as

$$2\mathbf{h} = \Sigma_{ij}a^{(ij)}\mathbf{y}_{ij}$$

in conformity with the expression for the analogous quantity in three dimensions.

The plane $\boldsymbol{\delta}\times\boldsymbol{\mu}$ of the indicatrix is polar to $\mathbf{h}$ with respect to Cone I, that is, it is perpendicular to $\Phi^{-1}\cdot\mathbf{h}$, as may also be seen by direct substitution in (94′).

The second fundamental forms $\psi_i$ of the projection of the surface on any 3-space containing the tangent plane **M** at a point and some normal $\mathbf{z}_i$ is $\psi_i = \mathbf{z}_i\cdot\Psi$. If we write,

$$\Psi = \Sigma\mathbf{z}_i\psi_i = \Sigma\mathbf{z}_i\mathbf{z}_i\cdot\Psi = \mathrm{I}\cdot\Psi,$$

*the mean curvature of the surface is seen to be the vector sum of the mean curvatures of the projections on the spaces determined successively by* $\mathbf{z}_i$, namely,

$$2\mathbf{h} = \Sigma_{ij}a^{(ij)}\mathbf{y}_{ij} = \Sigma_{ijk}(a^{(ij)}\mathbf{y}_{ij}\cdot\mathbf{z}_k)\mathbf{z}_k.$$

There is no need of letting $k$ vary over more than the values 1, 2, 3, as curvature phenomena are five dimensional. The expression for $G$ may be written

$$aG = \Omega_S = \Omega{:}\mathrm{I} = \Phi{:}\mathrm{I} = \mathbf{y}_{11}\cdot\mathrm{I}\cdot\mathbf{y}_{22} - \mathbf{y}_{12}\cdot\mathrm{I}\cdot\mathbf{y}_{21}$$
$$= \Sigma_k(\mathbf{y}_{11}\cdot\mathbf{z}_k\mathbf{z}_k\cdot\mathbf{y}_{22} - \mathbf{y}_{12}\cdot\mathbf{z}_k\mathbf{z}_k\cdot\mathbf{y}_{21}).$$

As the individual parentheses here are the values of $G$ for the projections of the surface it shows that: *The total curvature of a surface is the algebraic sum of the total curvatures of the orthogonal projections of the surface.*

Since $aG = \Phi{:}(\mathbf{z}_1\mathbf{z}_1 + \mathbf{z}_2\mathbf{z}_2 + \mathbf{z}_3\mathbf{z}_3)$ we may reduce $aG$ to a single term by choosing $\mathbf{z}_2$ and $\mathbf{z}_3$ on the cone $\mathbf{r}\cdot\Phi\cdot\mathbf{r} = 0$, i. e., upon Cone II. Then $aG = \mathbf{z}_1\cdot\Phi\cdot\mathbf{z}_1$. As $aG = \Phi_S$, this relation may be written as,

$$\Phi_S\mathbf{z}_1\cdot\mathbf{z}_1 = \mathbf{z}_1\cdot\Phi\cdot\mathbf{z}_1$$

or

$$\mathbf{z}_1\cdot(\Phi_S\mathrm{I} - \Phi)\cdot\mathbf{z}_1 = 0.$$

We therefore have another cone,

$$\mathbf{r}\cdot(\Phi_S\mathrm{I} - \Phi)\cdot\mathbf{r} = 0, \qquad \text{Cone III,} \qquad (96)$$

which is coaxial with Cones I and II and which has the property that if one normal $\mathbf{z}_i$ lies upon it, the other two may lie upon Cone II, and

$G$ will reduce to a single term. In other words: *It is possible in* $\infty^1$ *ways so to select three perpendicular normals* $\mathbf{z}_i$ *that one of the projections has the entire total curvature of the surface and the other two have zero total curvatures.*

If now the dyadics $\Phi$, $\Phi^{-1}$, $\Phi_S I - \Phi = \Xi$, be referred to their principal directions,

$$\begin{aligned} \Phi^{-1} &= \frac{\mathbf{ii}}{a^2} + \frac{\mathbf{jj}}{b^2} - \frac{\mathbf{kk}}{c^2}, \\ \Phi &= a^2\mathbf{ii} + b^2\mathbf{jj} - c^2\mathbf{kk}, \qquad (97) \\ \Xi &= (b^2 - c^2)\mathbf{ii} + (a^2 - c^2)\mathbf{jj} + (a^2 + b^2)\mathbf{kk}, \end{aligned}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, stand for $\mathbf{z}_1$, $\mathbf{z}_2$, $\mathbf{z}_3$. If $c^2$ is less than $a^2$ and $b^2$ the cone defined by $\Xi$ is not real and the above resolution of the surface is impossible in the real domain,— as must be expected when Cone II is so narrow as to have no vertical angles as great as 90°. In case $G = 0$, $\Phi_S = 0$ and Cones II and III coincide. Hence: *The condition* $G = 0$ *implies that Cone II is a cone circumscribed about a trirectangular trihedral angle.*

As Cone I is reciprocal to Cone II, Cone I in that case must be inscriptable in a trirectangular trihedral angle or: *when* $G = 0$, *the indicatrix must be tangent to three mutually perpendicular planes through the surface-point.* In the special case where the indicatrix lies in a plane through the surface point the condition requires that the conic subtend an angle of 90° at the surface point or that the surface point must lie upon a circle of radius $(a^2 + b^2)^{\frac{1}{2}}$ concentric with the indicatrix.

**39. The scalar invariants.** We have now interpretations for two fundamental invariants, $G$ and $h$, and the expressions of these invariants in terms of the coefficients $a_{ij}$ and $\mathbf{y}_{ij}$. The indicatrix and its position relative to the origin require for their determination, apart from the rotation in space, five invariant scalars as remarked by Levi (see note 27). The dyadic $\Phi$ has of course three invariants $\Phi_S$, $\Phi_{2S}$, $\Phi_3$ which are the coefficients in the characteristic equation for $\Phi$. Of these the last is $\Phi_3 = a^3(\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h})^2$ as in (101). *The geometrical meaning of* $\Phi_3$ *is, except for a factor, the square of the volume of the cone intercepted by the plane of the indicatrix from the infinite surface of Cone* I. Except for a factor this is Levi's invariant $\Delta_5$; $\Phi_S$ is his $\Delta_1$; and $h$ his $\Delta_2$.

The values of $\Phi_2$ and $\boldsymbol{\mu}\times\boldsymbol{\delta}$ may be found in terms of $\mathbf{y}_{ij}$ as follows. From (93) and (94),

$$\Phi = (\mathbf{h}\mathbf{h} - \boldsymbol{\mu}\boldsymbol{\mu} - \boldsymbol{\delta}\boldsymbol{\delta})a = \tfrac{1}{2}\mathbf{y}_{11}\mathbf{y}_{22} + \tfrac{1}{2}\mathbf{y}_{22}\mathbf{y}_{11} - \mathbf{y}_{12}\mathbf{y}_{21}, \tag{99}$$

$$\begin{aligned}\Phi_2 &= (\boldsymbol{\mu}\times\boldsymbol{\delta}\,\boldsymbol{\mu}\times\boldsymbol{\delta} - \boldsymbol{\delta}\times\mathbf{h}\,\boldsymbol{\delta}\times\mathbf{h} - \mathbf{h}\times\boldsymbol{\mu}\,\mathbf{h}\times\boldsymbol{\mu})a^2 \\ &= (-\tfrac{1}{4}\mathbf{y}_{22}\times\mathbf{y}_{11}\mathbf{y}_{22}\times\mathbf{y}_{11} + \tfrac{1}{2}\mathbf{y}_{11}\times\mathbf{y}_{12}\mathbf{y}_{12}\times\mathbf{y}_{22} + \tfrac{1}{2}\mathbf{y}_{12}\times\mathbf{y}_{22}\mathbf{y}_{11}\times\mathbf{y}_{12}),\end{aligned} \tag{100}$$

$$\Phi_3 = (\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h})^2a^3 = \tfrac{1}{4}(\mathbf{y}_{11}\times\mathbf{y}_{12}\times\mathbf{y}_{22})^2, \tag{101}$$

where $\Phi_2$ and $\Phi_3$ represent the Gibbs's double products.[38] Now

$$\Phi_2\times\mathbf{h} = (\boldsymbol{\mu}\times\boldsymbol{\delta}\,\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h})a^2, \qquad 2\mathbf{h} = \Sigma a^{(rs)}\mathbf{y}_{rs},$$

$$\Phi_2\times\mathbf{h} = (-\tfrac{1}{4}a^{(12)}\mathbf{y}_{22}\times\mathbf{y}_{11} + \tfrac{1}{4}a^{(11)}\mathbf{y}_{11}\times\mathbf{y}_{12} + \tfrac{1}{4}a^{(22)}\mathbf{y}_{12}\times\mathbf{y}_{22})(\mathbf{y}_{11}\times\mathbf{y}_{12}\times\mathbf{y}_{22}).$$

Choose,

$$\pm\, a^{\frac{1}{2}}\boldsymbol{\mu}\times\boldsymbol{\delta} = -\tfrac{1}{2}a^{(12)}\mathbf{y}_{22}\times\mathbf{y}_{11} + \tfrac{1}{2}a^{(11)}\mathbf{y}_{11}\times\mathbf{y}_{12} + \tfrac{1}{2}a^{(22)}\mathbf{y}_{12}\times\mathbf{y}_{22}.$$

Then,

$$\begin{aligned}a\boldsymbol{\mu}\times\boldsymbol{\delta}\,\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h} = {}&(-\tfrac{1}{2}a^{(12)}\mathbf{y}_{22}\times\mathbf{y}_{11} + \tfrac{1}{2}a^{(11)}\mathbf{y}_{11}\times\mathbf{y}_{12} + \tfrac{1}{2}a^{(22)}\mathbf{y}_{12}\times\mathbf{y}_{22}) \\ &(-\tfrac{1}{2}a^{(12)2} + \tfrac{1}{4}a^{(22)}a^{(11)} + \tfrac{1}{4}a^{(11)}a^{(22)})\mathbf{y}_{11}\times\mathbf{y}_{12}\times\mathbf{y}_{22},\end{aligned}$$

and the result checks.

The double sign which arises here has come in through the extraction of a root. We may obtain from (87) the value of $\boldsymbol{\mu}\times\boldsymbol{\delta}$ as follows;

$$\boldsymbol{\mu}\times\boldsymbol{\delta} = \frac{1}{2}\Sigma_{rs}\bar{\lambda}^{(r)}\lambda^{(s)}\mathbf{y}_{rs} \times [\Sigma_{pq}(\lambda^{(p)}\lambda^{(q)} - \bar{\lambda}^{(p)}\bar{\lambda}^{(q)})\mathbf{y}_{pq}].$$

The coefficient of $\mathbf{y}_{22}\times\mathbf{y}_{11}$ is

$$\bar{\lambda}^{(2)}\lambda^{(2)}(\lambda^{(1)2} - \bar{\lambda}^{(1)2}) - \bar{\lambda}^{(1)}\lambda^{(1)}(\lambda^{(2)2} - \bar{\lambda}^{(2)2}),$$

which by virtue of (61′) and (63) reduces to $a_{12}/a^{\frac{3}{2}}$. The sign of the term is therefore plus. In like manner the sign of $\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h}$ may be determined. Hence

$$\begin{aligned}2a^{\frac{3}{2}}\boldsymbol{\mu}\times\boldsymbol{\delta} &= (a_{12}\mathbf{y}_{22}\times\mathbf{y}_{11} + a_{22}\mathbf{y}_{11}\times\mathbf{y}_{12} + a_{11}\mathbf{y}_{12}\times\mathbf{y}_{22}), \\ 2a^{\frac{3}{2}}\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h} &= \mathbf{y}_{11}\times\mathbf{y}_{12}\times\mathbf{y}_{22}.\end{aligned} \tag{102}$$

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[38] See Gibbs-Wilson, *Vector Analysis*, p. 306. As we are using the progressive product $\Phi_3 = \frac{1}{6}\,\Phi{}^{\times}_{\times}\Phi{}^{\times}_{\times}\Phi$ instead of $\frac{1}{6}\,\Phi{}^{\times}_{\times}\Phi : \Phi$. See also Wilson, *Trans. Conn. Acad., New Haven*, **14**, 1–57 (1908).

The conditions $\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h} = 0$ and $\mathbf{y}_{11}\times\mathbf{y}_{12}\times\mathbf{y}_{22}$ are therefore equivalent as was to be expected. If we use an orthogonal system of curves for the parameter curves, $a_{12} = 0$, and $\boldsymbol{\mu}\times\boldsymbol{\delta}$ may be factored. If we use a minimum system, $a_{11} = a_{22} = 0$, and $\boldsymbol{\mu}\times\boldsymbol{\delta}$ reduces to $\mathbf{y}_{11}\times\mathbf{y}_{22}$ except for a factor. In general $\boldsymbol{\mu}\times\boldsymbol{\delta}$ may be factored in $\infty^3$ ways of which one simple case is,

$$2a^{\frac{3}{2}}\boldsymbol{\mu}\times\boldsymbol{\delta} = \left(a_{12}\frac{\mathbf{y}_{11} + \mathbf{y}_{22}}{a_{11} + a_{22}} - \mathbf{y}_{12}\right) \times (a_{22}\mathbf{y}_{11} - a_{11}\mathbf{y}_{22}).$$

The vertex of Cone II is located at the point,

$$\boldsymbol{\nu} = \frac{(\boldsymbol{\mu}\times\boldsymbol{\delta})\cdot(\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h})}{(\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h})^2}, \tag{103}$$

which may be expressed in terms of the $\mathbf{y}$'s if desired.

The invariant $[\boldsymbol{\mu}\times\boldsymbol{\delta}]^2$ which is proportional to the square of the area of the indicatrix is except for a factor Levi's invariant $\Delta_4$. The invariant

$$\Phi_{2S} = \mathbf{y}_{11}\times\mathbf{y}_{12}\cdot\mathbf{y}_{12}\times\mathbf{y}_{22} - \frac{1}{4}[\mathbf{y}_{22}\times\mathbf{y}_{11}]^2 \tag{104}$$

is, except for a factor, Levi's invariant $\Delta_3$. We have geometric interpretations for all the invariants except $\Phi_{2S}$. If we write

$$\Phi_{2S}/a^2 = [\boldsymbol{\mu}\times\boldsymbol{\delta}]^2 - [\boldsymbol{\delta}\times\mathbf{h}]^2 - [\mathbf{h}\times\boldsymbol{\mu}]^2, \tag{104'}$$

we have $\frac{1}{2}[\boldsymbol{\mu}\times\boldsymbol{\delta}]$ interpretable as the area of the triangle of which the conjugate radii $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$ are sides; $\frac{1}{2}[\boldsymbol{\delta}\times\mathbf{h}]$ as the area of the triangle of which $\boldsymbol{\delta}$ and $\mathbf{h}$ or $\boldsymbol{\delta}$ and $\boldsymbol{\alpha}$ are sides; $\frac{1}{2}[\mathbf{h}\times\boldsymbol{\mu}]$ as the area of the triangle of which $\mathbf{h}$ and $\boldsymbol{\mu}$ are sides. As $[\boldsymbol{\mu}\times\boldsymbol{\delta}]^2$ is itself an invariant $[\boldsymbol{\mu}\times\boldsymbol{\delta}]^2 - \Phi_{2S}/a^2$ is an invariant and is equal to four times the sum of the squares of the areas of the triangles on $\boldsymbol{\delta}$ and $\mathbf{h}$ and on $\mathbf{h}$ and $\boldsymbol{\mu}$.

We can therefore set up the following list of five scalar invariants,[39]

$$\mathbf{h}^2, \quad \Phi_S/a = G, \quad [\boldsymbol{\mu}\times\boldsymbol{\delta}]^2, \quad [\boldsymbol{\mu}\times\boldsymbol{\delta}]^2 - \Phi_{2S}/a^2, \quad [\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h}]^2.$$

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[39] To aid the reader to make the comparison between our notation and Levi's we give the following table of equivalents for his symbols $I$ and $J$.

$$I_{1010} = a_{11}, \quad I_{1001} = a_{12}, \quad I_{0101} = a_{22},$$
$$J_{2020} = \mathbf{y}_{11}{}^2, \quad J_{1111} = \mathbf{y}_{12}{}^2, \quad J_{0202} = \mathbf{y}_{22}{}^2,$$
$$J_{2002} = \mathbf{y}_{11}\cdot\mathbf{y}_{22}, \quad J_{2021} = \mathbf{y}_{11}\cdot\mathbf{y}_{12}, \quad J_{0211} = \mathbf{y}_{22}\cdot\mathbf{y}_{12}.$$

The condition on the surface due to the vanishing of these invariants is as follows:

(1) $\mathbf{h}^2 = 0$, minimum surfaces (§36),

(2) $G = 0$, developable surfaces (§41),

(3) $[\boldsymbol{\mu}\times\boldsymbol{\delta}]^2 - \Phi_{2S}/a^2 = 0$, surfaces with what Levi calls axial points, viz., three dimensional surfaces or surfaces formed by the tangents to a twisted curve (§43).

(4) $[\boldsymbol{\mu}\times\boldsymbol{\delta}]^2 = 0$, surfaces with perpendicular (Segre) characteristics — the indicatrix reduces to a linear segment — the simplest generalization of ordinary surfaces (§43).

(5) $[\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h}]^2 = 0$, surfaces possessing (Segre) characteristics — surfaces with what Levi calls planar points (§43).

Conditions (1), (3), (4) imply (5); condition (3) implies (4). We have already discussed minimum surfaces briefly; we shall take up the other types in some detail in later sections.

## CHAPTER III. SPECIAL DEVELOPMENTS IN SURFACE THEORY.

**40. The twisted curve surfaces and ruled surfaces.** As an illustration and application of the foregoing analysis, we may treat the case of the surface formed by the tangents to a twisted curve in $n$ dimensions. Let $\mathbf{y} = \mathbf{f}(u)$ be the equation of the curve, $u$ being the arc. The surface is then,

$$\begin{aligned} \mathbf{y} &= \mathbf{f}(u) + v\mathbf{f}'(u), \quad \mathbf{f}'\cdot\mathbf{f}' = 1, \quad \mathbf{f}'\cdot\mathbf{f}'' = 0, \\ d\mathbf{y} &= (\mathbf{f}' + v\mathbf{f}'')du + \mathbf{f}'dv, \\ ds^2 &= d\mathbf{y}\cdot d\mathbf{y} = (\mathbf{f}' + v\mathbf{f}'')^2du^2 + 2du\,dv + dv^2 \\ &= (1 + v^2/R^2)du^2 + 2du\,dv + dv^2, \end{aligned}$$

where $R = (\mathbf{f}''\cdot\mathbf{f}'')^{-\frac{1}{2}}$ is the radius of curvature of the curve. That *the surface is developable* follows from the familiar argument, namely: $ds$ does not depend upon the torsion of the curve and hence the surface is applicable upon the tangent surface to all curves for which $R$ is the same function of $u$, and a plane curve can be found satisfying this condition.

To calculate $\Psi$ two methods are available based on (75) and (76). The advantage of the first form (75) is that the expression $d\mathbf{y}\times d\mathbf{M}$ may be replaced by $(d\mathbf{y}\times dU\mathbf{M})/U$, where $U$ is any scalar function;—since $d\mathbf{y}$ lies in $\mathbf{M}$ and $d\mathbf{y}\times\mathbf{M} = 0$. Now

$$\begin{aligned} \mathbf{M} &= \frac{(\mathbf{f}' + v\mathbf{f}'')\times\mathbf{f}'}{\sqrt{[(\mathbf{f}' + v\mathbf{f}'')\times\mathbf{f}']^2}} = \frac{\mathbf{f}''\times\mathbf{f}'}{\sqrt{\mathbf{f}''\cdot\mathbf{f}''}} = R\mathbf{f}''\times\mathbf{f}', \\ d\mathbf{y}\times d\mathbf{M} &= R(\mathbf{f}' + v\mathbf{f}'')\times\mathbf{f}'''\times\mathbf{f}'du, \\ &= Rv\mathbf{f}''\times\mathbf{f}'''\times\mathbf{f}'du = -\mathbf{M}\times\Psi. \end{aligned}$$

Multiply by $\mathbf{M}$ as in the text and repeat the argument there given. Then

$$\begin{aligned} -\Psi &= R^2v(\mathbf{f}''\times\mathbf{f}')\cdot(\mathbf{f}''\times\mathbf{f}'''\times\mathbf{f}')du^2 \\ &= R^2v[\mathbf{f}''\cdot\mathbf{f}'''\mathbf{f}'' - \mathbf{f}''\cdot\mathbf{f}''\mathbf{f}''' + \mathbf{f}''\cdot\mathbf{f}''\mathbf{f}'\cdot\mathbf{f}'''\mathbf{f}']du^2, \\ \Psi &= v[\mathbf{f}''' + \mathbf{f}'/R^2 - \mathbf{f}''\cdot\mathbf{f}'''\mathbf{f}'']du^2, \end{aligned}$$

since

$$\mathbf{f}'\cdot\mathbf{f}''' + \mathbf{f}''\cdot\mathbf{f}'' = 0.$$

Hence comparing with $\Psi = \Sigma \mathbf{y}_{rs} dx_r dx_s$,

$$\mathbf{y}_{11} = v[\mathbf{f}''' + \mathbf{f}'/R^2 - \mathbf{f}''\cdot\mathbf{f}'''\mathbf{f}'']$$

$$\mathbf{y}_{12} = 0, \qquad \mathbf{y}_{22} = 0.$$

Also, comparing $ds^2$ with its standard form,

$$a_{11} = 1 + v^2/R^2, \quad a_{21} = 1, a_{22} = 1, \quad a = v^2/R^2,$$

$$a^{(11)} = R^2/v^2, \quad a^{(12)} = -R^2/v^2, \quad a^{(22)} = 1 + R^2/v^2,$$

$$2\mathbf{h} = \Sigma a^{(rs)}\mathbf{y}_{rs} = R^2 v^{-1}[\mathbf{f}''' + \mathbf{f}'/R^2 - \mathbf{f}''\cdot\mathbf{f}''\mathbf{f}''],$$

$$\Phi = \tfrac{1}{2}(\mathbf{y}_{11}\mathbf{y}_{22} + \mathbf{y}_{22}\mathbf{y}_{11}) - \mathbf{y}_{12}\mathbf{y}_{21} = 0.$$

The dyadic $\Phi$ vanished identically. Hence $\mathbf{hh} = \boldsymbol{\mu\mu} + \boldsymbol{\delta\delta}$, and $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$ must be collinear with $\mathbf{h}$. *The indicatrix for a twisted curve surface reduces to a line along the vector* $\mathbf{h}$, *extending from the surface* (vertex of degenerate Cone I) *to the end of* $2\mathbf{h}$. As $\Phi_S = 0$, the condition $G = 0$, is satisfied, as must be the case from the reasoning given at the outset.

By a similar method we may calculate the various quantities arising in the case of any surface expressed in parametric form as $\mathbf{y} = \mathbf{y}(u, v)$. Let

$$d\mathbf{y} = \mathbf{m}du + \mathbf{n}dv, \quad \mathbf{m} = \partial\mathbf{y}/\partial u, \quad \mathbf{n} = \partial\mathbf{y}/\partial v;$$

$$ds^2 = d\mathbf{y}\cdot d\mathbf{y} = \mathbf{m}^2du^2 + 2\mathbf{m}\cdot\mathbf{n}dudv + \mathbf{n}^2dv^2;$$

$$a_{11} = \mathbf{m}^2, \quad a_{12} = \mathbf{m}\cdot\mathbf{n}, \quad a_{22} = \mathbf{n}^2,$$

$$a = a_{11}a_{22} - a_{12}{}^2 = \mathbf{m}^2\mathbf{n}^2 - (\mathbf{m}\cdot\mathbf{n})^2 = (\mathbf{m}\times\mathbf{n})^2;$$

$$\mathbf{M} = \frac{\mathbf{m}\times\mathbf{n}}{\sqrt{a}} d\mathbf{y}\times d\mathbf{M} = -\mathbf{m}\times\mathbf{n}\times\frac{d\mathbf{m}du + d\mathbf{n}dv}{\sqrt{a}}.$$

Let $$d\mathbf{m} = \mathbf{p}du + \mathbf{q}dv, \quad d\mathbf{n} = \mathbf{q}du + \mathbf{r}dv,$$

$$\mathbf{p} = \partial^2\mathbf{y}/\partial u^2, \quad \mathbf{q} = \partial^2\mathbf{y}/\partial u\partial v, \quad \mathbf{r} = \partial^2\mathbf{y}/\partial v^2;$$

$$\Psi = a^{-1}(\mathbf{m}\times\mathbf{n})\cdot[(\mathbf{m}\times\mathbf{n})\times(\mathbf{p}du^2 + 2\mathbf{q}dudv + \mathbf{r}dv^2)];$$

$$y_{11} = a^{-1}(\mathbf{m}\times\mathbf{n})\cdot(\mathbf{m}\times\mathbf{n}\times\mathbf{p}), \quad y_{12} = a^{-1}(\mathbf{m}\times\mathbf{n})\cdot(\mathbf{m}\times\mathbf{n}\times\mathbf{q}),$$

$$y_{22} = a^{-1}(\mathbf{m}\times\mathbf{n})\cdot(\mathbf{m}\times\mathbf{n}\times\mathbf{r}).$$

The expansion of the products gives expressions like

$$(\mathbf{m}\times\mathbf{n})\cdot(\mathbf{m}\times\mathbf{n}\times\mathbf{p}) = \begin{vmatrix} \mathbf{m} & \mathbf{n} & \mathbf{p} \\ \mathbf{m}^2 & \mathbf{m}\cdot\mathbf{n} & \mathbf{m}\cdot\mathbf{p} \\ \mathbf{m}\cdot\mathbf{n} & \mathbf{n}^2 & \mathbf{n}\cdot\mathbf{p} \end{vmatrix};$$

such an expression represents the component of $\mathbf{p}$ perpendicular to $\mathbf{m}\times\mathbf{n}$ multiplied by the square of $\mathbf{m}\times\mathbf{n}$. The product

$$[(\mathbf{m}\times\mathbf{n})\cdot(\mathbf{m}\times\mathbf{n}\times\mathbf{p})]\cdot[(\mathbf{m}\times\mathbf{n})\cdot(\mathbf{m}\times\mathbf{n}\times\mathbf{r})] = (\mathbf{m}\times\mathbf{n})^2(\mathbf{m}\times\mathbf{n}\times\mathbf{p})\cdot(\mathbf{m}\times\mathbf{n}\times\mathbf{r}).$$

Hence

$$\Phi_S = (\mathbf{m}\times\mathbf{n}\times\mathbf{p})\cdot(\mathbf{m}\times\mathbf{n}\times\mathbf{r}) - (\mathbf{m}\times\mathbf{n}\times\mathbf{q})^2 = G\,a, \qquad (106)$$

$$Ga = \begin{vmatrix} \mathbf{m}^2 & \mathbf{m}\cdot\mathbf{n} & \mathbf{m}\cdot\mathbf{r} \\ \mathbf{m}\cdot\mathbf{n} & \mathbf{n}^2 & \mathbf{n}\cdot\mathbf{r} \\ \mathbf{m}\cdot\mathbf{p} & \mathbf{n}\cdot\mathbf{p} & \mathbf{p}\cdot\mathbf{r} \end{vmatrix} - \begin{vmatrix} \mathbf{m}^2 & \mathbf{m}\cdot\mathbf{n} & \mathbf{m}\cdot\mathbf{q} \\ \mathbf{m}\cdot\mathbf{n} & \mathbf{n}^2 & \mathbf{n}\cdot\mathbf{q} \\ \mathbf{m}\cdot\mathbf{q} & \mathbf{n}\cdot\mathbf{q} & \mathbf{q}^2 \end{vmatrix}. \qquad (106')$$

If the surface is a ruled surface the form

$$\mathbf{y} = \mathbf{f}(u) + v\mathbf{g}(u)$$

is a possible parametric form. Then

$$\mathbf{m} = \mathbf{f}' + v\mathbf{g}', \quad \mathbf{n} = \mathbf{g}, \quad \mathbf{q} = \mathbf{g}', \quad \mathbf{r} = 0,$$

$$Ga = -(\mathbf{m}\times\mathbf{n}\times\mathbf{q})^2 = -(\mathbf{f}'\times\mathbf{g}\times\mathbf{g}')^2.$$

Hence: *The total curvature of any ruled surface with real rulings is negative.* If the surface is developable, i. e., if $G = 0$, we have $\mathbf{f}'\times\mathbf{g}\times\mathbf{g}' = 0$ or $\mathbf{g}' = b\mathbf{f}' + c\mathbf{g}$, where $b$ and $c$ are functions of $u$ alone. Then,

$$\mathbf{M} = \frac{[\mathbf{f}' + v(b\mathbf{f}' + c\mathbf{g})]\times\mathbf{g} = (1+bv)\mathbf{f}'\times\mathbf{g}}{(1+bv)\,|\,\mathbf{f}'\times\mathbf{g}\,|} = \frac{\mathbf{f}'\times\mathbf{g}}{|\,\mathbf{f}'\times\mathbf{g}\,|}$$

is a function of the single variable $u$ and remains constant as $v$ changes, the tangent plane is tangent along the whole generator, and the surface is the tangent surface of a twisted curve. Hence: *All developable ruled surfaces are twisted curve surfaces.*

If the ruled surface is not developable we select as a simple canonical form that obtained by taking the directrix $\mathbf{y} = \mathbf{f}(u)$ orthogonal to the rulings and $u$ as the arc along this curve. Then,

$$\mathbf{f}'\cdot\mathbf{f}' = 1,\quad \mathbf{f}'\cdot\mathbf{g} = 0,\quad \mathbf{g}\cdot\mathbf{g} = 1,\quad \mathbf{g}\cdot\mathbf{g}' = 0,\quad \mathbf{f}'\cdot\mathbf{f}'' = 0,$$

$$\mathbf{m}\cdot\mathbf{n} = 0,\quad \mathbf{q}\cdot\mathbf{n} = 0,\quad \mathbf{n}^2 = 1,\quad a = \mathbf{m}^2;$$

$$\mathbf{y}_{11} = a^{-1}\begin{vmatrix} \mathbf{m} & \mathbf{n} & \mathbf{p} \\ \mathbf{m}^2 & 0 & \mathbf{m}\cdot\mathbf{p} \\ 0 & 1 & \mathbf{n}\cdot\mathbf{p} \end{vmatrix} = a^{-1}(-\,\mathbf{m}\cdot\mathbf{pm} + \mathbf{m}^2\mathbf{p} - \mathbf{m}^2\mathbf{n}\cdot\mathbf{pn}),$$

$$\mathbf{y}_{12} = a^{-1}\begin{vmatrix} \mathbf{m} & \mathbf{n} & \mathbf{q} \\ \mathbf{m}^2 & 0 & \mathbf{m}\cdot\mathbf{q} \\ 0 & 1 & 0 \end{vmatrix} = a^{-1}(\mathbf{m}^2\mathbf{q} - \mathbf{m}\cdot\mathbf{qm}),$$

$$\mathbf{y}_{22} = 0;$$

$$\Phi = \tfrac{1}{2}(\mathbf{y}_{11}\mathbf{y}_{22} + \mathbf{y}_{22}\mathbf{y}_{11}) - \mathbf{y}_{12}\mathbf{y}_{12}$$
$$= -\,a^{-2}(\mathbf{m}^2\mathbf{q} - \mathbf{m}\cdot\mathbf{qm})\,(\mathbf{m}^2\mathbf{q} - \mathbf{m}\cdot\mathbf{qm}).$$

Moreover, $a^{(11)} = 1/a,\quad a^{(12)} = 0,\quad a^{(22)} = 1,$

$$2\mathbf{h} = a^{-2}(-\,\mathbf{m}\cdot\mathbf{pm} + \mathbf{m}^2\mathbf{p} - \mathbf{m}^2\mathbf{n}\cdot\mathbf{pn}),$$

$$\Phi/a = \mathbf{hh} - \boldsymbol{\mu\mu} - \boldsymbol{\delta\delta},\quad \boldsymbol{\mu\mu} + \boldsymbol{\delta\delta} = \mathbf{hh} - \Phi/a.$$

$$\boldsymbol{\mu\mu} + \boldsymbol{\delta\delta} = \tfrac{1}{4}a^{-4}(-\,\mathbf{m}\cdot\mathbf{pm} + \mathbf{m}^2\mathbf{p} - \mathbf{m}^2\mathbf{n}\cdot\mathbf{pn})\,(-\mathbf{m}\cdot\mathbf{pm} + \mathbf{m}^2\mathbf{p} - \mathbf{m}^2\mathbf{n}\cdot\mathbf{pn})$$
$$+\,a^{-3}(\mathbf{m}^2\mathbf{q} - \mathbf{m}\cdot\mathbf{qm})\,(\mathbf{m}^2\mathbf{q} - \mathbf{m}\cdot\mathbf{qm}).$$

$$\boldsymbol{\mu}\times\boldsymbol{\delta} = \frac{-\,1}{2\mathbf{m}^7}(-\mathbf{m}\cdot\mathbf{pm} + \mathbf{m}^2\mathbf{p} - \mathbf{m}^2\mathbf{n}\cdot\mathbf{pn})\times(\mathbf{m}^2\mathbf{q} - \mathbf{m}\cdot\mathbf{qm}).$$

Hence $\boldsymbol{\mu}\times\boldsymbol{\delta}$ contains $\mathbf{h}$ and the indicatrix lies in a plane with the surface point, no matter how high the dimensionality of the space in which the surface lies. Hence: *A ruled surface is at each point of the four dimensional type and never of the general type,* i. e., *a ruled surface is made up of planar points,* in Levi's nomenclature.

The formulas will serve to investigate the whole surface. If we are interested only in the neighborhood of some ordinary point we may assume that the point lies on the trajectory $\mathbf{y} = \mathbf{f}(u)$, that is, $v = 0$. The formulas then simplify further; for

$$\mathbf{m} = \mathbf{f}',\quad \mathbf{p} = \mathbf{f}'',\quad \mathbf{m}\cdot\mathbf{p} = \mathbf{f}'\cdot\mathbf{f}'' = 0,\quad a = \mathbf{m}^2 = 1;$$

$$2\mathbf{h} = \mathbf{f}'' - \mathbf{g}\cdot\mathbf{f}''\mathbf{g},\quad \Phi = -\,(\mathbf{g}' - \mathbf{f}'\cdot\mathbf{g}'\mathbf{f}')\,(\mathbf{g}' - \mathbf{f}'\cdot\mathbf{g}'\mathbf{f}').$$

Moreover,[40]

$$\pm \boldsymbol{\mu} = \tfrac{1}{2}(\mathbf{f}'' - \mathbf{g}\cdot\mathbf{f}''\mathbf{g}), \quad \mp \boldsymbol{\delta} = \mathbf{g}' - \mathbf{f}'\cdot\mathbf{g}'\mathbf{f}',$$
$$\Phi_S = G = -(\mathbf{g}' - \mathbf{f}'\cdot\mathbf{g}'\mathbf{f}')^2 = -\boldsymbol{\delta}^2.$$

It is seen from these equations that: *The mean curvature* **h** *of a ruled surface is one-half the normal component of the curvature* **f**″ *of an orthogonal trajectory of the rulings; the indicatrix is a conic of which a pair of conjugate radii are the mean curvature* **h** *and the line* $\boldsymbol{\delta}$ *which is the normal component of* **g**′, *which gives the rate of turning of the rulings; the total curvature of the surface is the negative of the square of the normal component of* **g**′. *The ruled surface is a special type under the four-dimensional type in that the indicatrix passes through the surface point considered.* As the inverse of the pedal of an ellipse with respect to a point on the ellipse is a parabola, *the locus of points where consecutive normal planes (spaces) meet a given normal plane (space) is a parabola (parabolic cylinder) with its axis parallel to* **h**.

**41. Developable surfaces.** One particular parametric form for a general surface,

$$x = x, \quad y = y, \quad z_i = z_i(x, y),$$

which expresses the surface as the intersection of $n - 2$ cylinders $z_i = z_i(x, y)$, is often useful. In this case the vector coördinates of the surface and the differential element of arc are

$$\boldsymbol{\rho} = x\mathbf{i} + y\mathbf{j} + \Sigma z_i\mathbf{k}_i,$$
$$d\boldsymbol{\rho} = (\mathbf{i} + \Sigma \frac{\partial z_i}{\partial x}\mathbf{k}_i)dx + (\mathbf{j} + \Sigma \frac{\partial z_i}{\partial y}\mathbf{k}_i)dy, \tag{107}$$
$$d\boldsymbol{\rho}\cdot d\boldsymbol{\rho} = \left[1 + \Sigma\left(\frac{\partial z_i}{\partial x}\right)^2\right]dx^2 + 2\Sigma\frac{\partial z_i}{\partial x}\frac{\partial z_i}{\partial y}dxdy + \left[1 + \Sigma\left(\frac{\partial z_i}{\partial y}\right)^2\right]dy^2.$$

Let $p_i$, $q_i$ be the derivatives of $z_i$ with respect to $x$ and $y$. Then,

$$\mathbf{m} = \mathbf{i} + \Sigma p_i\mathbf{k}_i, \quad \mathbf{n} = \mathbf{j} + \Sigma q_i\mathbf{k}_i. \tag{107'}$$

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[40] The actual determination of a possible set of values for $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$ may be made when the values of $\boldsymbol{\mu\mu} + \boldsymbol{\delta\delta}$ and $\boldsymbol{\mu}\times\boldsymbol{\delta}$ are known. In this particular case $\boldsymbol{\mu}\times\boldsymbol{\delta} = -\frac{1}{2}\mathbf{c}\times\mathbf{d}$ where $\mathbf{c} = \mathbf{f}'' - \mathbf{g}'\cdot\mathbf{f}''\mathbf{g}$, $\mathbf{d} = \mathbf{g}' - \mathbf{f}'\cdot\mathbf{g}'\mathbf{f}'$ and $\mathbf{h} = \frac{1}{2}\mathbf{c}$. Then since $a = 1$, $\boldsymbol{\mu\mu} + \boldsymbol{\delta\delta} = \mathbf{hh} = \frac{1}{4}\mathbf{cc} + \mathbf{dd}$. If $\boldsymbol{\mu} = \mathbf{c}x$ and $\boldsymbol{\delta} = \mathbf{d}/x$, then $\boldsymbol{\mu\mu} + \boldsymbol{\delta\delta} = \frac{1}{4}x^2\mathbf{cc} + \mathbf{dd}/x^2$, and $x$ must be unity provided **c** and **d** have distinct directions as they must have since $\boldsymbol{\mu}\times\boldsymbol{\delta} \neq 0$.

If we use $r_i$, $s_i$, $t_i$, for the second derivatives of $z_i$ in accord with the usual notation, the quantities **p**, **q**, **r**, are

$$\mathbf{p} = \Sigma r_i \mathbf{k}_i, \quad \mathbf{q} = \Sigma s_i \mathbf{k}_i, \quad \mathbf{r} = \Sigma t_i \mathbf{k}_i. \tag{107''}$$

With these values, $\mathbf{y}_{11}$, $\mathbf{y}_{12}$, $\mathbf{y}_{22}$, etc. may be calculated.

We shall at this point merely calculate, from (106′),

$$\begin{aligned}
\Phi_S = Ga = &\begin{vmatrix} 1+\Sigma p_i^2 & \Sigma p_i q_i & \Sigma p_i t_i \\ \Sigma p_i q_i & 1+\Sigma q_i^2 & \Sigma q_i t_i \\ \Sigma p_i r_i & \Sigma q_i r_i & \Sigma r_i t_i \end{vmatrix} \\
&- \begin{vmatrix} 1+\Sigma p_i^2 & \Sigma p_i q_i & \Sigma p_i s_i \\ \Sigma p_i q_i & 1+\Sigma q_i^2 & \Sigma q_i s_i \\ \Sigma p_i s_i & \Sigma q_i s_i & \Sigma s_i^2 \end{vmatrix} \\
= \Sigma(r_i t_i - s_i^2) &\left[ \begin{vmatrix} 1+\Sigma p_i^2 & \Sigma p_i q_i \\ \Sigma p_i q_i & 1+\Sigma q_i^2 \end{vmatrix} - p_i^2(1+\Sigma q_i^2) \right. \\
&\left. - q_i^2(1+\Sigma p_i^2) + 2p_i q_i \Sigma p_i q_i \right] \qquad (108) \\
&+ \Sigma_{ij}(t_i r_j - s_i s_j)\left[- p_i p_j(1+\Sigma q_i^2) + p_i q_j \Sigma p_i q_i \right. \\
&\left. + q_i p_j \Sigma p_i q_i - q_i q_j(1+\Sigma p_i^2)\right].
\end{aligned}$$

In the particular case $n = 4$ where $i$ and $j$ run over the indices 1 and 2, the formula becomes,

$$\begin{aligned}
Ga = (r_1 t_1 - s_1^2)(1 + p_2^2 + q_2^2) + (r_2 t_2 - s_2^2)(1 + p_1^2 + q_1^2) \\
- (t_1 r_2 + r_2 t_1 - 2 s_1 s_2)(p_1 p_2 + q_1 q_2). \qquad (108')
\end{aligned}$$

The case $n > 4$ is much more complicated but consists of a sum of terms $rt - s^2$, with coefficients, and some supplementary terms.

If the surface is a twisted-curve surface its rulings will project into lines and hence each of its projections $z_i$ must be a twisted curve surface and the terms $rt - s^2$ vanish; but as $G = 0$ there are supplementary conditions to be satisfied, the condition in case $n = 4$ being

$$(t_1 r_2 + r_2 t_1 - 2 s_1 s_2)(p_1 p_2 + q_1 q_2) = 0.$$

But the surface may be developable, that is, $G = 0$, without making the individual terms $rt - s^2$ vanish. Indeed if in four dimensions we assume the projection $z_1 = z_1(x, y)$ at random, (108′) equated to zero becomes a partial differential equation of the second order for the other projection $z_2 = z_2(x,y)$, and any solution $z_2$ of this equation, taken with $z_1$, will define a developable surface in four dimensions. In case $n > 4$ we may assume at random $n - 3$ projections $z_1 = z_1(x, y)$, $z_2 = z_2(x, y), \ldots\ldots, z_{n-3} = z_{n-3}(x, y)$, and proceed to solve the differential equation obtained by setting (108) equal to zero for the projection $z_{n-2}$ which taken with the assumed $n - 3$ projections, will determine a developable. *In more than three dimensions developable 2-surfaces therefore are either* 1°, *ruled developables which are twisted curve surfaces, or* 2°, *non-ruled developable surfaces.*

As a particularly simple case of a non-ruled developable for $n = 4$ we may take

$$z_1 = \tfrac{1}{2}(x^2 + y^2), \qquad z_2 = xy.$$

This surface satisfies (108′) but the individual terms $rt - s^2$ do not vanish. If we turn the axes of $z_1$ and $z_2$ and of $x$ and $y$ through 45° in their respective planes and change the scale, the surface may take the form

$$z_1 = \tfrac{1}{2}x^2, \qquad z_2 = \tfrac{1}{2}y^2.$$

In this case each of the surfaces $z_1 = \frac{1}{2}x^2$ and $z_2 = \frac{1}{2}y^2$ taken as a three dimensional surface is developable. But the four dimensional surface is not a ruled surface. In other words the projections of a non-ruled developable may each be ruled developables. All surfaces of the type

$$z_1 = z_1(x), \qquad z_2 = z_2(y)$$

are developable, because the element of arc is

$$(1 + z_1'^2)dx^2 + (1 + z_2'^2)dy^2 = dX^2 + dY^2,$$
$$dX = \sqrt{1 + z_1'^2}\,dx, \quad dY = \sqrt{1 + z_2'^2}\,dy.$$

Such surfaces, however, are not in general ruled.

**42. Development of a surface about a point.** There is a great simplification in our formulas if we restrict ourselves to the neighborhood of a single point of the surface and take the tangent plane at that point as the $xy$-plane. (This is the method followed at length by Kommerell in the four dimensional case.) In general we have for the surface,

$$z_i = \tfrac{1}{2}(A_i x^2 + 2B_i xy + C_i y^2), \qquad i = 1, 2, \ldots n - 2,$$

up to infinitesimals of the third order. The $3n - 6$ constants are not geometrically independent because of the arbitrary choice of the direction of the axes in the $xy$-plane. There are only $3n - 7$ independent constants. There are $3(n-2) - 6 = 3n - 12$ degrees of freedom for a plane in $S_{n-3}$ and five degrees of freedom for an ellipse in the plane. The count of constants indicates, therefore, that the indicatrix may be any ellipse in the normal space. We may examine this proposition critically by reference to the general formulas (107), (107′), (107″).

For this case $\mathbf{m} = \mathbf{i}$, $\mathbf{n} = \mathbf{j}$, $a_{11} = 1$, $a_{12} = 0$, $a_{22} = 1$, $a = 1$.

$$a^{(11)} = 1, \quad a^{(12)} = 0, \quad a^{(22)} = 1, \quad \mathbf{p} = \Sigma A_i \mathbf{k}_i, \quad \mathbf{q} = \Sigma B_i \mathbf{k}_i,$$
$$\mathbf{r} = \Sigma C_i \mathbf{k}_i, \quad \mathbf{y}_{11} = \mathbf{p}, \quad \mathbf{y}_{12} = \mathbf{q}, \quad \mathbf{y}_{22} = \mathbf{r}, \quad 2\mathbf{h} = \Sigma(A_i + C_i)\mathbf{k}_i.$$

The center of the indicatrix may therefore be any point, and is the same point for any two surfaces for which $A_i + C_i$ are the same, $i = 1, 2, \ldots n - 2$. The plane $\boldsymbol{\mu}\times\boldsymbol{\delta}$ is determined by (102) as

$$2\boldsymbol{\mu}\times\boldsymbol{\delta} = \mathbf{y}_{12}\times(\mathbf{y}_{22} - \mathbf{y}_{11}) = \Sigma B_i \mathbf{k}_i \times \Sigma(C_j - A_j)\mathbf{k}_j.$$

As $A_j + C_j$ and $A_j - C_j$ are independent, $\boldsymbol{\mu}\times\boldsymbol{\delta}$ may be any plane in the normal $S_{n-2}$. The work may now be simplified by choosing $\mathbf{h}$ as the axis $z_1$ and by taking the axes $z_2$, $z_3$ in the space $\mathbf{h}\times\boldsymbol{\mu}\times\boldsymbol{\delta}$. The equations of the surface reduce to

$$z_1 = \tfrac{1}{2}[A_1 x^2 + 2B_1 xy + C_1 y^2], \qquad z_2 = \tfrac{1}{2}[A_2(x^2 - y^2) + 2B_2 xy],$$
$$z_3 = \tfrac{1}{2}[A_3(x^2 - y^2) + 2B_3 xy], \qquad z_i = 0, \qquad i = 4, 5, \ldots n - 2.$$

That $z_2$ and $z_3$ take these special forms is due to the fact (§38) that the mean curvature of each must vanish. A proper orientation of the $xy$ axes makes $B_1 = 0$. By properly choosing the axes $\mathbf{k}_2$, $\mathbf{k}_3$ we may make $B_2$ vanish. We have then as a canonical form for the surface,

$$z_1 = \tfrac{1}{2}[A_1 x^2 + C_1 y^2], \quad z_2 = \tfrac{1}{2}A_2(x^2 - y^2), \tag{109}$$
$$z_3 = \tfrac{1}{2}[A_3(x^2 - y^2) + 2B_3 xy], \quad z_i = 0, \quad i > 3.$$

Now, $2\mathbf{h} = (A_1 + C_1)\mathbf{k}_1$, $2\boldsymbol{\mu}\times\boldsymbol{\delta} = B_3\mathbf{k}_3\times[(C_1 - A_1)\mathbf{k}_1 - 2A_2\mathbf{k}_2]$.

If we set $A_1 = h + e$, $C_1 = h - e$, $A_2 = f$, $A_3 = A$, $B_3 = B$,

$$z_1 = \tfrac{1}{2}[h(x^2 + y^2) + e(x^2 - y^2)], \quad z_2 = \tfrac{1}{2}f(x^2 - y^2),$$
$$z_3 = \tfrac{1}{2}[A(x^2 - y^2) + 2Bxy], \quad z_i = 0, \quad i > 3. \tag{109′}$$

*This is a very useful standard form for the expansion of a surface near a given point.* Then $\boldsymbol{\mu}\times\boldsymbol{\delta} = B(e\mathbf{k}_1 + f\mathbf{k}_2)\times\mathbf{k}_3$, and only the ratio $e{:}f$ is effective in changing the plane $\boldsymbol{\mu}\times\boldsymbol{\delta}$. The equation therefore contains three constants after $\mathbf{h}$ and the plane $\boldsymbol{\mu}\times\boldsymbol{\delta}$ are satisfied, namely $A$, $B$, and $e$ or $f$, which may suffice to determine the indicatrix with its center and plane already fixed.

Using polar coordinates $(\rho, \theta)$ in the tangent plane, we have

$$z_1 = \tfrac{1}{2}\rho_2(h + e\cos 2\theta), \quad z_2 = \tfrac{1}{2}\rho^2 f\cos 2\theta, \tag{109''}$$
$$z_3 = \tfrac{1}{2}\rho^2(A\cos 2\theta + B\sin 2\theta), \quad z_i = 0, \quad i > 3.$$

The normal vector distance, of the surface curve in the direction $\theta$, above the tangent plane is therefore

$$\tfrac{1}{2}\rho^2[(h + e\cos 2\theta)\mathbf{k}_1 + f\cos 2\theta\mathbf{k}_2 + (A\cos 2\theta + B\,2\sin\theta)\mathbf{k}_3], \tag{110}$$

and the normal curvature $\boldsymbol{\alpha}$ is

$$\boldsymbol{\alpha} = (h + e\cos 2\theta)\mathbf{k}_1 + f\cos 2\theta\mathbf{k}_2 + (A\cos 2\theta + B\sin 2\theta)\mathbf{k}_3. \tag{111}$$

The vectors $\boldsymbol{\delta} = \boldsymbol{\alpha} - \mathbf{h}$ and $\boldsymbol{\mu}$, which is $\boldsymbol{\delta}$ advanced 45° in $\theta$, are

$$\boldsymbol{\delta} = (e\mathbf{k}_1 + f\mathbf{k}_2 + A\mathbf{k}_3)\cos 2\theta + B\mathbf{k}_3\sin 2\theta,$$
$$\boldsymbol{\mu} = -(e\mathbf{k}_1 + f\mathbf{k}_2 + A\mathbf{k}_3)\sin 2\theta + B\mathbf{k}_3\cos 2\theta. \tag{112}$$

The indicatrix reduces to a line when and only when $B = 0$ or $e = f = 0$. The former may be regarded as the general case. It appears then that $\boldsymbol{\delta}$ may describe any line in the normal $S_3$ and the range of $\boldsymbol{\delta}$ may be for any distance $(e^2 + f^2 + A^2)^{\frac{1}{2}}$ along that line. If the indicatrix does not reduce to a line, and if $u$, $v$ denote coordinates referred to the unit orthogonal vectors $\mathbf{k}_3$ and $\mathbf{k}' = (e\mathbf{k}_1 + f\mathbf{k}_2)/(e^2 + f^2)^{\frac{1}{2}}$, we have

$$u = A\cos 2\theta + B\sin 2\theta, \quad v = (e^2 + f^2)^{\frac{1}{2}}\cos 2\theta.$$

Let $e = \lambda\cos\varphi$, $f = \lambda\sin\varphi$. The plane $\boldsymbol{\mu}\times\boldsymbol{\delta}$ determines $\varphi$ but not $\lambda$. The equation of the indicatrix in its plane is then

$$\lambda^2 u^2 - 2\lambda Auv + (A^2 + B^2)v^2 = B^2\lambda^2. \tag{113}$$

Any ellipse may be written as

$$au^2 + 2buv + cv^2 = 1, \quad a > 0, \quad c > 0, \quad ac - b^2 > 0.$$

To determine the outstanding constants $\lambda$, $A$, $B$, so that the indicatrix takes this form we have merely to take

$$B^2 = \frac{1}{a}, \quad A^2 = \frac{b^2}{a(ac - b^2)}, \quad \lambda^2 = \frac{a}{ac - b^2},$$

and this choice is always possible. Hence we have shown that: *The indicatrix may be any ellipse or any segment of a straight line in the normal* $S_{n-2}$. (We are ordinarily more interested in the domain of reals than in the domain of complex numbers, and this theorem holds for reals.)

If we are working in the special case of four dimensions we have merely to set $f = 0$ throughout the work. The results are the same for the special case as for the general case,— the indicatrix may be any ellipse or segment of a line in the normal plane.

The dyadic $\Phi = (\mathbf{hh} - \boldsymbol{\mu\mu} - \boldsymbol{\delta\delta})a = \frac{1}{2}\mathbf{y}_{11}\mathbf{y}_{22} + \frac{1}{2}\mathbf{y}_{22}\mathbf{y}_{11} - \mathbf{y}_{12}\mathbf{y}_{21}$, is

$$\begin{gathered}\Phi = (h^2 - e^2)\mathbf{k}_1\mathbf{k}_1 - f^2\mathbf{k}_2\mathbf{k}_2 - (A^2 + \mathrm{B}^2)\mathbf{k}_3\mathbf{k}_3 \\ - fA(\mathbf{k}_2\mathbf{k}_3 + \mathbf{k}_3\mathbf{k}_2) - Ae(\mathbf{k}_1\mathbf{k}_3 + \mathbf{k}_3\mathbf{k}_1) - fe(\mathbf{k}_1\mathbf{k}_2 + \mathbf{k}_2\mathbf{k}_1), \quad (114) \\ \Phi_S = G = h^2 - e^2 - f^2 - A^2 - B^2.\end{gathered}$$

The vertex of Cone II is located at

$$\boldsymbol{\nu} = \frac{(\boldsymbol{\mu}\times\boldsymbol{\delta})\cdot(\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h})}{(\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h})^2} = \frac{B}{B}\,\frac{(f\mathbf{k}_1 - e\mathbf{k}_2)}{fh}, \qquad (115)$$

and retreats to infinity if $h$ vanishes unless special conditions are fulfilled. If $B = 0$ the vertex is indeterminate. The determinant of $\Phi$ reduces to $f^2h^2B^2$ and hence $\Phi$ becomes singular and Cone II degenerates when $f = 0$ or $h = 0$ or $B = 0$. It is however clear, from the equations (109′) of the surface, that if $fBh = 0$, the surface lies in four dimensions at the point considered. Hence for a true five dimensional surface in the neighborhood of a point, the indicatrix must be a true ellipse (cannot degenerate to a linear segment) in a plane not containing $\mathbf{h}$, and Cone II cannot degenerate nor its vertex retreat to infinity.

*Special cases.* It remains to discuss the case for a locally four dimensional surface

$$z_1 = \tfrac{1}{2}[h(x^2 + y^2) + e(x^2 - y^2)], \quad z_3 = \tfrac{1}{2}[A(x^2 - y^2) + 2Bxy].$$

Here $\boldsymbol{\mu}\times\boldsymbol{\delta} = Be\mathbf{k}_1\times\mathbf{k}_3$. This vanishes only when $B = 0$ or $e = 0$. As

$\boldsymbol{\delta} = (e\mathbf{k}_1 + A\mathbf{k}_3)\cos 2\theta + B\mathbf{k}_3 \sin 2\theta$, we see that $B = 0$ represents the general case. If $Be \neq 0$, the indicatrix (Conic I) is a true ellipse with central radius $\boldsymbol{\delta}$. Referred to its center the equation of the indicatrix is

$$e^2 z_3^2 - 2Aez_1z_3 + (A^2 + B^2)z_1^2 = B^2e^2,$$

as may be seen from (113). To find the locus of the intersection of consecutive normal planes we need the inverse of the pedal of the ellipse with respect to the origin. One observation may be made in advance: the conic (Conic II) which will be found must contain the origin in its interior.

The calculation of the inverse pedal may be carried through neatly by vectors. If $\Omega$ be the selfconjugate two dimensional dyadic that gives the conic referred to its center as $\boldsymbol{\delta}\cdot\Omega\cdot\boldsymbol{\delta} = 1$, $\Omega\cdot\boldsymbol{\delta}$ is the normal, and the equation of the tangent at $\boldsymbol{\delta}$, or at $\boldsymbol{\delta} + \mathbf{h}$ when referred to the origin, is

$$(\mathbf{r} - \mathbf{h} - \boldsymbol{\delta})\cdot\Omega\cdot\boldsymbol{\delta} = 0 \quad \text{or} \quad \mathbf{r}\cdot\Omega\cdot\boldsymbol{\delta} = 1 + \mathbf{h}\cdot\Omega\cdot\boldsymbol{\delta},$$

where $\mathbf{r}$ is the radius vector. Hence

$$\frac{\mathbf{r}\cdot\Omega\cdot\boldsymbol{\delta}}{1 + \mathbf{h}\cdot\Omega\cdot\boldsymbol{\delta}} = 1, \quad \text{and} \quad \mathbf{p} = \frac{\Omega\cdot\boldsymbol{\delta}}{1 + \mathbf{h}\cdot\Omega\cdot\boldsymbol{\delta}}$$

is the radius vector of the inverse of the pedal. Then

$$\boldsymbol{\delta} = \frac{\Omega^{-1}\cdot\mathbf{p}}{1 - \mathbf{h}\cdot\mathbf{p}} \quad \text{and} \quad \frac{\mathbf{p}\cdot\Omega^{-1}\cdot\mathbf{p}}{(1 - \mathbf{h}\cdot\mathbf{p})^2} = 1.$$

The inverse of the pedal is therefore,

$$\mathbf{p}\cdot\Omega^{-1}\cdot\mathbf{p} = 1 - 2\mathbf{h}\cdot\mathbf{p} + (\mathbf{h}\cdot\mathbf{p})^2.$$

Taking $\Omega$ from (114) with $f = 0$ we find $\Omega^{-1}$ at once and hence the desired locus (Conic II) is

$$(e^2 - h^2)z_1^2 + 2Aez_1z_3 + (A^2 + B^2)z_3^2 + 2hz_1 = 1. \qquad (116)$$

The discriminant of the first three terms is $h^2(A^2 + B^2) - B^2e^2$. The conic is an ellipse, parabola, or hyperbola according as this expression is negative, zero, or positive.[41] The conic breaks up into two lines if $Be = 0$, that is if the indicatrix is a linear segment.

The degenerate case $B = 0$ requires a little more investigation to find what happens to consecutive normal spaces. If we observe what happens as we pass from a true ellipse to a segment, we see that the points of intersection of consecutive normal planes bunch themselves more and more closely about the point $z_1 = 1/h$, $z_3 = -e/Ah$, which is the inverse of the foot of the perpendicular to the segment from the surface point. It therefore appears that the normal planes all pass through a common point $(0, 0, 1/h, -e/Ah)$ in this special case; the two nearby planes in the direction of the axes $x$, $y$ may be said to cut the normal plane in the lines $z_1 (h + e) + Az_3 = 1$ and $z_1(h + e) - Az_3 = 1$ respectively. These lines are those into which (116) factors and are perpendicular to the lines which join the surface point to the extremities of the indicatrix.

There is a special case under the case $B = 0$, namely that in which the indicatrix, now a segment, is collinear with $\mathbf{h}$. The surface is then three dimensional in the neighborhood of the point, or the point is axial in Levi's nomenclature. The common intersection of the consecutive normal planes has retreated to infinity and the locus reduces to two parallel straight lines which are the intersection of the consecutive normal planes in the $x$ and $y$ directions with the normal at the given point — thus consecutive normal planes do not in general meet that normal plane.

**43. Segre's Characteristics.** The points for which $B = 0$, that is, those where the indicatrix reduces to a linear segment have one property of importance in common with surfaces in three dimensions. For *if the indicatrix reduces to a linear segment, there are two directions on the surface, namely those corresponding to the ends of the linear segment, for which* $\boldsymbol{\mu} = 0$, *and these are orthogonal directions, and for them the normal curvature is a maximum or a minimum. If these lines be taken as parametric curves the second fundamental* (*vector*) *form and the first fundamental form reduce simultaneously to the sum of squares,*

$$\varphi = a_{11}dx_1^2 + a_{22}dx_2^2, \quad \Psi = \mathbf{y}_{11}dx_1^2 + \mathbf{y}_{22}dx_2^2.$$

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[41] Kommerell distinguishes these cases by saying that the surface point is elliptic, hyperbolic, or parabolic, but though this distinction may be useful in the case of surfaces lying in a 4-space and possibly at planar points in general, there is apparently no similar classification in general surface theory.

Further: the rate of change of the tangent plane squared is, from (73), (80), (82),

$$\frac{d\mathbf{M}}{ds} \times \frac{d\mathbf{M}}{ds} = (\boldsymbol{\alpha}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\mu}) \times (\boldsymbol{\alpha}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\mu}) = 2\boldsymbol{\xi}\times\boldsymbol{\eta}\times\boldsymbol{\mu}\times\boldsymbol{\alpha}. \tag{117}$$

As $\boldsymbol{\xi}\times\boldsymbol{\eta}$ and $\boldsymbol{\mu}\times\boldsymbol{\alpha}$ are completely perpendicular, the only possibility for the product to vanish is that $\boldsymbol{\mu}\times\boldsymbol{\alpha} = 0$. This will vanish when $\boldsymbol{\mu} = 0$ and hence: *If the indicatrix is a linear segment there are two directions for which the rate of change $d\mathbf{M}/ds$ is a simple plane vector.* In this case $\mathbf{M}$ and $d\mathbf{M}$ intersect in a line. *There are then only two directions in which consecutive tangent planes intersect in a line.*

If the indicatrix does not reduce to a linear segment the only way that $\boldsymbol{\mu}\times\boldsymbol{\alpha}$ can vanish is to have $\boldsymbol{\mu}$ and $\boldsymbol{\alpha}$ parallel or $\boldsymbol{\alpha}$ vanish. Now the latter alternative will happen when and only when the indicatrix (now an actual ellipse) passes through the surface point and in this case there is only a single direction in which $d\mathbf{M}$ is a simple vector. If $\mathbf{h}\times\boldsymbol{\mu}\times\boldsymbol{\delta} = 0$, that is, *if the surface at the point is four dimensional* (i. e., *planar*), *there are two directions on the surface for which* $\boldsymbol{\mu}\times\boldsymbol{\alpha} = 0$, *namely, those which make* $\boldsymbol{\alpha}$ *tangent to the indicatrix, for these directions, and only for these,* $d\mathbf{M}$ *is a simple plane and consecutive tangent planes intersect in a line.* These two directions cannot be perpendicular and may be imaginary, they are coincident in the case where $\boldsymbol{\alpha}$ may vanish. *If the surface at the point is five dimensional,* $d\mathbf{M}$ *is never a simple plane.*

It appears therefore that in no case above three dimensions can conjugate directions be defined by considering intersections of consecutive tangent planes.

We may express upon the $\mathbf{y}$'s the condition of degeneracy. First if the surface is four dimensional at the point, then at that point $\mathbf{y}_{11}$, $\mathbf{y}_{12}$, $\mathbf{y}_{22}$ must be coplanar and a linear relation

$$A\mathbf{y}_{11} + B\mathbf{y}_{12} + C\mathbf{y}_{22} = 0 \tag{118}$$

must subsist between these. Next if the ellipse collapses into a segment, the condition (102) for $\boldsymbol{\mu}\times\boldsymbol{\delta} = 0$ may be used to show that the normals, $\mathbf{y}_{11}/a_{11}$, $\mathbf{y}_{12}/a_{12}$, $\mathbf{y}_{22}/a_{22}$ are termino-collinear and the relation

$$Aa_{11} + Ba_{12} + Ca_{22} = 0$$

subsists between the coefficients $A$, $B$, $C$ in (118). Finally if the linear segment is collinear with $\mathbf{h}$, the normals are all collinear and must satisfy (118) and an additional relation

$$A'\mathbf{y}_{11} + B'\mathbf{y}_{12} + C'\mathbf{y}_{22} = 0. \tag{118'}$$

Segre[42] showed that if the coordinates of a surface satisfy the relation (118) at each point, there is traced on the surface a double system of curves, called characteristics, having the property that tangent planes to the surface in two infinitely near points in the direction of one of these characteristics will intersect in a line tangent to the other. Also an $S_{n-1}$ passing through the tangent plane will cut the surface in a curve having a node at the point of contact and such that the tangents at the node are separated harmonically by the tangents to the characteristics. The direction of the nodal tangents correspond to the points in which a line drawn through the surface point cuts the indicatrix. The considerations above given show that these surfaces are those for which $\mathbf{h}\times\boldsymbol{\mu}\times\boldsymbol{\delta} = 0$ (see §39).

If the equation (118) is of the parabolic type, that is if $B^2 - 4AC = 0$, the two characteristics will coincide. If this happens Moore[43] showed that the characteristics have the property that their tangents have three point contact with the surface. For this type the indicatrix passes through the surface point and consequently one of the nodal tangents always coincides with the tangent to the (double) characteristic.

Segre also showed that a surface whose coordinates satisfy two equations (118), (118′) either lies in a three space or else consist of the tangents to a twisted curve. These are the surfaces for which our invariant $(\boldsymbol{\mu}\times\boldsymbol{\delta})^2 - \Phi_{2S}/a^2$ of §39 vanishes, and the statement there made is thus substantiated.

If the indicatrix degenerates into a linear segment not passing through the surface point, the two characteristics are perpendicular, and this is the only case in which the characteristics are at right angles. If the linear segment passes through the surface point two cases arise. 1° If one end of the segment lies on the surface then at that point the surface has the character of a twisted curve surface. If the condition is satisfied identically the surface is a twisted curve surface. 2° If the segment does not have an end in the surface then at the point the surface has the character of a three dimensional surface which is not a developable. If this condition is satisfied identically the surface must lie in three dimensions.

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[42] Segre, Su una classa di superficie degl'iperspazi, ecc., *Atti di Torino*, 1907.

[43] C. L. E. Moore, Surfaces in hyperspace which have a tangent line with three point contact passing through each point, *Bull. Amer. Math. Soc.*, **18**, 1912.

As an application of these results we may show that a minimum ruled surface must always lie in three dimensions and consequently be the helicoid. For as the surface is ruled the indicatrix reduces to a segment which, as $\mathbf{h} = 0$, must pass through the surface point and indeed have that point for mid point and hence the surface must lie in three dimensions.

**44. Principal directions.** If we take the value of $\boldsymbol{\alpha}'$ from (89), we find, as the condition that $\boldsymbol{\alpha}'$ shall be maximum or minimum in magnitude.

$$\begin{aligned} 0 = \boldsymbol{\alpha}'\cdot d\boldsymbol{\alpha}' &= (\mathbf{h} + \boldsymbol{\mu}\sin 2\theta + \boldsymbol{\delta}\cos 2\theta)\cdot(\boldsymbol{\mu}\cos 2\theta - \boldsymbol{\delta}\sin 2\theta) = \boldsymbol{\alpha}'\cdot\boldsymbol{\mu}' \\ &= \mathbf{h}\cdot\boldsymbol{\mu}\cos 2\theta - \mathbf{h}\cdot\boldsymbol{\delta}\sin 2\theta + (\boldsymbol{\mu}^2 - \boldsymbol{\delta}^2)\sin 2\theta\cos 2\theta \\ &\qquad + (\boldsymbol{\mu}\cdot\boldsymbol{\delta})(\cos^2 2\theta - \sin^2 2\theta). \end{aligned}$$

If we let $x = \tan\theta$, the resulting equation in $x$ is

$$\begin{aligned} x^4(\boldsymbol{\mu}\cdot\boldsymbol{\delta} - \boldsymbol{\mu}\cdot\mathbf{h}) + 2x^3(\boldsymbol{\delta}^2 - \boldsymbol{\mu}^2 - \mathbf{h}\cdot\boldsymbol{\delta}) - 6x^2\boldsymbol{\mu}\cdot\boldsymbol{\delta} + 2x(\boldsymbol{\mu}^2 - \boldsymbol{\delta}^2 - \mathbf{h}\cdot\boldsymbol{\delta}) \\ + \boldsymbol{\mu}\cdot(\boldsymbol{\delta}\times\mathbf{h}) = 0. \end{aligned}$$

This is of the fourth degree and hence there are four directions of maximum or minimum for the magnitude of the normal curvature (Kommerell). Two of these directions must be real; for if we choose $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$, which may be any radii of the indicatrix, as the axes of the indicatrix, $\boldsymbol{\mu}\cdot\boldsymbol{\delta} = 0$, and the coefficients of the first and last terms are opposite in sign. If a single pair of these four directions are orthogonal, it must be possible to choose $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$ so that $x = 0$ and $x = \infty$ satisfy the equations, that is, so that $\boldsymbol{\mu}\cdot\boldsymbol{\delta} \pm \boldsymbol{\mu}\cdot\mathbf{h} = 0$ or $\boldsymbol{\mu}\cdot\boldsymbol{\delta} = \boldsymbol{\mu}\cdot\mathbf{h}$. This means that: *If two of the directions of maximum or minimum normal curvature are perpendicular one of the axes of the indicatrix must be perpendicular to* $\mathbf{h}$. If the four directions are perpendicular in pairs $x$ and $-1/x$ must satisfy the biquadratic and

$$\boldsymbol{\mu}\cdot\boldsymbol{\delta} - \boldsymbol{\mu}\cdot\mathbf{h} = \boldsymbol{\mu}\cdot\boldsymbol{\delta} + \boldsymbol{\mu}\cdot\mathbf{h}, \quad \boldsymbol{\mu}^2 - \boldsymbol{\delta}^2 - \mathbf{h}\cdot\boldsymbol{\delta} = \boldsymbol{\mu}^2 - \boldsymbol{\delta}^2 + \mathbf{h}\cdot\boldsymbol{\delta} \quad \text{or} \quad \boldsymbol{\mu}\cdot\mathbf{h} = \mathbf{h}\cdot\boldsymbol{\delta} = 0.$$

Hence: *If the directions of maximum or minimum normal curvature are perpendicular in pairs, the plane of the indicatrix must be perpendicular to* $\mathbf{h}$, *that is,* $\mathbf{h}$ *must be along the axis of Cone* I.

*Definition 1.* Kommerell calls the directions, which give a maximum or minimum magnitude to the normal curvature, principal directions and points out that for the principal directions (in the four dimensional case) the point of intersection of consecutive normal planes lies in the osculating plane of the normal section through that direction. This is clear from the configuration of the indicatrix when $\boldsymbol{\alpha}\cdot\boldsymbol{\mu} = 0$. For here the tangent $PT$ at the point $P$ of the indicatrix is perpendicular to $OP$ the vector from the surface point $O$ to $P$, and consequently $P'$ the inverse of the pedal of $P$, lies on $OP$. [He calls the conic which is the inverse of the pedal of the indicatrix (our Conic I) the characteristic (our Conic II) and shows that the principal directions correspond to the lines $OP'$ which may be drawn from $O$ perpendicular to the characteristic. These lines are the same as those perpendicular to the indicatrix.] In this way he generalizes one property of the principal directions in three dimensions. The generalization is far from perfect. For in the case of three dimensions consecutive normals do not intersect in general, but do intersect for principal directions, and the intersection lies in the osculating plane of the normal section through a principal direction. In the four dimensional case the normal planes in general intersect, but except for the principal directions the point of intersection does not lie in the normal $\boldsymbol{\alpha}$, and is not in the osculating plane of the normal section through the direction.

The result may be generalized to the general case of a surface in five (or more) dimensions. For if $O$ be the surface point, $F$ the foot of the perpendicular upon the plane of the indicatrix, and $P$ a point such that the tangent $PT$ is perpendicular to $OP$, then as $PT$ is perpendicular to $OF$, $PT$ is perpendicular to the plane $OFP$. Hence $FP$ is perpendicular to $PT$. Thus the points $P$ of the indicatrix which correspond to Kommerell's principal directions on the surface are those for which the radius $FP$ from the foot of the perpendicular $OF$ is perpendicular to the indicatrix. Moreover, if $F'$ be the inverse of $F$ all the normal three spaces pass through $F'$. Consecutive normal spaces intersect in a line through $F'$ perpendicular to the plane of $\boldsymbol{\alpha} = OP$ and $\boldsymbol{\mu} = PT$. Hence the intersection $F'P'$ of consecutive normal spaces cuts the line $OP$ in some point $P'$. Thus: *one of the* $\infty^1$ *points of intersection of consecutive normal spaces lies in the osculating plane of the normal section in case that section corresponds to a direction of maximum or minimum normal curvature.*

As Kommerell points out, in the special case of a surface which at $O$ is of the three dimensional type, the condition $\boldsymbol{\alpha}\cdot\boldsymbol{\mu} = 0$, breaks down into $\boldsymbol{\alpha} = 0$ and $\boldsymbol{\mu} = 0$. The principal directions corresponding to $\boldsymbol{\mu} = 0$ become the true principal directions while those corresponding to $\boldsymbol{\alpha} = 0$ become the asymptotic directions. In the four dimensional case when at $O$ the indicatrix reduces to a linear segment (not passing through $O$) the condition $\boldsymbol{\alpha}\cdot\boldsymbol{\mu} = 0$ breaks up into $\boldsymbol{\mu} = 0$ and $\boldsymbol{\alpha}\cdot\boldsymbol{\mu} = 0$ with $\boldsymbol{\mu} \neq 0$. The directions for which $\boldsymbol{\mu} = 0$ correspond to the extremities of the segment, which may be called the true principal directions and are perpendicular, while the directions for which $\boldsymbol{\alpha}\cdot\boldsymbol{\mu} = 0$ $(\boldsymbol{\mu} \neq 0)$ correspond to the directions which may be called asymptotic from analogy. These directions may be real coincident or imaginary, but in any case are bisected by the principal directions, since the two coincident points in which a line cuts the linear segment correspond to values of $\theta$ respectively greater than and less than those for which $\boldsymbol{\mu} = 0$ by equal amounts.

*Definition 2.* There are other ways of generalizing the principal directions to higher dimensions. For ordinary principal directions $\boldsymbol{\mu} = 0$, that is, $\boldsymbol{\mu}^2$ has a minimum. The lines for which $\boldsymbol{\mu}^2$ is a maximum corresponds to the bisectors of the principal directions. If we desire we can define as principal directions those for which $\boldsymbol{\mu}^2$ is a minimum or maximum (it is not important to distinguish between the two extremes when the indicatrix does not degenerate). Then the principal directions on the surface would be four in number, spaced equally at angles of 45° around the point on the surface and corresponding to the axes of the indicatrix.

*Definition 3.* There is another property which will define lines of curvature in ordinary space on all but minimal surfaces. If any direction $\lambda$ be drawn on the surface at a point, the change of the normal $d\mathbf{n}$ along that line has a definite direction. It is possible to find another direction $\lambda'$ such that the change $d\mathbf{n}'$ of the normal along that direction is perpendicular to $d\mathbf{n}$. In general $\lambda'$ is not perpendicular to $\lambda$. But for the principal directions $\lambda'$ and $\lambda$ are perpendicular. Thus: *The principal directions are the pair of perpendicular directions for which the differential changes of the normal are also perpendicular.* We shall examine the value of this (third) definition for principal directions in any number of dimensions. We may consider the equation

$$\frac{d\mathbf{M}}{ds_1}\cdot\frac{d\mathbf{M}}{ds_2} = 0, \tag{119}$$

which will connect two directions on the surface. Instead of setting up the relation in general we shall use formulas (73), (80), (82), (84) to express the condition that, for two perpendicular directions, the differential planes are orthogonal. We find

$$\frac{d\mathbf{M}}{ds}\cdot\frac{d\mathbf{M}}{d\bar{s}} = (\boldsymbol{\alpha}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\mu})\cdot(\boldsymbol{\mu}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\beta}) = \boldsymbol{\alpha}\cdot\boldsymbol{\mu} + \boldsymbol{\beta}\cdot\boldsymbol{\mu} = 2\mathbf{h}\cdot\boldsymbol{\mu} = 0. \tag{120}$$

*The condition for principal directions is therefore now* $\mathbf{h}\cdot\boldsymbol{\mu} = 0$; the directions on the surface are those for which $\boldsymbol{\mu}$ is perpendicular to $\mathbf{h}$. There is one line in the plane of the indicatrix that satisfies this condition on $\boldsymbol{\mu}$, namely the intersection of the plane of the indicatrix with the plane through the end of $\mathbf{h}$ perpendicular to $\mathbf{h}$. Two perpendicular directions on the surface are determined by the two opposite values of $\boldsymbol{\mu}$. Hence: *By definition 3 there are just two principal directions through each point of the surface, and these are orthogonal.* On a surface for which $\boldsymbol{\mu} = 0$ these two directions coincide with those previously called principal. In case $\mathbf{h} = 0$, the condition is satisfied for any direction on the surface, and in case $\mathbf{h}$ is not zero but is along the axis of Cone I, the condition is also satisfied identically. This last case may perhaps be likened to an umbilic in ordinary surface theory — for at an umbilic the principal directions are indeterminate.

The expression $d\mathbf{M}/ds = \boldsymbol{\alpha}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\mu}$ gives

$$(d\mathbf{M}/ds)^2 = \boldsymbol{\alpha}^2 + \boldsymbol{\mu}^2 = \mathbf{h}^2 + \boldsymbol{\delta}^2 + \boldsymbol{\mu}^2 + 2\mathbf{h}\cdot\boldsymbol{\delta}.$$

As the expressions $\mathbf{h}^2$ and $\boldsymbol{\mu}^2 + \boldsymbol{\delta}^2$ are invariants, the maximum and minimum values of $(d\mathbf{M}/ds)^2$ will fall where $\boldsymbol{\delta}$ has the greatest (positive or negative) projection on $\mathbf{h}$, that is at the point of tangency of planes tangent to the indicatrix and perpendicular to $\mathbf{h}$, and for this condition $\mathbf{h}\cdot\boldsymbol{\mu} = 0$. *The principal directions (definition 3) are therefore those for which $d\mathbf{M}/ds$ is a maximum or minimum in magnitude, as in the ordinary three dimensional case.* It may reasonably be asked whether such a condition as the maximum or minimum of $d\mathbf{M}/ds$ in magnitude is not more intimately connected with the surface than the similar conditions on the curvature of a normal section. Unfortunately the condition breaks down for the case $\mathbf{h} = 0$, but there are important theorems on principal directions in the three dimensional case which suggest that $\mathbf{h} = 0$ is a really exceptional case.[44]

It is not difficult to make a choice between the three generalizations just mentioned. The first, which follows Kommerell, gives four directions on the surface which are not perpendicular and which in the case of three dimensions reduce to the principal directions and the asymptotic directions. We do not ordinarily associate common properties to these two sets of directions. The second definition, suggested above, also gives four directions and in this case the reduction is to the principal directions combined with their bisectors. We do not usually investigate these directions or associate common properties to them and the principal directions. One great advantage of the third generalization is that we have, as principal, two and only two directions at each point and these directions are perpendicular.

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[44] See, for example, Eisenhart, *Differential Geometry*, p. 143.

The differential equations of the principal directions as defined by $\mathbf{h} \cdot \boldsymbol{\mu} = 0$ are, from (87),

$$\Sigma_{rs} \mathbf{h} \cdot \mathbf{y}_{rs} \overline{\lambda}^{(r)} \lambda^{(s)} = 0.$$

By (63) and $\lambda_r = \Sigma a_{rs} \lambda^{(s)}$ we may write

$$\Sigma_{rst} \mathbf{h} \cdot \mathbf{y}_{rs} a_{r+1,\, t} \lambda^{(t)} \lambda^{(s)} (-1)^{(r+1)} = 0,$$

or

$$\Sigma_{rst} \mathbf{h} \cdot \mathbf{y}_{rs} (-1)^{r+1} a_{r+1,\, t}\, dx_t dx_s = 0, \tag{121}$$

Written out at length we have

$$\mathbf{h} \cdot [(\mathbf{y}_{11} a_{21} - \mathbf{y}_{21} a_{11}) dx_1^2 + (\mathbf{y}_{11} a_{22} - \mathbf{y}_{22} a_{11}) dx_1 dx_2 \\ + (\mathbf{y}_{12} a_{22} - \mathbf{y}_{22} a_{12}) dx_2^2] = 0.$$

This equation is similar to the ordinary equation except that $\mathbf{y}_{rs}$ replaces $b_{rs}$ and the whole is multiplied by $\mathbf{h}$.

If the lines of curvature are taken as parametric lines,

$$\mathbf{h} \cdot \mathbf{y}_{11} a_{21} = \mathbf{h} \cdot \mathbf{y}_{21} a_{11}, \qquad \mathbf{h} \cdot \mathbf{y}_{12} a_{22} = \mathbf{h} \cdot \mathbf{y}_{22} a_{21}.$$

These equations, since $\mathbf{y}_{11} a_{22} - \mathbf{y}_{22} a_{11} \neq 0$, demand that $a_{12} = 0$ and $\mathbf{h} \cdot \mathbf{y}_{21} = 0$. *The condition that the lines of curvature be parametric is no longer* $a_{12} = 0$, $\mathbf{y}_{12} = 0$; *the normal* $\mathbf{y}_{12}$ *need merely be perpendicular to* $\mathbf{h}$. In all this work $\mathbf{h}$ may be replaced by its value $\Sigma_{rs} a^{(rs)} \mathbf{y}_{rs}$ if desired. Special considerations need to be developed for the case $\mathbf{h} = 0$.

**45. Asymptotic lines.** When we seek for a generalization for asymptotic lines we may consider the equation $\mathbf{h} \cdot \Psi = 0$, where $\Psi$ is the second fundamental form, as defining asymptotic lines in general.

Indeed equation (78), namely, $d\mathbf{M}\cdot d\mathbf{M} = -Gds^2 + 2\mathbf{h}\cdot\Psi$ contains an important property of asymptotic lines on ordinary surfaces; the asymptotic lines are those for which the rate of turning of the normal, in this case the torsion, is $\surd(-G)$. Along the asymptotic directions in the general case of $n$ dimensions, $d\mathbf{M}\cdot d\mathbf{M} = -Gds^2$, that is, the rate of turning of the tangent plane is $\surd(-G)$, if by analogy $(d\mathbf{M}/ds)^2$ may be called the rate of turning when successive planes do not intersect in a line.[45]

In the ordinary case $\boldsymbol{\alpha} = 0$ for the asymptotic lines. In the general case $\boldsymbol{\alpha}\cdot\mathbf{h} = 0$. To prove this consider [see (67)]

$$0 = \mathbf{h}\cdot\Psi = \mathbf{h}\cdot\Sigma_{rs}[\boldsymbol{\alpha}\lambda_r\lambda_s + \boldsymbol{\mu}(\lambda_r\bar{\lambda}_s + \bar{\lambda}_r\lambda_s) + \boldsymbol{\beta}\bar{\lambda}_r\bar{\lambda}_s]dx_r dx_s\,.$$

As $dx_r{:}dx_s = \lambda^{(r)}{:}\lambda^{(s)}$ for the curves defined by $\lambda$, these curves will be along the asymptotic directions when and only when $\mathbf{h}\cdot\boldsymbol{\alpha} = 0$, as the other terms vanish in the summation.

The condition $\mathbf{h}\cdot\boldsymbol{\alpha} = 0$ means that the curvature of the asymptotic line is perpendicular to $\mathbf{h}$ and consequently the osculating plane of the curve is perpendicular to $\mathbf{h}$. Conversely if the osculating plane is perpendicular to $\mathbf{h}$ then $\boldsymbol{\alpha}$ must be perpendicular to $\mathbf{h}$. Hence: *The asymptotic line is characterized by the property that its osculating plane is perpendicular to the mean curvature vector as in the three dimensional case.*

As in the case of principal directions (Definition 3), the asymptotic lines we have defined become illusory for minimal surfaces. *For surfaces, not minimal, the asymptotic lines cannot be orthogonal,* as may be seen from the configuration of the indicatrix.

That the condition $\mathbf{h}\cdot\boldsymbol{\alpha} = 0$ may be satisfied for a real direction on the surface, the plane $\pi$ through the surface point $O$ perpendicular to $\mathbf{h}$ must cut the indicatrix in real points. Now: *The asymptotic lines here defined for any surface are bisected by the principal directions* (Definition 3). For the plane $\pi$ is parallel to $\boldsymbol{\mu}$ if $\mathbf{h}\cdot\boldsymbol{\mu} = 0$ and cuts the plane of the indicatrix in a line parallel to $\boldsymbol{\mu}$. The two vectors $\boldsymbol{\alpha}$ go to points of the indicatrix which represent equal amounts of the surface angle $\theta$, above and below the directions for which $\mathbf{h}\cdot\boldsymbol{\mu} = 0$. If this plane $\pi$ is tangent to the indicatrix the asymptotic lines fall together. *The condition that the asymptotic directions fall together (and*

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[45] If we define the angle between two plane vectors, whether or not these be simple planes, by the formula $\cos\theta = \mathbf{M}\cdot\mathbf{N}/(\mathbf{M}^2\mathbf{N}^2)^{\frac{1}{2}}$ we have a real angle whenever the planes or complexes are real. If $\mathbf{M}$ is a unit plane, $\mathbf{N}$ a nearby unit plane $\mathbf{M} + \Delta\mathbf{M}$, then $2\mathbf{M}\cdot\Delta\mathbf{M} + (\Delta\mathbf{M})^2 = 0$ and by a familiar transformation we find $(d\theta/ds)^2 = (d\mathbf{M}/ds)^2$.

*coincide with one of the principal directions) is that* **h** *shall be normal to a tangent plane to Cone* I, *hence an element of Cone* II, that is,

$$\mathbf{h}\cdot\Phi\cdot\mathbf{h} = 0 = (\mathbf{h}\cdot\mathbf{h})^2 - (\mathbf{h}\cdot\boldsymbol{\mu})^2 - (\mathbf{h}\cdot\boldsymbol{\delta})^2.$$

In ordinary surface theory this reduces to $h^2G = 0$; but here **h**, $\boldsymbol{\mu}$, and $\boldsymbol{\delta}$ are not collinear and the condition is not satisfied by $G = 0$. Indeed,

$$\begin{aligned}\mathbf{h}\cdot\Phi\cdot\mathbf{h} - \mathbf{h}^2\Phi_S &= \mathbf{h}^2\boldsymbol{\mu}^2 - (\mathbf{h}\cdot\boldsymbol{\mu})^2 + \mathbf{h}^2\boldsymbol{\delta}^2 - (\mathbf{h}\cdot\boldsymbol{\delta})^2 \\ &= (\mathbf{h}\times\boldsymbol{\mu})^2 + (\mathbf{h}\times\boldsymbol{\delta})^2 = \mathbf{h}\cdot\Phi\cdot\mathbf{h} - \mathbf{h}^2Ga.\end{aligned}$$

We see therefore that surfaces for which $G = 0$ make $\mathbf{h}\cdot\Phi\cdot\mathbf{h}$ positive unless **h**, $\boldsymbol{\mu}$, $\boldsymbol{\delta}$ are collinear, that is unless the surface is three dimensional at the point in question. The condition $\mathbf{h}\cdot\Phi\cdot\mathbf{h} > 0$ means, however, that the plane $\pi$ perpendicular to **h** through $O$ does not cut the indicatrix, that is that the asymptotic directions on the surface are imaginary. Hence: *For all developables except the twisted curve surfaces the asymptotic directions are imaginary.*

The special cases which arise when the surface is four dimensional, with the indicatrix either an ellipse or a linear segment are not especially different from the general case.

The scalar form $\mathbf{h}\cdot\Psi$ which for the definition of asymptotic directions (in our sense) has taken the place of the scalar form $\psi$ (second fundamental form) in three dimensions may be used to define a conjugate system of curves upon the surface as in the ordinary three dimensional case. The asymptotic lines $\mathbf{h}\cdot\Psi = 0$ are then the double elements in the involution. It is easy to see that the lines of curvature (in our sense) are the pair of orthogonal elements in this involution. For if we use the lines of curvature (in our sense) as parametric curves the forms $\mathbf{h}\cdot\Psi$ and $\varphi = ds^2$ are simultaneously reduced to a sum of squares since $a_{12} = 0$ and $\mathbf{h}\cdot\mathbf{y}_{12} = 0$ in this case.

Kommerell's generalization of asymptotic directions in case $n = 4$ was to those directions which correspond to the infinite points of his characteristic (our Conic II, inverse pedal to the indicatrix), i. e., to directions which make the normal curvature $\boldsymbol{\alpha}$ tangent to the indicatrix, namely $\boldsymbol{\alpha}\times\boldsymbol{\mu} = 0$. On a surface in $S_4$ there are two such directions; but the generalization breaks down for $n > 4$ because the condition $\boldsymbol{\alpha}\times\boldsymbol{\mu} = 0$ cannot be satisfied.[46] We are therefore forced to conclude that neither the principal directions nor the asymptotic lines as defined by Kommerell are the best generalizations of corresponding lines on ordinary surfaces. That we have found other and better definitions may be attributed in part to the broader view point that we get by working in higher than four dimensional space, but must be credited in large measure to the suggestiveness of the method of attack developed by Ricci in his *Lezioni.*

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[46] Kommerell's second fundamental form, the vanishing of which determines his asymptotic directions, has therefore no relation at all to *general* surface theory, because his asymptotic directions do not exist in general. The second fundamental forms which we develop are vital to the theory.

Kommerell's type of asymptotic lines will exist only when $\boldsymbol{\alpha}\times\boldsymbol{\mu} = 0$, that is, only when the indicatrix lies in a plane through the surface point and the surface becomes four dimensional at the point. This condition has been discussed in §43. It will be seen that Kommerell's asymptotic lines are identical with Segre's characteristics; they are therefore important lines for those surfaces on which they exist. (Levi has asymptotic lines only in case the two characteristics coincide, their common direction being then called asymptotic.)

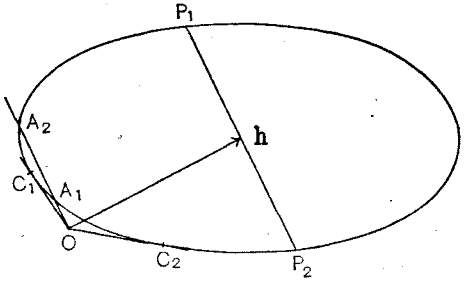

FIGURE 3.

In the four dimensional case the important lines on the surface are as follows: Two principal directions corresponding to the points $P_1$ and $P_2$ for which $\mathbf{h}\cdot\boldsymbol{\mu} = 0$; two asymptotic directions (in our sense) $A_1$ and $A_2$ for which $\mathbf{h}\cdot\boldsymbol{\alpha} = 0$; two characteristic directions $C_1$ and $C_2$ for which $\boldsymbol{\alpha}\times\boldsymbol{\mu} = 0$. The principal directions are orthogonal and bisect the asymptotic directions, but need not bisect the characteristic directions; the asymptotic and characteristic directions divide each other harmonically. If $O$ lies on the indicatrix, $A_1$, $C_1$, $C_2$ coincide. If the indicatrix reduces to a linear segment $P_1$ and $C_1$, $P_2$ and $C_2$ coincide.

**46. The Dupin indicatrix.** Another way of getting at the properties of a surface in ordinary space is by the Dupin indicatrix, which is the intersection of the surface by a tangent plane (or a plane parallel thereto). In five dimensions we must take a hyperplane (a four dimensional linear spread) to cut the surface. If we consider a hyperplane $uz_1 + vz_2 + wz_3 = 0$ tangent to the surface in the standard form (109′); the intersection is,

$$u[h(x^2 + y^2) + e(x^2 - y^2)] + vf(x^2 - y^2) + w[A(x^2 - y^2) + 2Bxy] = 0.$$

There are $\infty^2$ such hyperplanes. The discriminant of the quadratic form

$$[u(h + e) + vf + wA]x^2 + [u(h - e) - vf - wA]y^2 + 2wBxy] = 0 \quad (127)$$

is

$$\Delta = w^2B^2 + (ue + vf + wA)^2 - u^2h^2.$$

The equation $\Delta = 0$ determines a quadric cone. Hence: *There are* $\infty^1$ *normal directions* $u{:}v{:}w$ *(forming a quadric cone) such that the tangent hyperplanes normal to any of these directions cut the surface in coincident directions.*

If $u{:}v{:}w$ be the directions of an element of Cone II we have, from (114),

$$(h^2 - e^2)u^2 - f^2v^2 - (A^2 + B^2)w^2 - 2fAvw - 2Aeuw - 2feuv = 0,$$

as the equation of Cone II (with its vertex transferred to $O$). This is identical with $\Delta = 0$ except for sign. We see therefore that: *The tangent hyperplanes which are perpendicular to the elements of Cone* II *cut the surface in coincident lines; these hyperplanes are also the tangent hyperplanes to Cone* I. Hence we may state: *The tangent hyperplanes which cut the indicatrix in real points cut the surface in real directions; those which cut the indicatrix in imaginary points cut the surface in imaginary directions; and those tangent to the indicatrix cut the surface in coincident directions.*

Particular interest attaches to the hyperplane ($z_1 = 0$) perpendicular to $\mathbf{h}$. This cuts the surface in the directions $(h + e)x^2 + (h - e)y^2 = 0$. These directions are real, coincident, or imaginary according as $h < e$, $h = e$, or $h > e$. This locus will be called the (generalized) *Dupin indicatrix.*

The condition $\boldsymbol{\alpha}\cdot\boldsymbol{\mu} = 0$ and $\boldsymbol{\mu}\cdot\boldsymbol{\delta} = 0$ which give the first two generalizations of principal directions may be calculated from (111), (112), but exhibit no special properties relative to the axes used in standardizing the equation of the surface. The condition $\mathbf{h}\cdot\boldsymbol{\mu} = 0$, however, is satisfied by the $x$ and $y$ axes in the tangent plane when the form

(109′) is used. We may say therefore that: *The principal directions (third generalization) at a point coincide with the principal directions of the three dimensional surface obtained by projecting the surface on the three space determined by the tangent plane and the mean curvature, or, are along the axes of the (degenerate) conic in which a tangent $S_{n-1}$ normal to the mean curvature cuts the surface.* (The conic obtained as the intersection of any $S_{n-1}$ which is normal to any line in the plane of $\mathbf{h}$ and the perpendicular $\boldsymbol{\Phi}\cdot\mathbf{h}$ on the plane of $\boldsymbol{\mu}\times\boldsymbol{\delta}$ from $O$ has the same axes.)

In the special case $h = 0$ the conic (122) always has $\Delta > 0$, and is an hyperbola. There is no real hyperplane which cuts the surface in a double direction. Cone II becomes a cylinder with elements perpendicular to the plane $\boldsymbol{\mu}\times\boldsymbol{\delta}$ of the indicatrix. From (109), $\boldsymbol{\mu}\times\boldsymbol{\delta} = B(e\mathbf{k}_3\times\mathbf{k}_1 - f\mathbf{k}_2\times\mathbf{k}_3)$. The hyperplanes perpendicular to any direction in the plane of the indicatrix have $w = 0$ and cut the surface in the same locus $x^2 - y^2 = 0$,— except the particular one for which $u:v = f:-e$ which causes (121) to vanish identically and contains all directions on the surface. We may therefore define, if we choose, the directions of the $x$ and $y$ axes as principal directions and the orthogonal directions $x^2 - y^2 = 0$ as asymptotic lines on the surface at the point where $h = 0$.

A reference to (111) shows that $\mathbf{h}\cdot\boldsymbol{\alpha} = 0$ means $h + e\cos 2\theta = 0$. On comparison with (122) we see that the directions $\theta$ for which $h + e\cos 2\theta = 0$ are the asymptotic directions of the intersection of the surface with the tangent $S_{n-1}$ perpendicular to $\mathbf{h}$. Hence: *The asymptotic directions on a surface are the directions in which the surface is cut by a tangent hyperplane perpendicular to the mean curvature vector.* This gives added corroboration of the generalization of the Dupin indicatrix to the intersection of the surface and this particular $S_{n-1}$.

**47. A second standard form for a surface.** In the three dimensional theory the condition $G = 0$ is unchanged by the general linear transformation. This is no longer the case in higher dimensions. To discuss briefly projective properties of a surface we may proceed as follows. The general surface has the property that the tangent spaces $S_{n-1}$ which cut the surface in a double direction envelope a nondegenerate cone. This statement is projective and the analytic statement is $\mathbf{h}\times\boldsymbol{\mu}\times\boldsymbol{\delta} \neq 0$. The condition $(\mathbf{h}\times\boldsymbol{\mu}\times\boldsymbol{\delta})^2 = 0$ is therefore invariant under projection. (The condition $\mathbf{h}\times\boldsymbol{\mu}\times\boldsymbol{\delta} = 0$ is the condition for the existence of Segre's characteristics, and as Segre was discussing projective properties, the result stated is but a corollary to his work.) Now $(\mathbf{h}\times\boldsymbol{\mu}\times\boldsymbol{\delta})^2$ is Gibbs's invariant $\Phi_3$ or $|\,\Phi\,|$ for the dyadic $\Phi = \mathbf{hh} - \boldsymbol{\mu\mu} - \boldsymbol{\delta\delta} = \frac{1}{2}\mathbf{y}_{11}\mathbf{y}_{22} + \frac{1}{2}\mathbf{y}_{22}\mathbf{y}_{11} - \mathbf{y}_{12}\mathbf{y}_{12}$. We may therefore write as the projective invariant

$$\Phi_3 = |\,\Phi\,| = \tfrac{1}{4}(\mathbf{y}_{11}\times\mathbf{y}_{22}\times\mathbf{y}_{12})^2 = 0.$$

In case $\Phi_3 \neq 0$, we can find a second standard form for the development of a surface about a point. It has been shown that if we project a surface on the $S_3$ determined by the tangent plane and a normal parallel to an element of Cone II, the projection has total curvature null. By taking an element of Cone III and two perpendicular elements of Cone II as axes, the expansion to second order terms becomes

$$\begin{aligned} z_1 &= \tfrac{1}{2}(Ax^2 + 2Bxy + Cy^2), & G &= AC - B^2, \\ z_2 &= \tfrac{1}{2}(A_1x^2 + 2B_1xy + C_1y^2), & 0 &= A_1C_1 - B_1^2, \\ z_3 &= \tfrac{1}{2}(A_2x^2 + 2B_2xy + C_2y^2), & 0 &= A_2C_2 - B_2^2. \end{aligned}$$

We shall show that by a proper choice of the element of Cone III, *the standard form*

$$z_1 = \tfrac{1}{2}(Ax^2 + 2Bxy + Cy^2), \quad z_2 = \tfrac{1}{2}Dx^2, \quad z_3 = \tfrac{1}{2}Ey^2 \tag{123}$$

*may be found.* All that is necessary to prove this is to prove that the two double lines obtained from $z_2 = 0$ and $z_3 = 0$ may be made perpendicular. If we set $\xi = uh$, $\eta = ue + vf + wA$, $\zeta = wB$. The condition $\Delta = 0$ becomes $\zeta^2 + \eta^2 - \xi^2 = 0$. We have to find two directions $u$, $v$, $w$, such that

$$\zeta_1^2 + \eta_1^2 - \xi_1^2 = 0, \qquad \zeta_2^2 + \eta_2^2 - \xi_2^2 = 0, \qquad u_1u_2 + v_1v_2 + w_1w_2 = 0.$$

Furthermore the double lines $(\xi + \eta)x^2 + (\xi - \eta)y^2 + 2\zeta xy = 0$ must be perpendicular for the two series of $\xi$, $\eta$, $\zeta$. Hence, if $\rho$ be a factor,

$$\rho(\xi_2 - \eta_2) = \xi_1 + \eta_1, \qquad \rho(\xi_2 + \eta_2) = \xi_1 - \eta_1, \qquad \rho\zeta_2 = -\zeta_1,$$

or $\quad \zeta_1\xi_2 - \zeta_1\eta_2 + \zeta_2\xi_1 + \zeta_2\eta_1 = 0, \qquad \zeta_1\xi_2 + \zeta_1\eta_2 + \zeta_2\xi_1 - \zeta_2\eta_1 = 0,$

or $\quad \zeta_1\xi_2 + \zeta_2\xi_1 = 0, \qquad \zeta_1\eta_2 - \zeta_2\eta_1 = 0.$

Let $\xi_i/\zeta_i = \Xi_i$, $\eta_i/\zeta_i = H_i$; then the five equations are

$$H_1^2 - \Xi_1^2 + 1 = 0, \quad H_2^2 - \Xi_2^2 + 1 = 0, \quad \Xi_2 + \Xi_1 = 0, \quad H_2 - H_1 = 0$$

and

$$u_1u_2 + v_1v_2 + w_1w_2 = -\Xi_1{}^2\frac{f^2+e^2}{h^2} + H_1{}^2 - 2H_1\frac{A}{B} + \frac{f^2+A^2}{B^2} = 0.$$

These five equations are clearly consistent, $H_2{}^2 - \Xi_2{}^2 + 1 = 0$, being redundant. The actual solution could be carried out by finding $H_1$ first, then $\Xi_1$ and finally $\Xi_2$ and $H_2$. The solution is unique—the four apparently different solutions corresponding to changing the signs of $u$, $v$, $w$, and to interchanging the two sets. Hence (123) is established as a standard form.

The value of $\mathbf{h}$ is $2\mathbf{h} = (A+C)\mathbf{k}_1 + D\mathbf{k}_2 + E\mathbf{k}_3$, and of

$$\Phi = (AC - B^2)\mathbf{k}_1\mathbf{k}_1 + \tfrac{1}{2}CD(\mathbf{k}_1\mathbf{k}_2 + \mathbf{k}_2\mathbf{k}_1) + \tfrac{1}{2}AE(\mathbf{k}_1\mathbf{k}_3 + \mathbf{k}_3\mathbf{k}_1) + \tfrac{1}{2}DE(\mathbf{k}_2\mathbf{k}_3 + \mathbf{k}_3\mathbf{k}_2).$$

Here $\Phi_3 = \tfrac{1}{4}B^2E^2D^2$. If we carry out the linear transformation

$$x' = \alpha x, \quad y' = \beta y, \quad z_1' = \gamma z_1 + \delta z_2 + \epsilon z_3, \quad z_2' = \zeta z_2, \quad z_3' = \eta z_3, \tag{124}$$

the surface takes the form

$$z_1' = \frac{1}{2}\left(\frac{\gamma A + \delta D}{\alpha^2}x'^2 + \frac{2\gamma B}{\alpha\beta}x'y' + \frac{\gamma C + \epsilon E}{\beta^2}y'^2\right), \qquad z'_2 = \frac{1}{2}\frac{\zeta D}{\alpha^2}x'^2,$$
$$z'_3 = \frac{1}{2}\frac{\eta E}{\beta^2}y'^2.$$

The surface will be unaltered if the relations

$$\gamma = \alpha\beta, \quad \delta = A\alpha(\alpha - \beta)/D, \quad \epsilon = C\beta(\beta - \alpha)/E, \quad \zeta = \alpha^2, \quad \eta = \beta^2$$

are satisfied. There then $\infty^2$ transformations (124) which leave the surface unchanged in the neighborhood of the point $O$. Any of the $\infty^2$ transformations where

$$\gamma = \alpha\beta\frac{B'}{B}, \quad \delta = \alpha\frac{A'\beta\alpha - B'A\beta}{DB}, \quad \epsilon = \beta\frac{C'B\beta - B'C\alpha}{ED},$$
$$\zeta = \alpha^2\frac{D'}{D}, \quad \eta = \beta^2\frac{E'}{E}$$

will carry the surface into one in which the five coefficients are any quantities $A'$, $B'$, $C'$, $D'$, $E'$. The determinant of the transformation is $\Delta = \alpha^3\beta^3 B'D'E'/BDE$, and hence the restriction on the quantities is merely that no one of $\alpha$, $\beta$, $B'$, $D'$, $E'$, shall vanish. We see that $\Phi'_3 \neq 0$ if $\Phi_3 \neq 0$; but that $\Phi_3$ is not an invariant in the ordinary sense of projective geometry that $\Phi'_3 = \Delta^k\Phi_3$ — no more is $G$ in the usual surface theory.

**48. Surfaces of revolution.** In higher dimensions the simplest type of rotation is that parallel to a plane, all the normals to the plane remaining fixed. If then $x = x(s)$, $z_i = z_i(s)$, $i = 1, 2, \ldots$, be any twisted curve of which $s$ is the arc, a surface of revolution

$$x = x(s)\cos\theta, \quad y = x(s)\sin\theta, \quad z_i = z_i(s)$$

may be obtained by the revolution of the curve parallel to the $xy$-plane. The surface is made up of circles parallel to the plane with radii equal to the distance of the twisted curve from the $z$-space of $n$–2 dimensions. The parameters of the surface are $s$ and $\theta$; the parametric curves are orthogonal. Further

$$d\mathbf{y} = (\mathbf{i}x'\cos\theta + \mathbf{j}y'\sin\theta + \Sigma\mathbf{k}_i z_i')ds + (-\,\mathbf{i}x\sin\theta + \mathbf{j}x\cos\theta)d\theta.$$

$$\mathbf{m} = \mathbf{i}x'\cos\theta + \mathbf{j}x'\sin\theta + \Sigma\mathbf{k}_i z_i', \qquad \mathbf{n} = -\,\mathbf{i}x\sin\theta + \mathbf{j}x\cos\theta,$$

$$\mathbf{p} = \mathbf{i}x''\cos\theta + \mathbf{j}x''\sin\theta + \Sigma\mathbf{k}_i z_i'', \quad \mathbf{q} = -\,\mathbf{i}x'\sin\theta + \mathbf{j}x'\cos\theta,$$

$$\mathbf{r} = -\,\mathbf{i}x\cos\theta - \mathbf{j}x\sin\theta.$$

As $x'^2 + \Sigma z'^2 = 1$, we have $x'x'' + \Sigma z'z'' = 0$, and

$$\mathbf{m}^2 = 1, \quad \mathbf{m}\cdot\mathbf{n} = 0, \quad \mathbf{n}^2 = x^2, \quad \mathbf{m}\cdot\mathbf{p} = 0, \quad \mathbf{m}\cdot\mathbf{q} = 0,$$

$$\mathbf{m}\cdot\mathbf{r} = -\,xx', \quad \mathbf{n}\cdot\mathbf{p} = 0, \quad \mathbf{n}\cdot\mathbf{q} = xx', \quad \mathbf{n}\cdot\mathbf{r} = 0,$$

$$a_{11} = 1, \quad a_{12} = 0, \quad a_{22} = x^2, \quad a = x^2,$$

$$\mathbf{y}_{11} = \mathbf{p}, \quad \mathbf{y}_{12} = \mathbf{q} - x'\mathbf{m}/x = 0, \quad \mathbf{y}_{22} = \mathbf{r} + xx'\mathbf{m}.$$

The element of arc is $ds^2 + x^2d\theta^2$. It therefore appears that: *The surface of revolution is always applicable upon a surface of revolution in three dimensions* in which the directrix in the $xy$-plane is $x = x(s)$, $z = z(s)$. [The equation $z = z(s)$ is redundant and so is one of the $n - 2$ equations $z_i = z_i(s)$].

The value of $G$ is $\mathbf{q}\cdot\mathbf{r}/a = -\,x''/x$. The condition $G = 0$ for a developable is therefore $x'' = 0$ or $x = c_1 s + c_2$ which establishes between the differentials the relation $dx = c_1 ds$ or

$$(1 - c_1^2)dx^2 = c_1^2(dz_1^2 + dz_2^2 + \ldots + dz_{n-2}^2), \qquad c_1 < 1.$$

In case $n = 3$ the solution is immediate, viz. $z = mx + b$, a line. In case $n > 3$ we may assign to $n - 3$ of the variables $z_i$ any arbitrary values as functions of $x$ (provided that the sum of $dz^2$ is not too large if we desire a real surface). For instance if we consider the case $n = 4$ and let $z_1 = a_1 \cos bx$,

$$(1 - c_1^2 - c_1^2 a_1^2 b^2 \sin^2 bx) dx^2 = c_1^2 dz_2^2, \quad \text{or} \quad (1 - c_1^2) \cos^2 bx dx^2 = c_1^2 dz_2^2$$

if we choose $c_1^2 a_1^2 b^2 = (1 - c_1^2)$ to simplify the integration for a particular case. Then

$$z_2 = \frac{\sqrt{1 - c_1^2}}{c_1 b} \sin bx + C = a_1 \sin bx + C.$$

The curve $z_1 = a_1 \cos bx$, $z_2 = a_1 \sin bx$ is a circular helix about the axis of $x$ in the $xz_1z_2$ space. The four dimensional surface of revolution is

$$z_1 = a_1 \cos b \sqrt{x^2 + y^2}, \qquad z_2 = a_1 \sin b \sqrt{x^2 + y^2}.$$

We see therefore that: *The developables of revolution when $n > 3$ form an extended class of surfaces instead of reducing merely to the cones and cylinders.*

The value of $\mathbf{h}$ is given by

$$2\mathbf{h} = \mathbf{p} + (\mathbf{r} + xx'\mathbf{m})/x^2.$$

If we designate $\mathbf{i} \cos\theta + \mathbf{j} \sin\theta$ by $\mathbf{u}$, a unit vector,

$$\mathbf{p} = x''\mathbf{u} + \Sigma \mathrm{k}_i z_i'', \quad \mathbf{r} = -x\mathbf{u}, \quad \mathbf{m} = x'\mathbf{u} + \Sigma \mathbf{k}_i z_i'$$

and

$$2\mathbf{h} = \mathbf{u}(xx'' + x'^2)/x + \Sigma \mathbf{k}_i (z_i'' + x'z'/x).$$

The condition $\mathbf{h} = 0$ for a minimal surface therefore is

$$xx'' - 1 + x'^2 = 0, \quad z_i'' + x'z_i'/x = 0.$$

The last equation shows that $z_i'x = c_i$, the first that $x^2 = (s + b)^2 + a^2$. Hence

$$\frac{z_i + K_i}{c_i} = \cosh^{-1} \frac{x}{a}, \quad i = 1, 2, \ldots, n - 2,$$

with the condition $\Sigma c_i^2 = a^2$. From this it follows that

$$\frac{z_1 + K_1}{c_1} = \frac{z_2 + K_2}{c_2} = \ldots = \frac{z_{n-2} + K_{n-2}}{c_{n-2}}.$$

The curve therefore lies in a plane through the $x$ axis and some line in the $z$ space; it is the common catenary and the result is: *The only minimal surface of revolution is the ordinary catenoid.*[47]

As $a_{12} = 0$ and $\mathbf{y}_{12} = 0$, $\boldsymbol{\mu}\times\boldsymbol{\delta} = 0$. *All surfaces of revolution are of the type for which the indicatrix reduces to a linear segment.* Our lines of curvature coincide with Segre's characteristics and both lie along the circles and the various directions assumed by the directrix in the revolution.

**49. Note on a vectorial method of treating surfaces.** Another general method of dealing with the theory of surfaces upon a vector basis may be mentioned without going much into details. In the ordinary three dimensional case we set up the linear vector function $\Phi$ which expresses the differential normal $d\mathbf{n}$ in terms of the displacement $d\mathbf{r}$, i. e., $d\mathbf{n} = d\mathbf{r}\cdot\Phi$. As the properties of dyadics $\Phi$ are well known many properties of the surface may be found at once.[48]

In the general case the tangent plane $d\mathbf{M}$ is connected linearly with the displacement $d\mathbf{r}$. In fact if $d\mathbf{r} = \mathbf{m}\,du + \mathbf{n}\,dv$, a differentiating operator

$$\nabla = \mathbf{n}\cdot\mathbf{M}\frac{\partial}{\partial u} - \mathbf{m}\cdot\mathbf{M}\frac{\partial}{\partial v} \qquad (125)$$

may be written down which is invariant under a change of parameters. (This is obvious since

$$d\mathbf{r}\cdot\nabla = \mathbf{m}\cdot(\mathbf{n}\cdot\mathbf{M})du\frac{\partial}{\partial u} - \mathbf{n}\cdot(\mathbf{m}\cdot\mathbf{M})dv\frac{\partial}{\partial v} = du\frac{\partial}{\partial u} + dv\frac{\partial}{\partial v},$$

---

[47] A geometric proof may be given as follows. Since the surface is minimum $\mathbf{h} = 0$, and since the surface is of revolution the indicatrix reduces to a segment (see below). Consequently the minimum surface of revolution is one for which every point is axial with the center of the indicatrix (not its end) at the surface point. Hence the minimum surface of revolution must lie in three dimensions and in this case the surface is known to be a catenoid.

[48] For a brief discussion see Gibbs-Wilson, *Vector Analysis*, p. 411 ff.

which makes $d\mathbf{r}\cdot\nabla = d$. If desired it is possible to remove the condition that $\mathbf{M}$ be a unit tangent plane by writing

$$\nabla = \frac{\mathbf{n}\cdot\mathbf{M}}{\mathbf{M}^2}\frac{\partial}{\partial u} - \frac{\mathbf{m}\cdot\mathbf{M}}{\mathbf{M}^2}\frac{\partial}{\partial v},$$

where $\mathbf{M}$ is $\mathbf{m}\times\mathbf{n}$) We have then, in the general case where $n > 3$,

$$d\mathbf{M} = d\mathbf{r}\cdot\nabla\mathbf{M} = d\mathbf{r}\cdot\Lambda, \quad \Lambda = \nabla\mathbf{M}, \tag{126}$$

where $\mathbf{M}$ is the unit tangent plane, to correspond to $d\mathbf{n} = d\mathbf{r}\cdot\Phi$ in the particular case $n = 3$.

The dyadic $\Lambda$, however, is one in which the antecedent vectors in the dyads are 1-vectors and the consequent vectors are 2-vectors, i. e., planes, simple or otherwise,—

$$\Lambda = \mathbf{n}\cdot\mathbf{M}\frac{\partial\mathbf{M}}{\partial u} - \mathbf{m}\cdot\mathbf{M}\frac{\partial\mathbf{M}}{\partial v}. \tag{127}$$

Further

$$\frac{d\mathbf{M}}{ds} = \frac{d\mathbf{r}}{ds}\cdot\Lambda = \mathbf{t}\cdot\Lambda,$$

where $\mathbf{t}$ is a unit tangent 1-vector in any direction. The rate of change of the tangent plane in the direction $\mathbf{t}$ is therefore $\mathbf{t}\cdot\Lambda$.

The properties of a 1–2 dyadic such as $\Lambda$ are not well known and the development of the surface theory from this point of view is therefore hampered. Some points, however, are readily ascertained. First, there is an invariant or covariant line (1-vector) and an invariant space (3-vector) obtained from $\Lambda$ by the familiar processes of inserting the signs of scalar and vector products between the elements of the dyadic,— thus

$$\begin{aligned} \mathbf{l} &= (\mathbf{n}\cdot\mathbf{M})\cdot\frac{\partial\mathbf{M}}{\partial u} - (\mathbf{m}\cdot\mathbf{M})\cdot\frac{\partial\mathbf{M}}{\partial v} \\ \mathbf{S}_3 &= (\mathbf{n}\cdot\mathbf{M})\times\frac{\partial\mathbf{M}}{\partial u} - (\mathbf{m}\cdot\mathbf{M})\times\frac{\partial\mathbf{M}}{\partial v} \end{aligned} \tag{128}$$

By the transformation $(\mathbf{b}\cdot\mathbf{C})\cdot\mathbf{A} = -(\mathbf{C}\cdot\mathbf{A})\mathbf{b} + \mathbf{C}\cdot(\mathbf{b}\times\mathbf{A})$,

$$\mathbf{l} = -\mathbf{n}\mathbf{M}\cdot\frac{\partial\mathbf{M}}{\partial u} + \mathbf{M}\cdot\left((\mathbf{n}\times\frac{\partial\mathbf{M}}{\partial u}\right) + \mathbf{m}\mathbf{M}\cdot\frac{\partial\mathbf{M}}{\partial v} - \mathbf{M}\cdot\left(\mathbf{m}\times\frac{\partial\mathbf{M}}{\partial v}\right).$$

As $\mathbf{M}^2 = 1$, the first and third terms vanish; and as $\partial\mathbf{M}/\partial u$ and $\partial\mathbf{M}/\partial v$ are perpendicular to $\mathbf{M}$, so must the spaces $\mathbf{n}\times\partial\mathbf{M}/\partial u$ and $\mathbf{m}\times\partial\mathbf{M}/\partial v$ be perpendicular to $\mathbf{M}$, and the other two terms will vanish. Hence: *The vector invariant* $\mathbf{l}$ *of the dyadic* $\Lambda$ *vanishes.*

The invariant 3-vector $\mathbf{S}_3$ may be calculated. The work may be simplified by taking the parameter curves orthogonal with $u$ and $v$ equal to the arc along these curves (except for infinitesimals) in the neighborhood of any preassigned point.

Then $\mathbf{m} = \boldsymbol{\xi}$, $\mathbf{n} = \boldsymbol{\eta}$ and

$$\begin{aligned}\mathbf{S}_3 &= [\mathbf{n}\cdot(\boldsymbol{\xi}\times\boldsymbol{\eta})] \times \frac{d(\boldsymbol{\xi}\times\boldsymbol{\eta})}{ds} - [\boldsymbol{\xi}\cdot(\boldsymbol{\xi}\times\boldsymbol{\eta})] \times \frac{d(\boldsymbol{\xi}\times\boldsymbol{\eta})}{ds}\\ &= [\mathbf{n}\cdot(\boldsymbol{\xi}\times\boldsymbol{\eta})] \times [\boldsymbol{\alpha}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\mu}] - [\boldsymbol{\xi}\cdot(\boldsymbol{\xi}\times\boldsymbol{\eta})] \times [\boldsymbol{\mu}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\beta}]\\ &= \boldsymbol{\xi}\times\boldsymbol{\alpha}\times\boldsymbol{\eta} + \boldsymbol{\eta}\times\boldsymbol{\xi}\times\boldsymbol{\beta} = -(\boldsymbol{\xi}\times\boldsymbol{\eta}) \times (2\mathbf{h}) = -2\mathbf{M}\times\mathbf{h}.\end{aligned}$$

Hence: *The invariant 3-vector* $\mathbf{S}_3 = \Lambda_\times$ *is* $-2\,\mathbf{M}\times\mathbf{h}$, *the space of the tangent plane and the mean curvature, and of magnitude equal to the mean curvature.*

Other invariants of the dyadic

$$\Lambda = \boldsymbol{\xi}(\boldsymbol{\alpha}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\mu}) + \boldsymbol{\eta}(\boldsymbol{\mu}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\beta})$$

are the dyadics $\Lambda\cdot\Lambda_C$, $\Lambda_C\cdot\Lambda$ $\Lambda{:}\Lambda_C$, and so on, and the quantities obtained from them by inserting dots and crosses. For instance,

$$\Lambda_C\cdot\Lambda = (\boldsymbol{\alpha}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\mu})\,(\boldsymbol{\alpha}\times\boldsymbol{\xi} + \boldsymbol{\xi}\times\boldsymbol{\mu}) + (\boldsymbol{\mu}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\beta})\,(\boldsymbol{\mu}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\beta}),$$

and $$\mathbf{T}_4 = (\Lambda_C\cdot\Lambda)_\times = -2\mathbf{M}\times[(\boldsymbol{\alpha} - \boldsymbol{\beta})\times\boldsymbol{\mu}] = 4\mathbf{M}\times\boldsymbol{\mu}\times\boldsymbol{\delta},$$

$$\Lambda{:}\Lambda_C = 2(\boldsymbol{\delta}\boldsymbol{\delta} + \boldsymbol{\mu}\boldsymbol{\mu} - \mathbf{h}\mathbf{h}).$$

Hence $-\,\mathbf{M}\cdot\mathbf{S}_3 = 2\mathbf{h}$, $\mathbf{M}\cdot\mathbf{T}_4 = 4\boldsymbol{\mu}\times\boldsymbol{\delta}$, $-\Lambda{:}\Lambda_c = 2\Phi/a$ are the quantities, found directly from the fundamental dyadic $\Lambda$, which have been found of prime importance in the theory of surfaces.[49]

---

[49] The line of development here followed is the inverse of that which would be followed in developing the surface theory from $\Lambda$. It is for brevity that we choose merely to verify that the already known quantities $\mathbf{h}$, $\boldsymbol{\mu}\times\boldsymbol{\delta}$, and $\Phi$ of surface theory may be derived from $\Lambda$.

One of the first problems in discussing 1–2 dyadics of the type $\mathbf{aA} + \mathbf{bB}$ would be the establishment of a standard form. If $\mathbf{a}$ and $\mathbf{b}$ be replaced by linear combinations $x\mathbf{a}' + y\mathbf{b}'$, $x'\mathbf{a}' + y'\mathbf{b}'$ of two vectors in their plane, the dyadic becomes

$$\Lambda = \mathbf{a}'(x\mathbf{A} + x'\mathbf{B}) + \mathbf{b}'(y\mathbf{A} + y'\mathbf{B}) = \mathbf{a}'\mathbf{A}' + \mathbf{b}'\mathbf{B}',$$

where $\mathbf{A}'$, $\mathbf{B}'$ are linear combinations of $\mathbf{A}$ and $\mathbf{B}$. We may then consider that for the antecedents $\mathbf{a}'$, $\mathbf{b}'$ we have chosen unit normal vectors $\mathbf{i}$, $\mathbf{j}$ so that $\Lambda = \mathbf{iA} + \mathbf{jB}$. If a rotation is carried out on $\mathbf{i}$, $\mathbf{j}$, we have

$$\Lambda = \mathbf{i}'(\mathbf{A}\cos\varphi - \mathbf{B}\sin\varphi) + \mathbf{j}'(\mathbf{A}\sin\varphi + \mathbf{B}\cos\varphi) = \mathbf{i}'\mathbf{A}' + \mathbf{j}'\mathbf{B}'.$$

The condition $\mathbf{A}' \cdot \mathbf{B}' = 0$, i. e., the condition that the consequents be orthogonal is that $\varphi$ be determined from

$$\tan 2\varphi = 2\mathbf{A}\cdot\mathbf{B}/(\mathbf{A}^2 - \mathbf{B}^2),$$

which gives four values of $\varphi$ spaced at right angles. Hence: *We may reduce* $\Lambda$ *to the form*

$$\Lambda = \mathbf{iA} + \mathbf{jB}, \quad \mathbf{A}\cdot\mathbf{B} = 0, \tag{129}$$

*and this reduction is unique* (except for the indeterminateness of an interchange of $\mathbf{i}$ and $\mathbf{j}$ or a reversal of the sign of either). The reduction is wholly indeterminate when $\varphi$ is indeterminate, i. e., when $\mathbf{A}\cdot\mathbf{B} = 0$ and $\mathbf{A}^2 = \mathbf{B}^2$. *In the special reduced form* (129), *the directions* $\mathbf{i}$, $\mathbf{j}$ *are along the principal directions on the surface in case we use for principal directions the definition 3 introduced and preferred by us.*[50]

If $\mathbf{j}$ is chosen relative to $\mathbf{i}$ and $\mathbf{M}$ so that $\mathbf{M} = \mathbf{i}\times\mathbf{j}$, we have

$$\boldsymbol{\alpha} = \mathbf{i}\cdot\Lambda\cdot\mathbf{j}, \quad \boldsymbol{\beta} = -\mathbf{j}\cdot\Lambda\cdot\mathbf{i}, \quad \boldsymbol{\mu} = -\mathbf{i}\cdot\Lambda\cdot\mathbf{i} = \mathbf{j}\cdot\Lambda\cdot\mathbf{j}.$$

As $\mathbf{ij} - \mathbf{ji}$ is a dyadic independent of the directions of $\mathbf{i}$ in $\mathbf{M}$,

$$2\mathbf{h} = (\mathbf{ij} - \mathbf{ji}):\Lambda = \boldsymbol{\alpha} + \boldsymbol{\beta}$$

---

[50] Note the correspondence with three dimensions. If we have the dyadic $\mathbf{a}\boldsymbol{\alpha} + \mathbf{b}\boldsymbol{\beta} = \Phi$ where $d\mathbf{n} = d\mathbf{r}\cdot\Phi$, we may reduce to the form $\mathbf{i}\boldsymbol{\alpha} + \mathbf{j}\boldsymbol{\beta}$, $\boldsymbol{\alpha}\cdot\boldsymbol{\beta} = 0$, which is a reduction to the principal axes of $\Phi$, and have then $\mathbf{i}$, $\mathbf{j}$ along the principal directions, since $\Phi$ is self-conjugate.

is an invariant of $\Lambda$, as verified above in a different way. Further $2\delta = \mathbf{i}\cdot\Lambda\cdot\mathbf{j} + \mathbf{j}\cdot\Lambda\cdot\mathbf{i}$ may by a rotation of $\mathbf{i}, \mathbf{j}$ into $\mathbf{i}', \mathbf{j}'$ be seen to describe a conic, our indicatrix.

These brief remarks must suffice to indicate the ease and directness of the vector method of discussing surfaces through the use of the dyadic $\Lambda = \nabla\mathbf{M}$ which is determined by the relation $d\mathbf{M} = d\mathbf{r}\cdot\Lambda$, where $\nabla$ is a sort of surface differentiation built from analogy with the ordinary $\nabla$ and defined identically with it by the equation $d = d\mathbf{r}\cdot\nabla$.

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# *A CLASSIFICATION OF QUADRATIC VECTORS.*

By Frank Lauren Hitchcock.

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## INTRODUCTION.

In this paper it is shown that three ternary quadratic forms $X$, $Y$, and $Z$, homogeneous polynomials of the second degree in the variables $x$, $y$, and $z$, can in general be thrown into the form

$$X = vw_1 - v_1w + tx,\quad Y = wu_1 - w_1u + ty,\quad Z = uv_1 - u_1v + tz, \qquad \text{(I)}$$

where the seven letters: $u$, $v$, $w$; $u_1$, $v_1$, $w_1$; and $t$; — denote linear forms, that is, they are homogeneous polynomials of the first degree in $x$, $y$, $z$.

In the language of vector algebra, the scalars $X$, $Y$, $Z$ are the components of a vector $F(\rho)$, or simply $F\rho$. The linear forms $u$, $v$, and $w$ are components of a linear vector $\phi\rho$, and $u_1$, $v_1$, $w_1$ are components of a second linear vector $\theta\rho$. The linear form $t$ is regarded as the scalar product of a constant vector $\delta$ and the point-vector $\rho$. The above statement translates into the vector equation

$$F\rho = V\phi\rho\,\theta\rho + \rho S\delta\rho, \qquad \text{(II)}$$

or, in words, a quadratic vector function, homogeneous in $\rho$, can be expressed as the vector product of two linear vector functions, aside from a properly chosen scalar multiple of the point-vector.

The significance of this result lies, in part, in the fact that, in various problems depending on three quantics, the term in $\rho$ may be taken arbitrarily. For example, the differential equation

$$(yZ - zY)dx + (zX - xZ)dy + (xY - yX)\,dz = 0$$

in vector language becomes $S\rho F\rho d\rho = 0$, or $Sd\rho V\rho F\rho = 0$, and is thus independent of the term in $\rho$ and of the linear form $t$. The vector $V\rho F\rho$ may in general be factored into $V\rho V\phi\rho\theta\rho$.

It is also shown that the vector $\delta$, equivalent to the linear form $t$, may in general be determined in thirty-five ways. The three scalar equations (I), equivalent to the vector equation (II), ($X$, $Y$, and $Z$ being given and the right members to be determined), are in general equivalent to eighteen quadratic equations. The solution depends upon an equation of the seventh degree, which determines seven sets of values of $x$, $y$, and $z$, called axes of the quadratic vector. When these axes are found we can then find $\delta$ and $t$.

The thirty-five ways of finding $t$ are diminished in number by various special relations among the constituents of $X$, $Y$, and $Z$. These restrictions are simply expressible in terms of configurations of the axes. It is shown that at least one value of $t$ can be found, that is, that the eighteen quadratics have a solution, except in two cases.

Methods of finding $\delta$ or $t$ are worked out for all possible configurations of the axes. This is done by means of normal forms, or model vectors, including various classes of quadratic vectors.

The use made of the technical methods of vector algebra is slight in the first part of the paper, more extended when dealing with conditions of multiplicity among the axes. The results, while they were invariably obtained by vector algebra, can be verified in most instances by the reader who has only a slight acquaintance with these methods.

The classification of vectors under normal or type forms is worked out on the following scheme: —

Class I. Seven distinct axes. Type form (31).

Class II. At least one axis is of order two or more. These are subdivided into

1. Vectors having one or more double axes but no triple axis. The double axes may be one, two, or three, in number. Types Art. 16.

2. Vectors having at least one triple axis but no quadruple axis. Of these, there may be either one, or two, double axes. We may also have two distinct triple axes. Types Art. 25.

3. Vectors having an axis of order four or higher. These are of two distinct kinds, according as the multiple axis is —

Class 1°. A double element of all cones $V\rho F\rho = 0$. Type forms are discussed in Art. 27 and 37.

Class 2°. Not a double element for all cones $V\rho F\rho = 0$. Types Art. 30, including the two forms (185) and (195).

Class III. The number of axes is infinite. There are three special forms (57). The vector $V\rho F\rho$ possesses a scalar factor, and is said to be reducible. The equations (II) are not considered with reference to this class, the forms (57) being equally simple.

## PART ONE.

**1.** In many problems of Geometry and of Physics we meet with quantities which, occurring at the points of a portion of space, possess a definite direction, as well as magnitude or length along that direction. The velocity of a fluid at a point, and the force at a point due to the attraction of an assigned distribution of matter, are familiar examples. If such a directed quantity, or geometrical vector, be resolved along chosen axes of reference, it yields components $X$, $Y$, and $Z$, which are ordinary, or scalar, functions of $x$, $y$, and $z$, the coördinates of a point in space. Approaching the matter from the side of Algebra, both the independent variables and the component functions will, in general, be free to take on complex values.

If we agree upon the following four conventions,—

1. The vectors whose components are (1, 0, 0,), (0, 1, 0), and (0, 0, 1) are denoted, respectively, by $i$, $j$, and $k$.
2. Multiplication of a vector by a scalar means multiplication of all components of the vector by the scalar.
3. Vectors are added by adding their corresponding components.
4. Equality of vectors implies equality of corresponding components,—

it follows that we may write, for the vector $\rho$ from the origin to a point in space,

$$\rho = ix + jy + kz \qquad (1).$$

A vector function of $\rho$ may be denoted by $F(\rho)$, whence

$$F(\rho) = iX + jY + kZ \qquad (2).$$

My present object is to contribute something toward a theory of those vectors whose components are homogeneous polynomials of the second degree in the variables $x$, $y$, $z$. That is, $X$, $Y$, and $Z$ are ternary quadratic forms. The theory of my vectors will then be, up to a certain point, in close relation with the theory of a set of three forms, but the name vector implies also a definite order, $X$, $Y$, $Z$, among these forms; and we have, moreover, the above defined process of addition of vectors.

**2.** The classification which I propose to make of various types of quadratic vector depends upon the existence of directions called *axes*, which are directions of the point-vector $\rho$ such that we have simultaneously

$$yZ - zY = 0, \quad zX - xZ = 0, \quad xY - yX = 0 \tag{3}$$

Geometrically stated, an axis of $F(\rho)$ is a direction of $\rho$ such that $F(\rho)$ either vanishes in all its components or is parallel to $\rho$. Those axes, if any, for which $F(\rho)$ vanishes, may be called the *zeros* of $F(\rho)$.

It is well known that the number of sets of values of the ratios of $x$, $y$, and $z$ satisfying equations (3) is, when $X$, $Y$, and $Z$ are ternary forms of degree $p$, in general equal to $p^2 + p + 1$.[1] A quadratic vector form has therefore, in general, seven axes.

Of these seven axes, however, some may be in coincidence, giving multiple axes.

If, on the other hand, there are more than seven axes, there are an infinite number. For let there be eight or more axes, and suppose their number finite. Equations (3), if satisfied, will subsist after a change of the coördinate system. We may therefore suppose no axis to lie in the plane $z = 0$. The first two of equations (3) will have ten or more solutions, viz. the two lines of intersection of the plane $z = 0$ with the quadric cone $Z = 0$, and the eight or more axes of the vector. But two cubic equations with ten solutions have an infinite number. The third equation is a consequence of the first two wherever $z$ is not zero. Hence the three equations have a common linear or quadratic factor, and the number of axes cannot be finite.

These facts suggest a division of quadratic vectors into three types,—

I. General type. There are seven, and only seven distinct axes.

---

[1] For two proofs from very different points of view, see Darboux, "Memoire sur les Equations Differentieles Algébriques," Bul. Sci. Math. 13, (1878), p. 83, where the solutions of (3) give the singular points of a differential equation; and Clebsch, "Lecons sur la Geometrie," (Tr. Benoist), t. II p. 113 and t. III p. 435, (Vorl. u. Geom. Bd. I s. 390, 1001), where these equations are connected with a quadratic complex.

Again, if we let a point transformation in homogeneous coördinates be defined by the equations $x = X$, $y = Y$, $z = Z$, then (3) gives the fixed points, together with the singular points.

A short proof for the present case is as follows,—let the first two equations of (3) define two cubic curves. They intersect, in general, in nine points, of which two are the intersections of the line $z = 0$ and the conic $Z = 0$. If $z$ and $Z$ are not both zero the determinant $xY - yX$ must vanish. Hence at 9–2 points equations (3) hold simultaneously.

II. The number of distinct axes is less than seven, some being multiple.

III. The number of axes is infinite. It will be convenient to speak of this as the reducible type.

**3.** If the number of axes is infinite, there exists either a plane or a quadric cone (according as the common factor of (3) is linear or quadratic), such that any direction of $\rho$ in the plane or the cone gives an axis of the vector.

Conversely, that a quadratic vector have an infinite number of axes, it is sufficient that it have six distinct axes on a quadric cone, proper or degenerate. For suppose the number of axes finite. Let the coördinate system be so taken that no axis lies in the plane $z = 0$. If the first two equations (3) have six solutions on a quadric, the remaining three are linearly related.[2] That is, the seventh axis of the vector lies in the same plane with the two lines of intersection of $z = 0$ and $Z = 0$, contrary to hypothesis. Hence the number of axes cannot be finite.

**4.** The axes are not altered by adding to the vector a term of the form $t\rho$, where $t$ is a scalar. This is geometrically obvious, for if $\rho$ and $F(\rho)$ are parallel, extending $F(\rho)$ in the direction $\rho$ will not disturb the parallelism. Analytically, $t$ must, in the present case, be a linear form in $x, y, z$. If we write

$$t = px + qy + rz \tag{4}$$

$p$, $q$, and $r$ being constants, the addition of $t\rho$ to $F(\rho)$ is the same as putting $X + x(px + qy + rz)$ for $X$, with similar expressions put for $Y$ and $Z$. If we make these substitutions in (3), the equations are invariant.[3]

It is equally obvious that the axes of a vector are not altered if we multiply it by any non-vanishing scalar constant.

---

[2] Salmon, Higher plane curves, Art. 24, 1st Ed.

[3] This is directly seen if we introduce vector multiplication, (see part II, below), for equations (3) are equivalent to the vector equation $V\rho F\rho = 0$. We may change $F\rho$ into $F\rho + t\rho$ at will, since $\rho^2$ is a scalar.

The results of Arts. 2–4 are evidently applicable with slight modification to homogeneous vectors of any degree, the components being polynomials.

## VECTORS OF THE FIRST OR GENERAL TYPE.

5. **Theorem I.** A quadratic vector of type I is completely determined by its axes, aside from a constant non-vanishing multiplier $h$ and an additive term $t\rho$.

The truth of the theorem would, on the proviso that the axes are independent, appear from a count of the scalar constants involved. For the three quadratic forms $X$, $Y$, $Z$ involve eighteen scalars. The multiplier $h$, and the three scalars which determine $t$, leave fourteen. The seven axes, if independent, involve fourteen.

That the axes may, in fact, be assigned arbitrarily, I shall show by expressing $F(\rho)$ in terms of its axes by means of a vector equation.

The theorem further implies that, no matter what special relations exist among the axes of $F(\rho)$, provided they are distinct and of finite number, any other vector having the same axes may be written $hF(\rho) + t\rho$.

To prove the theorem, let any seven axes be chosen, distinct, with no six on any quadric cone. Let vectors in these seven directions be denoted by $\beta_1, \beta_2, \ldots \beta_7$. We may, without loss of generality, suppose that $\beta_1$, $\beta_2$, and $\beta_3$ are not coplanar, and, at the same time, that $\beta_4$, $\beta_5$, and $\beta_6$ are not coplanar. For the seven axes cannot all lie in the same plane, because such a plane, with any other plane, would constitute a degenerate quadric cone, contrary to hypothesis. Accordingly, let any three non-coplanar axes be called $\beta_1$, $\beta_2$, and $\beta_3$. The remaining four axes cannot all lie in the same plane, for, with the plane of $\beta_1$ and $\beta_2$, it would constitute a quadric cone. Let any three which are not coplanar be taken from the four and called $\beta_4$, $\beta_5$, and $\beta_6$.

I shall now determine the constants $p$, $q$, and $r$, of the linear form $t$, so that $\beta_1$, $\beta_2$, and $\beta_3$ shall be zeros of the vector $F(\rho) + t\rho$; that is, so that we have, simultaneously,

$$F(\beta_1) + t\beta_1 = 0, \quad F(\beta_2) + t\beta_2 = 0, \quad F(\beta_3) + t\beta_3 = 0. \tag{5}$$

Let the components of $\beta_1$, $\beta_2$, $\beta_3$ be given by

$$\beta_1 = ib_{11} + jb_{12} + kb_{13}, \quad \beta_2 = ib_{21} + jb_{22} + kb_{23},$$
$$\beta_3 = ib_{31} + jb_{32} + kb_{33}.$$

Let the determinant of the components be denoted by (123), i. e.

$$(123) = \begin{vmatrix} b_{11}, & b_{12}, & b_{13} \\ b_{21}, & b_{22}, & b_{23} \\ b_{31}, & b_{32}, & b_{33} \end{vmatrix} \tag{6}$$

This determinant is not zero, i. e., by hypothesis $\beta_1, \beta_2, \beta_3$ are not coplanar. That these $\beta$'s are axes of $F(\rho)$ is equivalent to writing

$$F(\beta_1) = c_1\beta_1; \ \mathrm{F}(\beta_2) = c_2\beta_2; \ F(\beta_3) = c_3\beta_3 \tag{7}$$

$c_1, c_2, c_3$ being constants; whence (4) and (5) yield

$$\begin{aligned} c_1 + pb_{11} + qb_{12} + rb_{13} &= 0 \\ c_2 + pb_{21} + qb_{22} + rb_{23} &= 0 \\ c_3 + pb_{31} + qb_{32} + rb_{33} &= 0 \end{aligned} \tag{8}$$

three linear equations in the three unknowns $p, q, r$. Since (123) does not vanish, the solution is uniquely possible. Let the values of $p, q, r$ thus determined be $p_0, q_0, r_0$, and write

$$F_0(\rho) = F(\rho) + t_0\rho$$

The relations (5) are then equivalent to

$$F_0(\beta_1) = 0, \quad F_0(\beta_2) = 0, \quad F_0(\beta_3) = 0.$$

The vector $F_0(\rho)$ is most simply expressed in a new coördinate system given by writing $\rho = \beta_1 x_1 + \beta_2, x_2 + \beta_3, x_3$, equivalent to

$$x_1 = \frac{(23\rho)}{(123)}, \quad x_2 = \frac{(31\rho)}{(123)}, \quad x_3 = \frac{(12\rho)}{(123)}, \tag{9}$$

where $(23\rho)$ denotes the determinant[4]

$$\begin{vmatrix} b_{21}, & b_{22}, & b_{23} \\ b_{31}, & b_{32}, & b_{33} \\ x \ , & y \ , & z \end{vmatrix}$$

with similar meaning for $(31\rho)$ and $12\rho)$. If we write $\beta_1$ for $\rho$ we have simultaneously $x_2 = 0$ and $x_3 = 0$. Since $F_0(\beta_1)$ vanishes in all its components, no component can, in the new coördinate system, contain any term in $x_1^2$. Similarly, no component can contain any term in $x_2^2$ nor in $x_3^2$. If the original components of $F_0(\rho)$ be called $X_0$, $Y_0$, and $Z_0$, that is

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[4] Using scalar products of three vectors, we may write $x_1 = \dfrac{S\beta_2\beta_3\rho}{S\beta_1\beta_2\beta_3}$ and similarly for $x_2$ and $x_3$.

$$F_0(\rho) = iX_0 + jY_0 + kZ_0,$$

the change of coördinates may be analytically represented by

$$\begin{aligned} X_0 &= X + x(p_0x + q_0y + r_0z) = a_{11}x_2x_3 + a_{12}x_3x_1 + a_{13}x_1x_2 \\ Y_0 &= Y + y(p_0x + q_0y + r_0z) = a_{21}x_2x_3 + a_{22}x_3x_1 + a_{23}x_1x_2 \\ Z_0 &= Z + z(p_0x + q_0y + r_0z) = a_{31}x_2x_3 + a_{32}x_3x_1 + a_{33}x_1x_2, \end{aligned} \qquad (10)$$

where the nine $a$'s are constants to be determined. If three vectors $\alpha_1$, $\alpha_2$, and $\alpha_3$ be defined by

$$\alpha_1 = ia_{11} + ja_{21} + ka_{31}, \quad \alpha_2 = ia_{12} + ja_{22} + ka_{32}, \quad \alpha_3 = ia_{13} + ja_{23} + ka_{33}, \qquad (11)$$

it is obvious that the transformation (10) is equivalent to

$$F_0(\rho) = \alpha_1x_2x_3 + \alpha_2x_3x_1 + \alpha_3x_1x_2 \qquad (12)$$

Taking this result as one step in the demonstration of theorem I, we note that the form of the right member is determined when the choice of axes $\beta_1$, $\beta_2$, and $\beta_3$ has been made. In other words, any two vectors, alike in having $\beta_1$, $\beta_2$, $\beta_3$ for axes can, by a proper choice of $p$, $q$, and $r$, be thrown into the form (12), and will then differ in the vectors $\alpha$ but not otherwise.

Consider next the disposition to be made of the $\alpha$'s that $\beta_4$, $\beta_5$, and $\beta_6$ may be axes of $F_0(\rho)$ and therefore of $F(\rho)$. If $F_0(\beta_4)$ is a scalar multiple of $\beta_4$, the determinant of the coefficients of the three vectors $\beta_4$, $\beta_5$ and $F_0(\beta_4)$ vanishes. Abbreviate this determinant by $(45F_{04})$. Similarly $(45F_{05})$ vanishes if $F_0(\beta_5)$ is parallel to $\beta_5$. Advancing cyclically the subscripts 4, 5, 6, we have in this manner six necessary conditions

$$\begin{aligned} &(45F_{04}) = 0, \quad (56F_{05}) = 0, \quad (64F_{06}) = 0, \\ &(45F_{05}) = 0, \quad (56F_{06}) = 0, \quad (64F_{04}) = 0, \end{aligned} \qquad (13)$$

They are also sufficient; for, pairing the six relations in a different manner,

$$\begin{aligned} &(45F_{04}) = 0, \quad (56F_{05}) = 0, \quad (64F_{06}) = 0, \\ &(64F_{04}) = 0, \quad (45F_{05}) = 0, \quad (56F_{06}) = 0, \end{aligned}$$

we see that the two equations of the first column require that $F_0(\beta_4)$ shall, at the same time, lie in the plane of $\beta_4$, $\beta_5$, and the plane of $\beta_6$,

$\beta_4$. That is, $F_0(\beta_4)$ coincides with $\beta_4$ aside from a scalar factor. The sufficiency of the conditions we may, if we prefer, show analytically, by using components $b_{41}$, $b_{42}$, $b_{43}$, for $\beta_4$, etc.; and remembering that (456) does not vanish.

From (12) and (9) we have

$$(123)^2 F_0(\rho) = a_1(31\rho)\,(12\rho) + a_2(12\rho)\,(23\rho) + a_3(23\rho)\,(31\rho), \qquad (14)$$

By writing respectively $\beta_4$ and $\beta_5$ for $\rho$ in this last equation, we obtain the two equations of the first column of (13) in the form [5]

$$\left.\begin{aligned}(45a_1)\,(314)\,(124) + (45a_2)\,(124)\,(234) + (45a_3)\,(234)\,(314) = 0\\ (45a_1)\,(315)\,(125) + (45a_2)\,(125)\,(235) + (45a_3)\,(235)\,(315) = 0\end{aligned}\right\}(15)$$

two linear homogeneous equations in the three quantities $(45a_1)$, $(45a_2)$, and $(45a_3)$. The two-row determinants from the matrix of the coefficients cannot all be zero. For, identically,

$$\begin{vmatrix}(124)\,(234), & (234)\,(314)\\ (125)\,(235), & (235)\,(315)\end{vmatrix} \equiv (234)\,(235)\,[(124)\,(315) - (125)\,(314)]$$
$$\equiv (234)\,(235)\,(415)\,(123) \qquad (16)$$

whence, with similar transformations [6] for the other two-row determinants, and remembering that (123) does not vanish, we see that the simultaneous vanishing of the three determinants would be equivalent to

$$(234)\,(235)\,(415) = 0,\ (314)\,(315)\,(425) = 0,\ (124)\,(125)\,(435) = 0 \qquad (17)$$

We cannot have (234) and (314) both zero, for $\beta_3$ and $\beta_4$ were taken distinct in direction. We may, however, satisfy the first two conditions by assuming $(234) = (315) = 0$. But (124) cannot vanish with (234), (125) cannot vanish with (315), and (435) cannot vanish with (315), since no four axes can lie in one plane. Similarly we may exclude all other combinations of vanishing factors, one from each of the three equations.

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[5] Using products of vectors, we multiply both sides of (14) by the quaternion $\beta_4\beta_5$ and take scalars.

[6] Using scalar products, the bracketed terms $= S\beta_1\beta_2(\beta_4 S\beta_3\beta_1\beta_5 - \beta_5 S\beta_3\beta_1\beta_4)$ $= S\beta_1\beta_2 V(V\beta_3\beta_1 V\beta_5\beta_4) = S\beta_1\beta_2\beta_3 S\beta_1\beta_5\beta_4 =$ (123) (154).

Equations (15) therefore determine the three quantities $(45\alpha_1)$, $(45\alpha_2)$, and $(45\alpha_3)$, in terms of a constant of proportionality, which I shall denote by $k_6$, as follows,

$$(45\alpha_1)=k_6\begin{vmatrix}(124)\,(234), & (234)\,(314)\\(125)\,(235), & (235)\,(315)\end{vmatrix}=k_6(123)\,(234)\,(235)\,(415), \quad (18)$$

$$(45\alpha_2)=k_6\begin{vmatrix}(234)\,(314), & (314)\,(124)\\(235)\,(315), & (315)\,(125)\end{vmatrix}=k_6(123)\,(314)\,(315)\,(425), \quad (19)$$

$$(45\alpha_3)=k_6\begin{vmatrix}(314)\,(124), & (124)\,(234)\\(315)\,(125), & (125)\,(235)\end{vmatrix}=k_6(123)\,(124)\,(125)\,(435), \quad (20)$$

These equations are changed into one another by cyclic advancement of the numbers 1, 2, and 3. By advancing cyclically the numbers 4, 5, 6 we may obtain two other sets, of three equations each, sufficient to determine the quantities

$$(56\alpha_1),\quad (56\alpha_2),\quad (56\alpha_3);\text{ and }(64\alpha_1),\quad (64\alpha_2),\quad (64\alpha_3);$$

respectively in terms of two other constants of proportionality $k_4$ and $k_5$.

These nine relations enable us to write $(F_0(\rho)$ in terms of $\beta_1, \beta_2 \ldots \beta_6$ and the constants $k_4$, $k_5$, $k_6$, by means of a vector equation. Consider the determinant of the coefficients of the three vectors $\beta_4$, $\beta_5$, and $F_0(\rho)$, which we may abbreviate $(45F_0\rho)$. By (14) we have

$$(123)^2\,(45F_0\rho)=(45\alpha_1)\,(31\rho)\,(12\rho)+(45\alpha_2)\,(12\rho)\,(23\rho) \\ +(45\alpha_2)\,(23\rho),\,(31\rho), \quad (21)$$

whence, comparing with (15), we have

$$(123)^2\,(45F_0\rho)=k_6\begin{vmatrix}(31\rho)\,(12\rho), & (12\rho)\,(23\rho), & (23\rho)\,(31\rho)\\(314)\,(124), & (124)\,(234), & (234)\,(314)\\(315)\,(125), & (125)\,(235), & (235)\,(315)\end{vmatrix}. \quad (22)$$

If we agree to write

$$P(\rho)=i\,(31\rho)\,(12\rho)+j\,(12\rho)\,(23\rho)+k\,(23\rho)\,(31\rho), \quad (23)$$

we may conveniently denote the determinant on the right of (22) as $(P_4P_5P\rho)$, since it is the determinant of the coefficients of the three vectors $P(\beta_4)$, $P(\beta_5)$ and $P(p)$. By advancing the numbers 4, 5, and 6 we obtain two similar relations and have

$$\begin{aligned}(123)^2\,(45F_0\rho) &= k_6\,(P_4P_5P\rho),\\ (123)^2\,(56F_0\rho) &= k_4\,(P_5P_6P\rho),\\ (123)^2\,(64F_0\rho) &= k_5\,(P_6P_4P\rho),\end{aligned} \tag{24}$$

To collect results, note the identity (which we may obtain by writing out determinants),

$$X_0(456) \equiv b_{41}(56F_0\rho) + b_{51}(64F_0\rho) + b_{61}(45F_0\rho),$$

where $b_{41}$, $b_{51}$ and $b_{61}$ are the first or $x$-components of $\beta_4$, $\beta_5$, $\beta_6$. If this identity, with two similar identities[7] for $Y_0$ and $Z_0$, be multipled respectively by $i$, $j$, and $k$, and the results added, the vectorial identity is obtained

$$(456)\,F_0(\rho) = \beta_4(56F_0\rho) + \beta_5(64F_0\rho) + \beta_6(45F_0\rho), \tag{25}$$

Values determined by equations (24), (equivalent to the six equations (13), necessary and sufficient that $\beta_4$, $\beta_5$, and $\beta_6$ shall be axes), introduced in (25), give

$$(123)^2\,(456)\,F_0(\rho) = k_4\beta_4(P_5P_6P\rho) + k_5\beta_5(P_6P_4P\rho) + k_6\beta_6(P_4P_5P\rho), \tag{26}$$

The form of this result shows that, on the one hand $F_0(\beta_1)$, $F_0(\beta_2)$, and $F_0(\beta_3)$ vanish, (because $P(\beta_1)$, $P(\beta_2)$, and $P(\beta_3)$ vanish), while on the other hand we have

$$F_0(\beta_4) = \beta_4\cdot\frac{k_4(P_4P_5P_6)}{(123)^2\,(456)}, \tag{27}$$

with similar expressions for $F_0(\beta_5)$ and $F_0(\beta_6)$. As a step in the demonstration of theorem I, we note that two vectors, alike in having $\beta_1, \beta_2, \ldots \beta_6$ for axes, can be thrown into the form (26), and will then differ in the constants $k_4$, $k_5$, $k_6$, but not otherwise.

It remains to dispose of $k_4$, $k_5$ and $k_6$ so that $\beta_7$ shall be an axis. Let $\beta_7$ be expressed in terms of $\beta_4$, $\beta_5$, and $\beta_6$ by an identity like (25), viz.

$$(456)\,\beta_7 = \beta_4(567) + \beta_5(647) + \beta_6(457), \tag{28}$$

If $\beta_7$ is an axis, $F_0(\beta_7) = h\beta_7$ where $h$ is some constant; whence, writing $\beta_7$ for $\rho$ in (26),

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[7] The three are equivalent to the well-known vector identity, ($\beta$, $\lambda$, $\mu$, $\nu$, being any four vectors), $\beta S\lambda\mu\nu \equiv \lambda S\mu\nu\beta + \mu S\nu\lambda\beta + \nu S\lambda\mu\beta$.

$$(123)^2(456)k\beta_7 = k_4\beta_4(P_5P_6P_7) + k_5\beta_5(P_6P_4P_7) + k_6\beta_6(P_4P_5P_7), \quad (29)$$

Comparing corresponding components of $(\beta_7)$ in (28) and (29) we have, as necessary and sufficient conditions that $\beta_7$ shall be an axis,

$$k_4 = \frac{h(567)\,(123)^2}{(P_5P_6P_7)},\ k_5 = \frac{h(647)\,(123)^2}{(P_6P_4P_7)},\ k_6 = \frac{h(457)\,(123)^2}{(P_4P_5P_7)}, \quad (30)$$

Allowing for the moment that none of the denominators vanish, we may introduce these results in (26) and have, finally,

$$F_0(\rho) = \beta_4\cdot\frac{h(567)\,(P_5P_6P\rho)}{(456)\,(P_5P_6P_7)} + \beta_5\cdot\frac{h(647)\,(P_6P_4P\rho)}{(456)\,(P_6P_4P_7)} + \beta_6\,\frac{h(457)\,(P_4P_5P\rho)}{(456)\,(P_4P_5P_7)}, \quad (31)$$

If $\beta_7$ be written for $\rho$, the right side reduces to $h\beta_7$ by the identical relation (28). $h$ cannot be zero for $F_0(\rho)$ would vanish and $F(\rho)$ would reduce to the term $t\rho$, contrary to the hypothesis that $F(\rho)$ is of type I. $h$ is otherwise arbitrary and two vectors alike in possessing the axes $\beta_1, \beta_2, \ldots \beta_7$ can differ in the constants $h$, $c_1$, $c_2$, and $c_3$, that is, in regard to $h$ and the form $t$, but not otherwise.

I shall now show that none of the denominators in (30) can vanish if the choice of $\beta_7$ is consistent with the hypothesis that no six axes lie on a quadric cone; whence the seven axes of (31) are assignable in any manner consistent with that hypothesis. Consider the determinant on the right of (22), or $(P_4P_5P\rho)$. Expanding by the elements of the first row, and developing the minors as in (18), (19), and (20), we have

$$\begin{aligned}(P_4P_5P\rho) = \{&(31\rho)\,(12\rho)\,(234)\,(235)\,(415)\\ &+ (12\rho)\,(23\rho)\,(314)\,(315)\,(425)\\ &+ (23\rho)\,(31\rho)\,(124)\,(125)\,(435)\,\}\,(123)\end{aligned}$$

In the first term on the right, in place of the product of the two factors $(31\rho)\,(234)$, write, identically, $(314)\,(23\rho) + (123)\,(34\rho)$. Then employ successively the two identities

$$(235)\,(415) + (315)\,(425) = (345)\,(125) \text{ and } (31\rho)\,(124) - (12\rho)\,(314) = (123)\,(41\rho)$$

and we have

$$(P_4P_5P\rho) = (123)^2 \{ (12\rho)(34\rho)(235)(415) - (125)(345)(23\rho)(41\rho) \} \tag{32}$$

The expression in braces may be regarded as a function of the four vectors $\beta_1$, $\beta_2$, $\beta_3$, $\beta_4$, taken in cyclic order, and of the two vectors $\rho$ and $\beta_5$. It may be abbreviated $C_{1234}(5, \rho)$. It is homogeneous and quadratic in each of the six vectors, and vanishes if any two coincide. Therefore, if equated to zero, it denotes a quadric cone through the five vectors $\beta$. If we write $\beta_7$ instead of $\rho$, the result cannot vanish, since, by hypothesis, no six axes lie on a quadric cone. That is, the third denominator in (30) does not vanish. Similarly, neither $(P_5P_6P_7)$ nor $(P_6P_4P_7)$ can vanish. Theorem I is therefore proved.

**6.** We may regard (31) as a normal or model form for a vector of type I. A variety of results follow immediately, either by inspection, or by using identities like (32).

For example, if a quadratic vector $F(\rho)$ has three axes which are coplanar, it may, by the addition of a properly chosen term $\rho(px + qy + rz)$ be reduced to a binomial. For let the coplanar axes be numbered 4, 5, 7, and proceed as in (31). The last term vanishes, and $F_0\rho$ is a sum of scalar multiples of two axes. The resulting binomial vector has $\beta_1$, $\beta_2$, $\beta_3$, and $\beta_6$ for zeros.

Conversely, if $F(\rho)$, being of type I, has four zeros, it has the other three axes coplanar. For choose three of the zeros to be the $\beta_1$, $\beta_2$, $\beta_3$ of the foregoing discussion. Determine $F_0(\rho)$ as in (31). $\beta_7$ cannot be a zero, for, as has been shown, $h$ cannot be zero. If either $\beta_4$, $\beta_5$, or $\beta_6$ is a zero, we have $k_4$, $k_5$, or $k_6$, respectively, zero, entailing the vanishing of either (457), (567), or (647). That is, a set of three axes lie in a plane, not including one of the four zeros.

Again, let there be three coplanar axes, and number them 1, 2, and 7. By expanding as in (32) we see that each of the denominators in the model form (31) becomes a product of three-row determinants.

If a vector of type I has two sets of coplanar axes, the two sets must have one, and only one, axis in common, since no six lie on any quadric cone. Suppose (127) = (457) = 0. (31) becomes, aside from a scalar factor,

$$\beta_4 \cdot \frac{(567) \cdot C_{1235}(6, \rho)}{(126)(356)(517)} + \beta_5 \cdot \frac{(647) \cdot C_{1236}(4, \rho)}{(124)(364)(617)} \tag{33}$$

where $C_{1235}(6, \rho)$ vanishes on the cone through $\beta_1$, $\beta_2$, $\beta_3$, $\beta_5$ and $\beta_6$, and is obtained as in (32). There is nothing to prevent us from interchanging the two sets of axes numbered 1, 2, 3 and 4, 5, 6, whence (33) becomes

$$\beta_1 \cdot \frac{(237) \cdot C_{4562}\,(3, \rho)}{(345)\,(236)\,(247)} + \beta_2 \cdot \frac{(317) \cdot C_{4562}\,(1, \rho)}{(145)\,(631)\,(347)} \tag{34}$$

and the two vectors (33) and (34) can, by theorem I, differ only in a scalar factor and a term $(px + qy + rz)\rho$. As an example of the striking relations that hold between the constants $p$, $q$, $r$ and the axes, let $p$, $q$, and $r$ be determined so that $t\rho$ added to (33) gives a vector equal or parallel to (34). If we write

$$\delta = ip + jq + kr \tag{35}$$

the vector $\delta$ thus determined is at right angles to both the axes $\beta_3$ and $\beta_6$ which do *not* enter into either of the coplanar sets, a consequence of the fact that $\beta_3$ and $\beta_6$ are zeros of both vectors (33) and (34). More generally, if $F_1(\rho) = hF_2(\rho) + t\rho$, and if $F_1$ and $F_2$ have a common zero $\beta$, $t$ must vanish if $\rho$ has the direction $\beta$, i. e. $\delta$ is at right angles to $\beta$.[8]

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[8] Darboux has pointed out, (loc. cit.) the importance of linear relations of the type (127) = 0 in the solution of differential equations. For example, that the solution of the equation

$$(yZ - zY)dx + (zX - xZ)dy + (xY - yX)dz = 0$$

may be made to depend on that of a Riccati equation, it is necessary and sufficient that we have, in the language of this present paper, three sets of coplanar axes, with one common axis, e. g. (127) = (457) = (367) = 0. If there are four sets of coplanar axes the equation can be integrated by quadratures. On the other hand, if we have three coplanar sets, but not with one common axis, e. g. (127) = (457) = (134) = 0, no general solution of the equation has been obtained.

Another application of the ideas developed in the text is to point transformations. If we regard $x_1$, $x_2$, $x_3$, of (9) as plane homogeneous coördinates, (31) gives the most general quadratic transformation having three singular points $\beta_1$, $\beta_2$, $\beta_3$, and four fixed points $\beta_4$, $\beta_5$, $\beta_6$, $\beta_7$.

## PART II. REDUCIBLE VECTORS.

**7.** To obtain typical forms for vectors of the third class, which it will be convenient to consider next, much is gained in simplicity by introducing vector multiplication. If we adopt the Hamiltonian laws for $i$, $j$, and $k$,—

$$ij = k,\ jk = i,\ ki = j;\ ji = -k,\ kj = -i,\ ik = -j,$$
$$i^2 = j^2 = k^2 = -1.$$

it is well known that vector multiplication is distributive with respect to addition and is associative. It is obviously not commutative. The product of two vectors is, in general, partly a scalar, and partly a vector. These two parts of the product are denoted respectively, by the selective symbols $S$ and $V$. We may verify by direct multiplication that equations (3) are equivalent to the vector equation

$$V\rho F(\rho) = 0 \qquad (36)$$

It was shown in Art. 2 that, if the number of axes of $F(\rho)$ is infinite, the left members of equations (3) have a common factor, that is, $V\rho F(\rho)$ consists of a scalar factor multiplied into a vector of lower degree.

I shall now show that reducible quadratic vectors may be thrown into one of three typical forms, according as they possess

(a) A proper cone of axes, every element of the cone being an axis of the vector.

(b) A single plane of axes, every direction in the plane being an axis of the vector.

(c) Two planes of axes, which, as a special case, may be in coincidence.

In each of these cases, there will, in general, be one or more discrete axes not in the plane or cone of axes.

**8.** The above subdivision of reducible vectors follows readily from certain properties of homogeneous vectors.

**Theorem II.** If a vector $F_n(\rho)$, whose components are homogeneous polynomials in $x$, $y$, and $z$ of degree $n$, satisfies the identity

$$S\rho F_n(\rho) \equiv 0 \qquad (37)$$

it can be written as $V\rho F_{n-1}(\rho)$, where $F_{n-1}(\rho)$ is a vector of degree $n-1$.

For let the components of $F_n(\rho)$ be $X$, $Y$, and $Z$. The identity (37) is equivalent to

$$xX + yY + zZ \equiv 0. \tag{38}$$

When $y$ and $z$ vanish together, $x$ does not in general vanish, hence $X$ must vanish. Therefore $X$, as a polynomial in $x$, $y$, and $z$, can contain no term in $x^n$. We may therefore write $X = yw + zv$ where $v$ and $w$ are scalar polynomials of degree $n - 1$. Similarly,

$$Y = zu + xw' \quad \text{and} \quad Z = xv' + yu'\cdot$$

(38) becomes

$$yz(u + u') + zx(v + v') + xy(w + w') \equiv 0. \tag{39}$$

When $x = 0$ neither $y$ nor $z$ are generally zero, hence $u + u'$ vanishes all over the plane $x = 0$. With similar reasoning for $y$ and $z$ we may write

$$u + u' = px, \quad v + v' = qy, \quad w + w' = rz, \tag{40}$$

where, in the case $n = 1$, the factors $p$, $q$, and $r$ are necessarily zero, since $u$, $u'$, etc. are constants, but for larger values of $n$ we may have $p$, $q$, and $r$ polynomials of degree $n - 2$. From (39) we now obtain

$$p + q + r \equiv 0 \tag{41}$$

By eliminating $u'$, $v'$, $w'$, and $p$, we have

$$X = yw + zv, \quad Y = z(u + rx) - xw, \quad Z = -vx - y(u + rx) \tag{42}$$

If, therefore, we write

$$F_{n-1}(\rho) = iP + jQ + kR = i(u + rx) + j(-v) + kw, \tag{43}$$

we find by actual multiplication

$$\begin{aligned} V\rho F_{n-1}(\rho) &= i(yw + zv) + j(zu + zrx - xw) - k(vx + yu + yrx) \\ &= F_n(\rho), \text{ by (42).} \end{aligned}$$

The vector $F_{n-1}(\rho)$ is not uniquely determined, since we may add to it an arbitrary vector term of the form $\rho t$, where $t$ is a scalar polynomial of degree $n-2$.

**Theorem III.** If a vector $F_n(\rho)$, whose components are homogeneous polynomials in $x$, $y$, and $z$ of degree $n$, satisfies the identity

$$V\rho F_n(\rho) \equiv 0 \tag{44}$$

it can be written in the form $\rho t$, where $t$ is a scalar polynomial of degree $n-1$.

Proof. Identity (44) implies that all directions of $\rho$ are axes of the vector, or that equations (3) become identities for the vector in question. It follows that $X$ vanishes all over the plane $x = 0$, and we may write $X = tx$ where $t$ is a polynomial of degree $n-1$. Similarly, $Y = ty$ and $Z = tz$, the factor $t$ being the same in all three cases, by (3). This proves the theorem.[9]

**9.** Returning now to the case of a reducible quadratic vector $F(\rho)$, if the common factor of the left members of (3) is a quadratic polynomial which is irreducible, we have

$$V\rho F(\rho) = q\phi\rho \tag{45}$$

where $q$ is the quadratic scalar and $\phi\rho$ is a vector of the first degree in $\rho$. If we multiply both sides of (45) by $\rho$ and take scalars we have

$$S\rho\phi\rho = 0, \tag{46}$$

because $S\cdot\rho V\rho F(\rho) = S\cdot\rho^2 F(\rho) = 0$. Therefore by theorem II

$$\phi\rho = V\alpha\rho \tag{47}$$

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[9] Theorems similar to II and III may be proved by Euler's theorem for any vectors whose components are homogeneous of the same degree. In general, if $F(\rho)$ is of degree $n$, we have the identity

$$F(\rho) \equiv V\rho F_{n-1}(\rho) + \nabla s,$$

where $s$ is a scalar function of degree $n+1$ and $\nabla$ is the differential operator

$$i\frac{\delta}{\delta x} + j\frac{\delta}{\partial y} + k\frac{\delta}{\partial z}$$

From this, theorem II follows at once, $s$ being zero. The vector $F_{n-1}(\rho)$ may always be taken parallel to $V\nabla F(\rho)$. See Phil. Mag., **29** (May 1915), p. 704.

where $\alpha$ is a constant vector. We may thus write (45) in the form

$$V\rho \{ F(\rho) - q\alpha \} = 0 \tag{48}$$

By theorem III the vector in braces is a scalar multiple of $\rho$, and it is of the second degree, giving

$$F(\rho) = q\alpha + \rho t \tag{49}$$

where $t$ is a linear form in $x$, $y$, and $z$. It is evident that $\alpha$ is an axis of the right member of (49). The cone $q = 0$ is a cone of axes. (49) may be regarded as a normal form for type (a) of reducible quadratic vectors. In vectorial language, a scalar quadratic form may always be written $S\rho\theta\rho$, where $\theta\rho$ is a linear vector, and a linear form $t$ may always be written $S\delta\rho$ where $\delta$ is a constant vector. (49) then becomes

$$F(\rho) = \alpha S\rho\theta\rho + \rho S\delta\rho. \tag{50}$$

No change occurs in the order of reasoning in case the quadratic form $s$, that is $S\rho\theta\rho$, is reducible to a product of linear factors. By Art. 3, if a quadratic vector possesses two sets of three coplanar axes, the six axes being distinct, it is a reducible vector. Any vector in either of the two planes containing the sets of three must be an axis of the vector, which may be written, as a normal form (c),

$$F(\rho) = \alpha S\beta_1\rho\beta_2\rho + \rho S\delta\rho \tag{51}$$

where $\beta_1$ and $\beta_2$ are constant vectors normal to these two planes, giving $S\beta_1\rho$ and $S\beta_2\rho$ two linear forms.

If the common factor of the left members of (3) is a linear polynomial, we shall have, instead of (45),

$$V\rho F(\rho) = S\beta\rho \cdot G(\rho) \tag{52}$$

where $S\beta\rho$ is the linear factor and $G(\rho)$ is, consequently, a quadratic vector. Multiplying both sides by $\rho$ and equating the scalar parts,

$$S\rho G(\rho) = 0, \tag{53}$$

whence by theorem II

$$G(\rho) = V\rho\phi\rho, \tag{54}$$

where $\phi\rho$ is a linear vector. We may thus write (52) in the form

$$V\rho[F(\rho) - \phi\rho \cdot S\beta\rho] = 0 \tag{55}$$

By theorem III the vector in brackets is a scalar multiple of $\rho$, and it is of the second degree, giving

$$F(\rho) = \phi\rho \cdot S\beta\rho + \rho S\delta\rho, \tag{56}$$

where $S\delta\rho$ is, as before, a linear form. It is evident that the three axes of the linear vector $\phi\rho$ are axes of the right member and that any vector in the plane $S\beta\rho = 0$ is an axis. It is also clear that this type (*b*) of reducible quadratic vector contains, as does (*a*), the more special type (*c*) as a limiting case, since $\phi\rho$ may itself be reducible, i. e., have an infinite number of axes. By Art. 3, a sufficient condition that a quadratic vector shall be of type (*b*) is that it shall possess four distinct axes in the same plane.

Finally, if the left members of (3) vanish identically, theorem III shows that the quadratic vector $F(\rho)$ is of the form $\rho S\delta\rho$, and it may then be regarded as a limiting case of either (*a*), (*b*), or (*c*). Collecting results, any reducible vector $F(\rho)$ of the second degree may be written in one of the three type forms

$$\begin{array}{ll} (a) & aS\rho\theta\rho + \rho S\delta\rho \\ (b) & \phi\rho S\beta\rho + \rho S\delta\rho \\ (c) & aS\beta_1\rho S\beta_2\rho + \rho S\delta\rho \end{array} \tag{57}$$

**10.** The following negative theorem is occasionally useful,—

**Theorem IV.** If, for a given quadratic vector $F(\rho)$, seven distinct axes can be found such that no six lie on a quadric cone, and if $V\rho F(\rho)$ does not vanish identically, $F(\rho)$ is not reducible.

Proof. If the vector is reducible of type (a), it cannot consist merely of its last term $\rho S\delta\rho$, since, by hypothesis, $V\rho F(\rho)$ does not vanish identically. Its only axes are the vector $a$, with the cone of axes $S\rho\theta\rho = 0$. Whence it is not possible to choose seven not having six on this cone. Similar reasoning applies to (c). If the vector is reducible of type (b), we may suppose $\phi$ to possess not more than three distinct axes, for if so it could be written as (c).[10] Whence it is impossible to choose seven distinct axis not having four in the plane $S\beta\rho = 0$. This plane, with the plane of two other axes, constitutes a quadric cone.

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[10] For $V\rho F(\rho) = S\beta\rho \cdot V\rho\phi\rho$. If $\phi\rho$ has more than three distinct axes, $V\rho\phi\rho$ has a linear factor, by reasoning parallel to that of Art. 3.

There are no other possibilities; that is, it is never possible to choose, for a reducible quadratic vector, seven distinct axes without six on a quadratic cone, (excluding the quadratic vector $\rho S\delta\rho$). This is the theorem.

When the axes are not already known, we may test $F(\rho)$ for reducibility by resolving $V\rho F(\rho)$ into scalar components in any convenient manner, and examining these scalars for common factors according to any of the well-known geometric or algebraic processes for detecting reducible polynomials.

When, by any method, a scalar factor has been found for $V\rho F(\rho)$, we throw $F(\rho)$ into the proper type form, by the processes of Art. 9.

It is of value, in theoretical investigations, to have tests for reducibility not requiring resolution into components. These are always possible. For example, if $F(\rho)$ is of type (a) or type (c), $V\rho F(\rho)$ is always in the plane at right angles to $\alpha$. Hence if $\rho_1$, $\rho_2$, and $\rho_3$ are any three values of $\rho$ we must have

$$S(V\rho_1 F\rho_1)(V\rho_2 F\rho_2)(V\rho_3 F\rho_3) = 0 \qquad (58)$$

The further study of these tests leads naturally to the use of differential operators, and lies beyond the purpose of this paper.

## PART III. VECTORS WITH MULTIPLE AXES.

**11.** I shall now suppose that a quadratic vector is given possessing three known axes, distinct and diplanar. With the notation and ideas of the first part of this paper, we may suppose the vector to be thrown into the form (12), by the addition of a term $\rho S\delta\rho$, that is $t\rho$.

Let the three vectors $\alpha_1$, $\alpha_2$, $\alpha_3$ be expressed in terms of the three axes $\beta_1$, $\beta_2$, $\beta_3$ by identities like (25), e. g.

$$\alpha_1 S\beta_1\beta_2\beta_3 = \beta_1 S\beta_2\beta_3\alpha_1 + \beta_2 S\beta_3\beta_1\alpha_1 + \beta_3 S\beta_1\beta_2\alpha_1, \qquad (59)$$

the scalar of the product of three vectors being, sign excepted, the determinant of their components. If we adopt the notation

$$A_{11} = \frac{S\beta_2\beta_3\alpha_1}{S\beta_1\beta_2\beta_3}, \quad A_{21} = \frac{S\beta_3\beta_1\alpha_1}{S\beta_1\beta_2\beta_3}, \quad A_{12} = \frac{S\beta_2\beta_3\alpha_2}{S\beta_1\beta_2\beta_3}, \text{ etc.}, \qquad (60)$$

we shall have $F_0(\rho)$ in the form

$$\begin{aligned} F_0(\rho) = {} & \beta_1(A_{11}x_2x_3 + A_{12}x_3x_1 + A_{13}x_1x_2) \\ & + \beta_2(A_{21}x_2x_3 + A_{22}x_3x_1 + A_{23}x_1x_2) \\ & + \beta_3(A_{31}x_2x_3 + A_{32}x_3x_1 + A_{33}x_1x_2), \end{aligned} \qquad (61)$$

where the nine $A$'s are constants to be determined.

If $\beta_1$ is a double axis, the cubic cones (3) have the same tangent plane at the element $\beta_1$, or else have a double line at $\beta_1$. By taking polars,[11] the vector $V\rho F(\rho)$ gives

$$V\rho F_0(\beta_1) + V\beta_1\beta_2(A_{22}x_3 + A_{23}x_2) + V\beta_1\beta_3(A_{32}x_3 + A_{33}x_2), \qquad (62)$$

because $x_2$ and $x_3$ vanish when $\beta_1$ is put for $\rho$. But $F_0(\beta_1)$ vanishes. Hence, that we should have, at most, one polar plane for the three cubic cones (3), it is necessary and sufficient that the determinant

$$\begin{vmatrix} A_{22}, & A_{23} \\ A_{32}, & A_{33} \end{vmatrix} \qquad (63)$$

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[11] That is, differentiating $V\rho F\rho$ and putting $\beta_1$ for $\rho$ after the differentiation; we have, as the polar vector, $V\rho^1 F\rho + V\rho\left\{\beta_1(A_{11}x_2^1x_3 + A_{11}x_2x_3^1) + \dots \text{etc.}\right\}$, which, writing $\beta_1$ for $\rho$, and dropping accents, gives (62).

shall vanish. $\beta_1$ is thus a multiple axis of $F(\rho)$ when, and only when, this condition is satisfied.

We may now suppose $\beta_4$, $\beta_5$, and $\beta_6$ to be three more axes, as in Art. 5. If we assume, as before, that no quadric cone can be passed through the six vectors $\beta_1 \ldots \beta_6$, and that $\beta_4$, $\beta_5$ and $\beta_6$ are diplanar, the investigation of Art. 5 is valid through (27). The vanishing of (63) must therefore be equivalent to a relation between the constants of proportionality $k_4$, $k_5$, and $k_6$. To obtain this relation we have first to write $\alpha_2$ in terms of $\beta_4$, $\beta_5$, $\beta_6$, by an identity like (28),

$$(456)\alpha_2 = \beta_4(56\alpha_2) + \beta_5(64\alpha_2) + \beta_6(45\alpha_2) \tag{64}$$

For $A_{22}$ we then have

$$\begin{aligned}(123)\,(456)A_{22} &= (456)\,(31\alpha_2), \text{ by } (60),\\ &= (314)\,(56\alpha_2) + (315)\,(64\alpha_2) + (316)\,(45\alpha_2), \text{ by } (64),\\ &= (314)k_4(123)\,(315)\,(316)\,(526)\\ &+ (315)k_5(123)\,(316)\,(314)\,(624)\\ &+ (316)k_6(123)\,(314)\,(315)\,(425),\end{aligned} \tag{65}$$

by using the value of $(45\alpha_2)$ from (19), with two similar expressions for $(56\alpha_2)$ and $(64\alpha_2)$. The result may be most simply expressed by taking a vector $\kappa$ such that its components along $V\beta_5\beta_6$ etc. are $k_4$, $k_5$, $k_6$, that is

$$(456)\kappa = k_4 V\beta_5\beta_6 + k_5 V\beta_6\beta_4 + k_6 V\beta_4\beta_5 \tag{66}$$

We then have, multiplying both sides by $\beta_2$ and taking scalars,

$$\begin{aligned}(456)S\kappa\beta_2 &= S(k_4\beta_2\beta_5\beta_6 + k_5\beta_2\beta_6\beta_4 + k_6\beta_2\beta_4\beta_5)\\ &= -\,[k_4(562) + k_5(642) + k_6(452)]\end{aligned}$$

because the scalar of the product of three vectors is the negative of their determinant. (65) may now be written

$$A_{22} = -\,(314)\,(315)\,(316)S\kappa\beta_2 \tag{67}$$

By similar reasoning

$$A_{33} = -\,(124)\,(125)\,(126)S\kappa\beta_3 \tag{68}$$

To obtain $A_{23}$, it is more practicable to use, in (20), the determinant form of $(45\alpha_3)$, with two similar determinants for $(56\alpha_3)$ and $(64\alpha_3)$; we have

$$\begin{aligned} A_{23}(123)\,(456) &= (31\alpha_3)\,(456), \text{ by } (60), \\ &= (314)\,(56\alpha_3)+(315)\,(64\alpha_3)+(316)\,(45\alpha_3), \text{ identically,} \end{aligned}$$

whence, using (20) as above indicated, the last expression is the same as the determinant

$$\begin{vmatrix} k_4(314), & (314)\,(124), & (124)\,(234) \\ k_5(315), & (315)\,(125), & (125)\,(235) \\ k_6(316), & (316)\,(126), & (126)\,(236) \end{vmatrix} \tag{69}$$

If we expand by the elements of the third column we find

$$A_{23}(123)\,(456)=(124)\,(234)\,(315)\,(316)\,[k_5(126)-k_6(125)]+\ldots+\ldots; \tag{70}$$

but, by (66), $k_5 = -S\kappa\beta_5$ and $k_6 = -S\kappa\beta_6$, giving

$$\begin{aligned} k_5(126)-k_6(125) &= -S\kappa\beta_5(126) + S\kappa\beta_6(125) \\ &= +S\kappa[\beta_5 S\beta_1\beta_2\beta_6 - \beta_6 S\beta_1\beta_2\beta_5] \\ &= S\kappa V(V\beta_1\beta_2 V\beta_6\beta_5), \text{ identically,} \\ &= S\kappa[\beta_1 S\beta_2\beta_6\beta_5 - \beta_2 S\beta_1\beta_6\beta_5] \\ &= S\kappa\beta_1(256) - S\kappa\beta_2(156); \end{aligned} \tag{71}$$

the second and the third terms on the right of (70) may be similarly transformed, being obtained from the first term by advancing the numbers 4, 5, 6. If we collect the coefficients of $S\kappa\beta_1$ we therefore have

$$\begin{aligned} (124)\,(234)\,(315)\,(316)\,(256) + (125)\,(235)\,(316)\,(314)\,(264) \\ + (126)\,(236)\,(314)\,(315)\,(245). \end{aligned}$$

In the first of these three terms, make the identical transformations

$$(234)\,(315) = (123)\,(345) + (235)\,(314)$$

and in the third term,

$$(236)\,(315) = (123)\,(365) + (235)\,(316).$$

If the resulting five terms are grouped into those with the factor (123) and those without it we have

$$(123)\,[(124)\,(345)\,(316)\,(256) + (126)\,(365)\,(314)\,(245)]$$
$$+ (314)\,(316)\,(235)\,\{(124)\,(256) + (125)\,(264) + (126)\,(245)\}.$$

The expression in braces vanishes, for it is identically equal to

$$(122)\,(456)$$

by a transformation like (64). The coefficient of (123) may be written

$$(453)\,(613)\,(562)\,(142) - (452)\,(612)\,(563)\,(143) \qquad (72)$$

by a mere rearrangement. But this is the same as $C_{4561}$ (23) by the notation of Art. 6.

Collecting the coefficients of $-S\kappa\beta_2$ we have precisely similar transformations to make, except that, in the last factor of every term, $\beta_1$ is written for $\beta_2$, i. e. 1 for 2. Thus the coefficient of (123) in the result is

$$(124)\,(345)\,(316)\,(156) + (126)\,(365)\,(314)\,(145),$$

while the other terms contain the factor (121) (456) and vanish. By a slight rearrangement, the two above terms may be written

$$(453)\,(613)\,(561)\,(142) - (451)\,(612)\,(563)\,(143) \qquad (73)$$

which may be regarded as derivable from (72), symbolically, by the operation $1 \cdot \frac{\delta}{\delta 2}$; this is the same as saying, geometrically, that a quadric cone through the vectors $\beta_4$, $\beta_5$, $\beta_6$, $\beta_1$, and $\beta_3$ is denoted by $C_{4561}(\rho, 3) = 0$, and that the polar of $\beta_1$ with respect to $C$ be taken at $\beta_2$. If the tangent plane to this cone at $\beta_1$, obtained by polarization of $C$, be denoted by $T_{4561}(\rho_1, 3) = 0$, we shall naturally write (73) as $T_{4561}({}_{12}, 3)$. The relation between $T$ and $C$ is most easily expressed by the operator $\nabla$, thus

$$T_{4561}({}_{12}, 3) = S\beta_1\nabla' \cdot C_{4561}(2', 3), \qquad (74)$$

where, as indicated by the accents, $\nabla$ operates on $\beta_2$ alone, or, if we prefer, $\rho$ is written for $\beta_2$ before the operation. These results enable us to write, from (70), (cancel (123)),

$$A_{23}\,(456) = S\kappa\beta_1 \cdot C_{4561}(2, 3) - S\kappa\beta_2 \cdot T_{4561}({}_{12}, 3) \qquad (75)$$

In a similar manner we may obtain any other $A$ with double subscript in terms of $\kappa$. Thus

$$A_{32}\,(456) = S\kappa\beta_1 \cdot C_{4561}(3, 2) - S\kappa\beta_3 \cdot T_{4561}({}_{13}, 2), \tag{76}$$

where, as before, $T$ may be obtained from $C$ by writing $\rho$ for $\beta_3$ and operating by $-S\beta_1\nabla$, and afterwards writing $\beta_3$ for $\rho$.

Before putting for the $A$'s their values in the determinant (63), it will be well for the sake of symmetry of form, to transform $A_{22}$ as follows,

$$\begin{aligned}(456)\,A_{22} &= -\,(456)\,(314)\,(315)\,(316) S\kappa\beta_2, \quad \text{by} \quad (67),\\ &= S\kappa\beta_2 \cdot (314)\,(316)[(451)\,(563) - (453)\,(561)], \text{ identically},\\ &= S\kappa\beta_2 \cdot [(453)\,(613)\,(561)\,(143) - (451)\,(613)\,(563)\,(143)],\\ &= S\kappa\beta_2 \cdot T_{4561}({}_{31}, 3), \quad \text{by} \quad (72),\end{aligned} \tag{77}$$

where $T_{4561}({}_{31}, 3)$ denotes the result of polarizing $C_{4561}(\rho, 3)$ with respect to $\beta_1$ and $\beta_3$. It is evident that any '$T$' which is a function of five vectors only can be similarly transformed. Thus

$$(456)\,A_{33} = S\kappa\beta_3 \cdot T_{4562}({}_{12}, 1) \tag{78}$$

The determinant (63) may now be written

$$\begin{vmatrix} S\kappa\beta_2 \cdot T_{4561}({}_{31}, 3); & S\kappa\beta_1 \cdot C_{4561}(2, 3) \\ & \quad - S\kappa\beta_2 \cdot T_{4561}({}_{12}, 3) \\ S\kappa\beta_1 \cdot C_{4561}(3, 2) - S\kappa\beta_3 \cdot T_{4561}({}_{13}, 2); & S\kappa\beta_3 \cdot T_{4562}({}_{12}, 1) \end{vmatrix} \tag{79}$$

whose vanishing determines that $\beta_1$ shall be a double axis, and clearly requires that $\kappa$ shall lie on a quadric cone. The constant $C$, and its derived constant '$T$,' are found at once when the six axes are assigned.

**12.** A second method for obtaining a general condition for a multiple axis is to start with (26), which, by (32), becomes

$$(456) F_0(\rho) = k_4\beta_4 C_{1235}(6, \rho) + k_5\beta_5 C_{1236}(4, \rho) + k_6\beta_6 C_{1234}(5, \rho). \tag{80}$$

We may make $\beta_4$ a double axis by so choosing $k_4$, $k_5$, and $k_6$ that the vector $V\rho F(\rho)$, polarized[12] with respect to $\beta_4$ and equated to zero, yields not more than one distinct scalar equation; for this is the same as saying that the cubic cones (3) are either tangent at $\beta_4$ or have a double line there. The resulting vector equation is

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[12] That is, forming the polar vector as in the note to Art. 11, we write $\beta_4$ for $\rho$ after the operation. The easiest way to form the polar vector in this case is to multiply (80) by $\rho$ and operate by $S\rho'\nabla$.

$$V\beta_4\beta_5\cdot k_5T_{1236}(4, {}_4\rho) + V\beta_4\beta_6\cdot k_6T_{1234}(5, {}_4\rho) + V\rho\beta_4\cdot k_4C_{1235}(6, 4) = 0. \tag{81}$$

If this is equivalent to one scalar equation, and if we put for $\rho$, in succession, any two distinct vectors, the coefficients of $V\beta_4\beta_5$ and of $V\beta_4\beta_6$ in the two results must be in proportion. Take as the two vectors $\beta_5$ and $\beta_6$. The proportionality of the coefficients is given by the vanishing of the determinant whose elements are these coefficients, viz.

$$\begin{vmatrix} k_5T_{1236}(4, {}_{45}) - k_4C_{1235}(6, 4), & k_6T_{1234}(5, {}_{45}) \\ k_5T_{1236}(4, {}_{46}) \qquad\qquad , & k_6T_{1234}(5, {}_{46}) - k_4C_{1235}(6, 4) \end{vmatrix} \tag{82}$$

whose vanishing determines that $\beta_4$ shall be a double axis. As a verification, we may note that if we write $k_4 = -S\kappa\beta_4$, etc., this result differs from (79) only in the numbering of the axes.

**13.** The two methods above given for obtaining a multiple axis depended upon applying the theory of polars to the function $C$ of six vectors. The determinants (79) and (82) are, in fact, symmetrical functions of the five single axes, and give a general relation between $\kappa$, the double axis, and the other five. They may be transformed in many ways by identities similar to those already used. It is desirable, however, to have a normal form for a quadratic vector with a multiple axis in which the tangent plane to the cones (3) at the double axis shall appear explicitly. This may easily be found as follows,—Start with the general normal form (31). Write $\beta_4 = m\alpha + n\beta_5$. By this substitution $(P_4P_5P\rho)$ becomes, with the aid of (32),

$$\begin{aligned} m_2(P\alpha P_5P\rho) + mn(123)_2 \{ (12\rho)\,(35\rho)\,(235)\,(\alpha 15) \\ - (125)\,(3\alpha 5)\,(23\rho)\,(51\rho)\} \end{aligned}$$

The expression in braces is quadratic in $\beta_1$, $\beta_2$, $\beta_3$ and $\rho$. It expresses, by its vanishing, a quadric cone through $\beta_1$, $\beta_2$, $\beta_3$, and $\beta_5$, with the tangent plane at $\beta_5$ given by $(5\alpha\rho) = 0$; to prove this, take the polar with respect to $\beta_5$,

$$(125)\,(35\rho)\,(235)\,(\alpha 15) - (125)\,(3\alpha 5)\,(235)\,(51\rho)$$

but this is equal to (125) (235) (351) $(5\alpha\rho)$ by the identity

$$(35\rho)\,(\alpha 15) - (3\alpha 5)\,(51\rho) = (351)\,(5\alpha\rho).$$

If we therefore agree to write, in keeping with the notation already used,

$$C_{1235}({}_5\alpha, \rho) = (12\rho)\,(35\rho)\,(235)\,(\alpha 15) - (125)\,(3\alpha 5)\,(23\rho)\,(51\rho), \qquad (83)$$

the writing of $m\alpha + n\beta_5$ for $\beta_4$ gives

$$(P_4P_5P\rho) = m^2(P\alpha P_5P\rho) + mn\,(123)^2C_{1235}({}_5\alpha, \rho)$$

and, similarly,

$$(P_4P_5P_7) = m^2(P\alpha P_5P_7) + mn\,(123)^2C_{1235}({}_5\alpha, 7).$$

The factors (457) and (456) become $m$ $(\alpha 57)$ and $m$ $(\alpha 56)$. Hence the coefficient of $\beta_6$ in (31) becomes

$$\frac{(\alpha 57)\,[m(P\alpha P_5P\rho) + n(123)^2C_{1235}({}_5\alpha, \rho)]}{(\alpha 56)\,[m(P\alpha P_5P_7) + n(123)^2C_{1235}({}_5\alpha, \rho)]}$$

which, if $m$ approaches zero, approaches the limit

$$\frac{(\alpha 57)C_{1235}({}_5\alpha, \rho)}{(\alpha 56)C_{1235}({}_5\alpha, 7)} \qquad (84)$$

Expressions like the right of (83), while of geometrical significance, are sometimes less convenient than determinants, (or scalar products), like $(P\alpha P_5P_7)$. We might have kept the latter form of work by writing at the start, (by (23)),

$$P(m\alpha + n\beta_5) = m^2P\alpha + n^2P_5 + mnP\alpha_5, \qquad (85)$$

where $P\alpha_5$ has been written for

$$P\alpha_5 = i\,[(31\alpha)\;(125) + (315)\;(12\alpha)] + j\,[(12\alpha)\;(235) + (125)\;(23\alpha)] + k\,[(23\alpha)\,(315) + (235)\,(31\alpha)] \qquad (86)$$

If we attach a similar meaning to any other $P$ with double subscript (i. e., the result of polarizing $P(\rho)$ with respect to two vectors), the coefficient of $\beta_6$ in (31) approaches, by the same reasoning as before, the limit

$$\frac{(\alpha 57)\ (P\alpha_5 P_5 P\rho)}{(\alpha 56)\ (P\alpha_5 P_5 P_7)} \tag{87}$$

The factor $(P\alpha_5 P_5 P\rho)$ differs from $C_{1235}({}_5\alpha, \rho)$ only in the presence of a factor $(123)^2$. (Cf. (32)).

Considering the remaining terms of (31), the coefficient of $\alpha$ after the substitution of $m\alpha + n\beta_5$ for $\beta_4$ is

$$\frac{(567)\ (P_5 P_6 P\rho)}{(\alpha 56)\ (P_5 P_6 P_7)}$$

By the aid of (85), the coefficient of $\beta_5$ may be written

$$\frac{n(567)\ (P_5 P_6 P\rho)}{m(\alpha 56)\ (P_5 P_6 P_7)} + \frac{[m(6\alpha 7) - n(567)]\,[m^2(P\alpha P_6 P\rho) + mn(P\alpha_5 P_6 P\rho) + n^2(P_5 P_6 P\rho)]}{m(\alpha 56)\qquad [m^2(P\alpha P_6 P_7) + mn(P\alpha_5 P_6 P_7) + n^2(P_5 P_6 P_7)]}$$

which, if we let $m$ approach zero, approaches the limit

$$\frac{(567)\ [(P_5 P_6 P\rho)\ (P\alpha_5 P_6 P_7) - (P_5 P_6 P_7)\ (P\alpha_5 P_6 P\rho)] + (6\alpha 7)\ (P_5 P_6 P_7)\ (P_5 P_6 P\rho)}{(\alpha 56)\ (P_5 P_6 P_7)^2}$$

a result rendered more compact by the identity

$$(P_5 P_6 P\rho)\ (P\alpha_5 P_6 P_7) - (P_5 P_6 P_7)\ (P\alpha_5 P_6 P\rho) = (P_5 P_6 P\alpha_5)\ (P_6 P_7 P\rho)$$

Collecting results, we find that as $m$ approaches zero, (31) approaches the limiting form

$$\frac{k_7\alpha(567)\ (P_5 P_6 P\rho)}{(\alpha 56)\ (P_5 P_6 P_7)} + \frac{k_7\beta_5[(567)\ (P_5 P_6 P\alpha_5)\ (P_6 P_7 P\rho) + (6\alpha 7)\ (P_5 P_6 P_7)\ (P_5 P_6 P\rho)]}{(\alpha 56)\ (P_5 P_6 P_7)^2} + \frac{k_7\beta_6(\alpha 57)\ (P\alpha_5 P_5 P\rho)}{(\alpha 56)\ (P\alpha_5 P_5 P_7)}, \tag{88}$$

a normal form for a quadratic vector having the vectors $\beta_1$, $\beta_2$, $\beta_3$, $\beta_6$, and $\beta_7$ as ordinary axes, but $\beta_5$ a double axis, the cones (3) being tangent to the plane $(\alpha 5\rho) = 0$ along the vector $\beta_5$. We may, if we wish, verify directly by polarization that $V\rho F\rho = 0$ gives at most one scalar equation when operated on by $S\rho'\nabla$, if $\rho = \beta_5$. The cones (3) do not, in general, have a double line at $\beta_5$. When, however, we choose one axis of coördinates along $\beta_5$, at least one of these cones passes twice through the double axis.

**14.** A fourth method for multiple axes, while in some respects less convenient, in that the tangent plane to (3) at the double axis is less explicitly contained in the result, brings the present discussion into close relation to the theory of point transformations; — instead of (23) take

$$Q(\rho) = i(12\rho)^2 + j(12\rho)\,(2\alpha'\rho) + k(2\alpha'\rho)\,(\alpha'1\rho) \tag{89}$$

This vector evidently has $\beta_1$ for an axis. By polarization, it is obvious that it has $\beta_2$ for a double axis, with the plane $(2\alpha'\rho) = 0$ tangent to the cones (3) or meeting them twice at $\beta_2$. If, therefore, we replace $P$ by $Q$ in the normal form (31) we shall have $\beta_1$, $\beta_4$, $\beta_5$, $\beta_6$, and $\beta_7$ as single axes, $\beta_2$ as a double axis, and $(2\alpha'\rho) = 0$ the tangent plane to (3) at $\beta_2$. Furthermore, the result will be the most general quadratic vector satisfying these conditions, aside from an additive term $\rho S\delta\rho$, for it can otherwise differ from (88) only in the numbering of the axes. Any vector with a multiple axis differs from another with the same multiple axis only in the direction of the tangent plane to (3) at that axis, a multiplicative constant, and the term $\rho S\delta\rho$; provided the five single axes also coincide.

**15.** These methods for multiple axes may be employed simultaneously to obtain two, or three, distinct double axes. Thus to form a vector having $\beta_1$, $\beta_6$, and $\beta_7$ for single axes, $\beta_2$ and $\beta_5$ for double axes, we have only to write $Q$ instead of $P$ in (88). The vector $\alpha$ may, in (89), be the same vector as in (88), i. e., it may be the line of intersection of the tangent planes to the cones (3) at $\beta_2$ and $\beta_5$, — with the obvious restriction that this line does not itself coincide with an axis. In general we may, if we wish, take $\alpha$ and $\alpha'$ any two vectors such that, with the five axes, no six vectors lie on a quadric cone.

More symmetrically, let the vector have $\beta_1$, $\beta_2$, $\beta_3$, for single axes, and two other double axes. By writing $\beta_7 = m\alpha + n\beta_6$ in (88) and letting $m$ approach zero we easily find

$$\alpha(P_5P_6P\alpha_5)\,(P_5P_6P\alpha_6)\,(P_5P_6P\rho) + \beta_5(P_5P_6P\alpha_5)^2\,(P_6P\alpha_6P\rho) \\ + \beta_6(P_5P_6P\alpha_6)^2\,(P\alpha_5P_5P\rho), \tag{90}$$

as a normal form for a quadratic vector with two double axes, $\beta_5$ and $\beta_6$, which may be renumbered at our convenience. The vector $\alpha$ is here the line of intersection of the tangent planes to (3) at the double axes.

The symmetrical form (90) is possible only when the three similar axes are not in the same plane. If they are coplanar, some of the methods previously described may be used instead, e. g., the function $Q$ may be used.[13]

For three double axes we might write $Q$ for $P$ in this last result, it being now necessary to take, in general, $\alpha$ distinct from the $\alpha'$ of (89). For a symmetrical formula, however, we shall best return to (61), and impose upon the nine $A$'s conditions that $\beta_1$, $\beta_2$, $\beta_3$ shall all be double axes, viz. that the determinant (63) shall vanish together with two others obtained by advancing subscripts. If we wish the single axis to appear explicitly, we shall most easily begin with the general normal form (31), writing $m_1\beta_4 + n_1\beta_1$ instead of $\beta_4$, $m_2\beta_5 + n_2\beta_2$ instead of $\beta_5$, and $m_3\beta_6 + n_3\beta_3$ instead of $\beta_6$. As $m_1$, $m_2$, and $m_3$ approach zero we have the limit

$$\frac{\beta_1(237)\,(P_{25}P_{36}P\rho)}{(123)\,(P_{25}P_{36}P_7)} + \frac{\beta_2(317)\,(P_{36}P_{14}P\rho)}{(123)\,(P_{36}P_{14}P_7)} + \frac{\beta_3(127)\,(P_{14}P_{25}P\rho)}{(123)\,(P_{14}P_{25}P_7)}, \qquad \underline{(91)}$$

as a normal form for a quadratic vector having $\beta_1$, $\beta_2$, $\beta_3$, for double axes, $\beta_7$ for a single axis, and the planes $(14\rho) = 0$, $(25\rho) = 0$, and

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[13] As a simple example leading to a vector of the type (91), let it be required to investigate whether the equations

$$\frac{dx}{xy} = \frac{dy}{yz} = \frac{dz}{zx}$$

can be integrated by quadratures. The integration depends upon that of the partial differential equation $SF\rho\nabla u = 0$ where

$$F\rho = ixy + jyz + kzx$$

and hence upon

$$S\rho d\rho F\rho = 0.$$

We easily find that $i, j$, and $k$ are double axes of $F\rho$ and that $i + j + k$ is the single axis. The tangent planes to (3) at $i, j$, and $k$ are found by taking polars of $V\rho F\rho$ thus,—

$$S\rho'\nabla\cdot V\rho F\rho = V\rho' F\rho + V\rho[i(x'y + xy') + j(y'z + yz') + k(z'x + zx')] = 0,$$

which, putting $\rho = i$, $y = z = 0$, $x = 1$, gives the single scalar equation $z' = 0$. Similarly we have $x' = 0$ and $y' = 0$ at $j$ and $k$, respectively. Thus the vector $F\rho$ is a limiting form of a type having three sets of coplanar axes, in the three coördinate planes, the sets not possessing an axis common to all. Hence the equation is not reducible to quadratures by any known process, (Cf. note to Art. 6), nor even to a Riccati equation.

$(36\rho) = 0$, the tangent planes to the cones (3) at the double axes. Any other vector having the same axes and the same tangent planes can differ from the above at most by a multiplicative constant and an additive term $\rho S\delta\rho$.

With regard to the case of three double axes (91) gives the most general form of such a quadratic vector. For, by its method of derivation, it is always possible whenever the three double axes are not coplanar. But no irreducible vector can have three distinct multiple axes in the same plane: a fact we would perhaps guess from the stand-point of Art. 3, if the same ideas apply to vectors with multiple axes; for the plane of the three double axes, taken twice, would constitute a quadric cone. It is more conclusive to prove directly. The following method of attack is, moreover, applicable to a variety of cases.

Let two axes of a quadratic vector be $\beta_1$ and $\beta_2$. Let $\beta_3$ be some third vector, not necessarily an axis, but such that (123) does not vanish. Let a suitable term $\rho S\delta\rho$ be added to the vector so as to make zeros of $\beta_1$ and $\beta_2$, (by two equations like (8)). With the notation (9), the resulting vector can have no terms in $x_1^2$ or in $x_2^2$, but may have terms in $x_3^2$. If we now expand as in (61), writing $B_1$, $B_2$, and $B_3$, for the coefficients of $x_3^2$, we shall have

$$\begin{aligned}&\beta_1(A_{11}x_2x_3 + A_{12}x_3x_1 + A_{13}x_1x_2 + B_1x_3^2)\\ +\,&\beta_2(A_{21}x_2x_3 + A_{22}x_3x_1 + A_{23}x_1x_2 + B_2x_3^2)\\ +\,&\beta_3(A_{31}x_2x_3 + A_{32}x_3x_1 + A_{33}x_1x_2 + B_3x_3^2).\end{aligned} \tag{92}$$

This is evidently a form to which any quadratic vector with two known axes can be reduced with ease.

The necessary and sufficient conditions that this vector possess a third axis coplanar with $\beta_1$ and $\beta_2$, but distinct, are

$$A_{33} = 0, \quad A_{13} \text{ not zero}, \quad A_{23} \text{ not zero}. \tag{93}$$

For directions in the plane of $\beta_1$ and $\beta_2$, but not along $\beta_1$ nor $\beta_2$, are given by

$$x_3 = 0, \quad x_1 \text{ not zero}, \quad x_2 \text{ not zero} \tag{94}$$

If $x_3 = 0$, and (92) lies in the plane of $\beta_1$ and $\beta_2$, we must thus have necessarily $A_{33} = 0$. The direction of (92) is then

$$\beta_1A_{13} + \beta_2A_{23}. \tag{95}$$

which is therefore an axis; coinciding with $\beta_1$ or $\beta_2$ if either $A_{13}$ or $A_{23}$ vanish. Therefore conditions (93) are both necessary and sufficient.

To introduce double axes we now form the polar vector of $V\rho\, F\rho$ by operating with $S\rho'\nabla$ or its equivalent, yielding

$$\begin{aligned}
&V\rho' \{ \beta_1(A_{11}x_2x_3 + A_{12}x_3x_1 + A_{13}x_1x_2 + B_1x_3^2) \\
&\quad + \beta_2(A_{21}x_2x_3 + A_{22}x_3x_1 + A_{23}x_1x_2 + B_2x_3^2) \\
&\quad + \beta_3(A_{31}x_2x_3 + A_{32}x_3x_1 + \qquad\quad + B_3x_3^2) \} \\
&+ V\rho \{\beta_1[A_{11}(x'_2x_3 + x_2x'_3) + A_{12}(x'_3x_1 + x_3x'_1) + A_{13}(x'_1x_2 + x_1x'_2) \\
&\hspace{22em} + 2B_1x'_3x_3] \\
&\quad + \beta_2[\mathrm{A}_{21}(x'_2x_3 + x_2x'_3) + A_{22}(x'_3x_1 + x_3x'_1) + A_{23}(x'_1x_2 + x_1x'_2) \\
&\hspace{22em} + 2B_2x'_3x_3] \\
&\quad + \beta_3[A_{31}(x'_2x_3 + x_2x'_3) + A_{32}(x'_3x_1 + x_3x'_1) + \\
&\hspace{18em} + 2B_3x'_3x_3]. \qquad (96)
\end{aligned}$$

That a direction $\rho$ shall be a double axis is the same as saying that the polar vector equated to zero shall yield at most one scalar equation. At $\beta_1$ we have $x_2 = x_3 = 0$, leading to the vanishing of the determinant (63), independently of the $B$'s. But this leads to $A_{32} = 0$, because $A_{23}$ is not zero, while $A_{33}$ vanishes. In a similar manner we have $A_{31} = 0$. That is, all three $A$'s of the third line of (92) are zero.

To make the direction $\beta_1A_{13} + \beta_2A_{23}$ a double axis, we write, in the polar vector, $\rho = \beta_1A_{13} + \beta_2A_{23}$, $x_1 = A_{13}$, $x_2 = A_{23}$, $x_3 = 0$. We have also $\rho' = \beta_1x'_1 + \beta_2x'_2 + \beta_3x'_3$. Substituting these values and dropping accents we must have

$$\begin{aligned}
&V\{\beta_1A_{13} + \beta_2A_{23}\}\ \{\beta_1[A_{11}A_{23}x_3 + A_{12}A_{13}x_3 + A_{13}(A_{23}x_1 + A_{13}x_2)] \\
&\hspace{10em} + \beta_2[A_{21}A_{23}x_3 + A_{22}A_{13}x_3 + A_{23}(A_{23}x_1 + A_{13}x_2)] \\
&+ V\{\beta_1x_1 + \beta_2x_2 + \beta_3x_3\}\ \{\beta_1A_{13}^2A_{23} + \beta_2A_{13}A_{23}^2\} = 0 \qquad (97)
\end{aligned}$$

equivalent to a single scalar equation in $\rho$. The coefficient of $V\beta_3\beta_1$ is $x_3A_{13}^2A_{23}$, so that if neither $A_{13}$ nor $A_{23}$ vanish the required scalar equation must be $x_3 = 0$. This is the same as saying that the tangent plane to the cones (3) at the required axis must be the plane of $\beta_1$ and $\beta_2$, (in so far as these cones do not possess double lines there). The coefficient of $V\beta_2\beta_3$ yields nothing new. In the coefficient of $V\beta_1\beta_2$, on the other hand, we have terms in $x_3$ together with

$$A_{13}A_{23}(A_{23}x_1 - A_{13}x_2). \qquad (98)$$

That the coefficient of $V\beta_1\beta_2$ shall be a multiple of $x_3$ is thus impossible. This completes the proof.

**16.** From the foregoing results on double axes it appears that any quadratic vector having less than seven distinct axes but no triple axis may be written in one of the three following normal forms,—

(a) If there is but one double axis, write $Q$ for $P$ in (31). The tangent plane to (3) at the double axis can pass, at most, through one of the single axes. Any other single axis may be taken as a zero of $Q(\rho)$. The remaining four single axes cannot all lie in the same plane. Any three which are diplanar may be numbered 4, 5, and 6, to correspond with (31).

(b) If there are just two double axes, write $Q$ for $P$ in (88). Choose either double axis to be a zero of $Q$. The tangent plane to (3) at this double axis can pass at most through one of the single axes. Choose either of the other single axes to be the second zero of $Q(\rho)$. At the double axis not already taken, the tangent plane to (3) can pass, at most, through one of the two remaining single axes. The one through which it passes, (if either), must be taken as $\beta_7$. Otherwise, the choice of numbering is arbitrary.

(c) If there are three double axes, (91) is always possible.

**17.** It remains to consider vectors with less than six distinct axes, one of which is of multiplicity three or greater. For triple axes we have at our disposal a variety of methods, analogous to those used above for double axes. We may, for example, assign a relation to connect the constants $A$ of (91) in order that $\beta_1$ may be a triple axis.

It may well happen that all four elements of the determinant (63) are zero. If so, the cones (3) all have a double line at $\beta_1$, which is consequently a quadruple axis. I shall assume, for the present, that such is not the case.

This possibility excluded, the cubic cone

$$S\lambda\rho F\rho = 0, \tag{99}$$

where $\lambda$ is a constant vector, (so that $S\lambda\rho F\rho$ is linearly related to the left members of (3)), will not have a double line at $\beta_1$ for all values of $\lambda$. To say that $\beta_1$ is a triple axis of $F\rho$ is therefore equivalent to saying that all cones obtained by varying $\lambda$, exclusive of those with double lines at $\beta_1$, will osculate along $\beta_1$. Or again, all these cones give the same curvature for normal sections at any point on an element $\beta_1$. The most direct way to express this condition is to say that $d\nu$ is independent of $\lambda$, where $\nu$ is a vector of unit length normal to the cones at a point on an element $\beta_1$, and $d\rho$ is any vector in the tangent plane. If $d\rho$ is parallel to $\rho$, $d\nu$ vanishes, because we are dealing with cones; it is sufficient, therefore, to consider any other one direction of $d\rho$ in the tangent plane, preferably the direction $V\rho\nu$, the direction of maximum curvature. If we write $d\nu = \chi d\rho$, $V\rho\nu$ is an axis of the linear vector function $\chi$, and $\chi V\rho\nu = gV\rho\nu$, where $g$ is the maximum curvature of a normal section of the cone.[14] But this curvature is equal to the "divergence" of the unit vector $\nu$ at the point on the cone. For, by definition of divergence,

$$\text{div}\ \nu = -S\nabla\nu = -S\nu\chi\nu - Su\chi u - S\epsilon\chi\epsilon, \tag{100}$$

where $u$ and $\epsilon$ are unit vectors along $\rho$ and $V\rho\nu$, respectively. But $S\nu\chi\nu$ vanishes, because the differential of a unit vector is always perpendicular to the unit vector itself. And $\chi u$ vanishes because we deal with cones. As above, $\chi\epsilon = g\epsilon$, giving, (because $\epsilon^2 = -1$),

$$g = -S\nabla\nu \tag{101}$$

The necessary and sufficient conditions for a triple axis may accordingly be stated: if $\beta$ is a triple axis of $F\rho$, a unit vector normal to the cone (99) and its divergence have at most one determinate direction and one determinate numerical value, respectively, independent of $\lambda$.

**18.** The direction of the normal is found by operating with $\nabla$ on $S\lambda\rho F\rho$. If we put, for convenience

$$\sigma = \nabla S\lambda\rho F\rho \tag{102}$$

so that

$$\nu = \frac{\sigma}{T\sigma} \tag{103}$$

we have, from (101),

$$\begin{aligned} g &= -S\nabla\left(\frac{\sigma}{T\sigma}\right) \\ &= -\frac{S\nabla\sigma}{T\sigma} + \frac{S\sigma\nabla T\sigma}{T^2\sigma} \\ &= \frac{\sigma^2 S\nabla\sigma + S\sigma\varphi\sigma}{T^3\sigma}, \end{aligned} \tag{104}$$

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[14] For a more detailed examination of $\chi$, see Phil. Mag., June, 1902, p. 576, and Feb., 1903, p. 187.

because $\nabla T\sigma = \phi'\nu$, when $d\sigma = \phi d\rho$.[15] The numerator of (104) may be written in a number of remarkable forms. It is, for example, the Hessian of the ternary form $S\lambda\rho F\rho$, multiplied by a factor independent of $F\rho$. This follows from the fact that $H$ is Hamilton's $m$-invariant for the function $\phi$; that is, if $i$, $j$, $k$, are ANY three diplanar vectors,

$$H = \frac{S\phi i\phi j\phi k}{Sijk} \tag{105}$$

Choose, as three convenient vectors, $\rho$, $\sigma$, and $V\rho\sigma$. Then

$$H = \frac{S\phi\rho\phi\sigma\phi V\rho\sigma}{S\rho\sigma V\rho\sigma} \tag{106}$$

By a well-known expansion [16] we have

$$\phi V\rho\sigma = -S\nabla\sigma\cdot V\rho\sigma - V\phi\rho\sigma - V\rho\phi\sigma \tag{107}$$

If we write this value for $\phi V\rho\sigma$, and multiply out, remembering that $\phi\rho$ is parallel to $\sigma$ because we deal with homogeneous functions, while $S\rho\sigma$ vanishes on the cone, we have $H$ equal to the numerator of (104) aside from a factor which is a constant multiple of $\rho^2$.

More important for our present purpose than this connection with the Hessian, is the fact that the numerator of (104) can be obtained by differential operations performed directly upon the vector $V\rho F\rho$. For, taking the first term of this numerator,

$$S\nabla\sigma = \nabla^2 S\lambda\rho F\rho = S\lambda\nabla^2\rho F\rho, \tag{108}$$

because $\nabla^2$ is a scalar and commutative. As to the second term of the same numerator,

$$S\sigma\phi\sigma = -\sigma^2 S\nu\phi\nu, \text{ identically.} \tag{109}$$

If we let $\gamma$ be the direction of $\nu$ at the point on the cone, we have

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[15] Proof: $dT^2\sigma = -d\sigma^2 = -2S\sigma d\sigma = -2S\sigma\phi d\rho$. Hence $\nabla T^2\sigma = 2\phi'\sigma = 2T\sigma\nabla T\sigma$, and, dividing by $2T\sigma$, we have $\nabla T\sigma = \phi' U\sigma = \phi'\nu$.

[16] Hamilton, Elements of Quaternions, Art. 350. Hamilton's $m''$ is the same as $-S\nabla\sigma$, and $\phi$ is self-conjugate.

$$-S\gamma\phi\gamma = -S\nu\phi\nu = \frac{d^2}{dh_\gamma}S\lambda\rho F\rho,$$

the second derivative of the ternary form $S\lambda\rho F\rho$ along the normal, more conveniently written $S^2\gamma\nabla \cdot S\lambda\rho F\rho$. By the commutative property of $S^2\gamma\nabla$ we thus have

$$S\sigma\phi\sigma = \sigma^2 S\lambda S^2\gamma\nabla \cdot \rho F\rho, \tag{111}$$

These results substituted in (104) give, as the greatest curvature of a normal section of the cone at a point where the normal is in the direction $\gamma$,

$$g = S\lambda \frac{\gamma^2\nabla^2 - S^2\gamma\nabla}{T\sigma}\rho F\rho \tag{112}$$

where the factor $\gamma^2$ is introduced for homogeneity, in order that $\gamma$ need not be a unit vector. This new numerator thus defines a differential operation upon $V\rho F\rho$.

We may now introduce the conditions for a triple axis. First, $\sigma$ is, in direction, independent of $\lambda$, hence is of the form $\gamma S\eta\lambda$, when $\beta$, the axis, is written for $\rho$. Therefore

$$T\sigma = T\gamma S\eta\lambda, \tag{114}$$

and, in order that $\lambda$ may cancel from the expression for $g$, it is necessary and sufficient that

$$(\gamma^2\nabla^2 - S^2\gamma\nabla)V\rho F\rho$$

shall be parallel to $\eta$ when $\beta$ is put for $\rho$ after the differentiation.

But this condition may be still further simplified. Let $\alpha$ be the direction which $V\rho\gamma$ takes when $\beta$ is written for $\rho$, so that $\alpha$, $\beta$, and $\gamma$ form a rectangular system. Therefore, if they are taken of unit length,

$$\nabla^2 = -S^2\alpha\nabla - S^2\beta\nabla - S^2\gamma\nabla$$

while $S^2\beta\nabla$ vanishes if $\beta$ be put for $\rho$ after the differentiation, because $\beta$ is an axis. This gives

$$\gamma^2\nabla^2 - S^2\gamma\nabla = +S^2\alpha\nabla,$$

when applied to $V\rho F\rho$ at a point on the element $\beta$. Now if $dF\rho = \Phi(\rho, d\rho)$,

$$\begin{aligned} S^2\alpha\nabla \cdot V\rho F\rho &= S\alpha\nabla \cdot [V\alpha F\rho + V\rho\Phi(\alpha, \rho)] \\ &= 2V\alpha\Phi(\alpha, \rho) + 2V\rho F\alpha, \end{aligned}$$

and writing $\beta$ for $\rho$, we have twice the vector

$$V\alpha\Phi(\alpha, \beta) + V\beta F\alpha \tag{114}$$

which is the polar vector of $V\rho F\rho$ with regard to $\alpha$ at $\beta$.

Returning now to the direction $\eta$, this is the direction which the polar vector of $V\rho F\rho$ takes at $\beta$, and is perpendicular to $\beta$. Also, $\Phi(\alpha, \beta)$ is parallel to $\beta$. If we agree to write

$$\eta = V\beta\pi, \quad S^2\alpha\nabla \cdot V\rho F\rho = V\beta\tau, \tag{115),}$$

$\beta$ being put for $\rho$ after the differentiation, the parallelism of these two vectors is expressed by

$$VV\beta\pi V\beta\tau = 0$$

which, by a simple expansion, reduces to

$$S\beta\pi\tau = 0, \tag{116}$$

which is both necessary and sufficient that $g$ shall be independent of $\lambda$.
We may sum up the foregoing investigation of triple axes in the rule:—
Let the polar vector of $V\rho F\rho$ be $V\rho' F\rho + V\rho\Phi(\rho', \rho)$. If $\beta$ is a double axis, and $\beta$ be written for $\rho$, the polar vector takes the form $S\gamma\rho' V\beta\pi$; while if $\alpha$ be written for $\rho$ and $\beta$ for $\rho'$, the polar vector takes the form $V\beta\tau$. The necessary and sufficient condition for $\beta$ to be a triple axis is $S\beta\pi\tau = 0$.

**19.** It now becomes a simple matter to apply this rule to (61). If $\beta_1$ be a double axis, we have, as already shown, $A_{22}A_{33} - A_{23}A_{32} = 0$. If $\beta_1$ be written for $\rho$, the polar vector, by (62), becomes

$$V\beta_1\beta_2(A_{22}x_3 + A_{23}x_2) + V\beta_1\beta_3(A_{32}x_3 + A_{33}x_2) \tag{117}$$

The normal direction is thus the normal to the plane determined by

$$A_{22}x_3 + A_{23}x_2 = A_{32}x_3 + A_{33}x_2 = 0 \tag{118}$$

By (9) these equations are equivalent to

$$A_{22}S\beta_1\beta_2\rho + A_{23}S\beta_3\beta_1\rho = A_{32}S\beta_1\beta_2\rho + A_{33}S\beta_3\beta_1\rho = 0 \qquad (119)$$

Let $\gamma$ and $\beta_1$ be supposed unit vectors; and let $c_2$ and $c_3$ be two constants defined by

$$A_{22}V\beta_1\beta_2 + A_{23}V\beta_3\beta_1 = c_2\gamma, \quad A_{32}V\beta_1\beta_2 + A_{33}V\beta_3\beta_1 = c_3\gamma \quad (120)$$

The equations which determine the tangent plane then become

$$c_2S\gamma\rho = c_3S\gamma\rho = 0 \qquad (121)$$

and the polar vector for $\beta_1$ becomes, by (117),

$$V\beta_1(c_2\beta_2 + c_3\beta_3)S\gamma\rho \qquad (122)$$

The required vector $\pi$ is thus given by

$$\pi = c_2\beta_2 + c_3\beta_3 \qquad (123)$$

The direction of $\alpha$ is $V\beta_1\gamma$, which is certainly determined by

$$c'_2V\beta_1(A_{22}V\beta_1\beta_2 + A_{23}V\beta_3\beta_1) + c'_3V\beta_1(A_{32}V\beta_1\beta_2 + A_{33}V\beta_3\beta_1), \quad (124)$$

since, under the present hypothesis, the four $A$'s are not all zero; $c_2{}^1$ and $c_3{}^1$ being any two new constants such that (124) does not vanish.

To find $\tau$ we have, by the rule, to write $\beta_1$ for $\rho$, (instead of for $\rho'$), in the polar vector, which then becomes

$$V\beta_1F\rho + V\rho\beta_1(A_{12}x_3 + A_{13}x_2) + V\rho(c_2\beta_2 + c_3\beta_3)S\gamma\rho \qquad (125)$$

On writing $\alpha$ for $\rho$, $S\gamma\rho$ vanishes; and the remaining terms are at right angles to $\beta_1$. It is therefore obvious that, in (124), we may neglect any component along $\beta_1$. Multiplying out, (124) gives

$$(\beta_1{}^2 - \beta_1S\beta_1)[(c'_2A_{22} + c'_3A_{32})\beta_2 - (c'_2A_{23} + c'_2A_{33})\beta_3] \quad (126)$$

By dropping the component along $\beta_1$, we see that we may use, instead of the true value of $\alpha$, the simpler vector

$$(c'_2A_{22} + c'_3A_{32})\beta_2 - (c'_2A_{23} + c'_3A_{33})\beta_3 \qquad (127)$$

The required vector $\tau$ is the result of writing the above vector for $\rho$ in

$$F\rho - \rho(A_{12}x_3 + A_{13}x_2), \tag{128}$$

namely, the vector into which $\beta_1$ is multiplied in (125), $S\gamma\rho$ vanishing. By (9), the substitution of $\rho$ for (127) is equivalent to

$$x_1 = 0, \quad x_2 = c'_2A_{22} + c'_3A_{32}, \quad x_3 = -c'_2A_{23} - c'_3A_{33}. \tag{129}$$

This gives, for $F\alpha$,

$$F\alpha = -(c'_2A_{22} + c'_3A_{32})(c'_2A_{23} + c'_3A_{33})(A_{21}\beta_2 + A_{31}\beta_3),$$

neglecting the $\beta_1$ component. Substituting values in (128), and collecting coefficients, the condition $S\beta_1\pi\tau = 0$ may be arranged as

$$\begin{aligned} &c'^2_2\{c_2(A_{23}^2A_{12} - A_{23}A_{13}A_{22} + A_{31}A_{22}A_{23}) - c_3(A_{22}^2A_{13} - A_{22}A_{12}A_{23} \\ &\qquad\qquad + A_{21}A_{22}A_{23})\} \\ &+c'_2c'_3\{c_2(A_{22}A_{33} + A_{23}A_{32})(+A_{31} - A_{13}) + 2A_{23}A_{12}A_{23} \\ &\qquad -c_3(A_{22}A_{33} + A_{23}A_{32})(+A_{21} - A_{12}) + 2A_{22}A_{32}A_{13}\} \\ &+c'^2_3\{c_2(A_{33}^2A_{12} - A_{33}A_{13}A_{32} + A_{31}A_{32}A_{33}) - c_3(A_{32}^2A_{13} - A_{32}A_{12}A_{33} \\ &\qquad\qquad + A_{21}A_{32}A_{33})\} = 0. \end{aligned} \tag{130}$$

The three expressions in braces are easily seen to be equivalent, when none of the four elements of (63) are zero, in virtue of the equations

$$\frac{c_2}{c_3} = \frac{A_{22}}{A_{32}} = \frac{A_{23}}{A_{33}} \tag{131}$$

In any case, all three expressions in braces must vanish, since the choice of constants $c'_2$ and $c'_3$ is arbitrary.

Considering various cases that may arise, the vanishing of any one of the four elements of (63) entails the vanishing of one of the constants $c_2$ or $c_3$, provided the vector $F\rho$ is not reducible. For example, suppose $A_{33} = 0$. The vanishing of (63) entails $A_{23}A_{32} = 0$. If $A_{23} = 0$, the vanishing of the third line of (130) gives $A_{13} = 0$ and $F\rho$ is reducible; whence we have $A_{32} = 0$ and so $c_3 = 0$. Similarly we may show that the vanishing of either $A_{32}$, $A_{22}$, or $A_{23}$, entails the vanishing of one of the constants $c_2$ or $c_3$. Accordingly we have only three possibilities,—

(a). Neither $c_2$ nor $c_3$ is zero. The three expressions in braces are equivalent.

(b) $c_2 = 0$, that is, $A_{22} = A_{23} = 0$. The first and second expressions in braces vanish of themselves. The third gives

$$A_{32}(A_{32}A_{13} - A_{12}A_{33} + A_{21}A_{33}) = 0. \tag{132}$$

(c) $c_3 = 0$, that is $A_{32} = A_{33} = 0$. The second and third expressions in braces vanish of themselves. The first gives

$$A_{23}(A_{23}A_{12} - A_{13}A_{22} + A_{31}A_{22}) = 0. \tag{133}$$

There are no other possible cases for irreducible quadratic vectors, these conditions are, then, necessary and sufficient for a triple axis $\beta_1$. In subcases (b) and (c), the meaning of the condition is, geometrically, as follows: let $V\rho F\rho = 0$ define three cubic cones by separation into components along $V\beta_1\beta_2$, $V\beta_2\beta_3$, and $V\beta_3\beta_1$. The second of these always has a double line at $\beta_1$. If $c_2 = 0$, the first also has $\beta_1$ for a double line. The condition (132) then requires than the tangent plane to the third cone shall touch one sheet of the first. It is easy to show that it also touches one sheet of the second, whence a triple axis. We have a similar meaning for (133).

**20.** We may note in passing that the rule developed in Art. 18 for detecting double and triple axes is applicable to vectors of any degree,— with, however, one important modification. In the quadratic case, $S^2\alpha\nabla V\rho F\rho$ can be obtained from the polar vector. When $F\rho$ is of higher degree the rule may read:

### General rule for Double and Triple Axes of Vectors.

If $\beta$ is a double axis of a vector $F\rho$, (homogeneous in $\rho$), the derived vector $S\rho'\nabla\cdot V\rho F\rho$ takes the form $S\gamma\rho'V\beta\pi$ when $\beta$ is written for $\rho$ after the differentiation; and the second derived vector $S^2\beta\gamma\nabla\cdot V\rho F\rho$ takes the form $V\beta\tau$ when $\beta$ is written for $\rho$ after the differentiation. The axis $\beta$ is of multiplicity higher than two if (and only if) $S\beta\pi\tau = 0$.

These two formal conditions may be combined in one vector equation as follows. Let $d\rho$ and $\delta\rho$ be two independent differentials of $\rho$. The vector

$$V(\delta V\rho F\rho)\,(d^2V\rho F\rho) \tag{134}$$

must vanish if, after the differentiation, $\beta$ is written for $\rho$, and $d\rho$ is any vector perpendicular to $\nabla S\rho F\rho\delta\rho$, (i. e. $d\rho$ is any vector tangent to the cones (3)). The vanishing is independent of the value of $\delta\rho$, corresponding to the arbitrary constants of (130).

The reader familiar with Hamilton's theory of linear vectors will perceive the condition for a double axis to be identical with the condition that the vector function $\nabla S\delta\rho\rho F\rho$, linear in $\delta\rho$, shall have two roots of its symbolic cubic equal to zero.

**21.** To obtain a normal form for a quadratic vector having $\beta_1$ as a triple axis, we may apply the method of limits to (31). Let $\beta_4$ be replaced by $m\beta_1 + n\beta_4$ and let $n$ approach zero. In the limit we obtain, somewhat after the manner of (91),

$$\beta_1 \frac{h(567)\,(P_5P_6P\rho)}{(156)\,(P_5P_6P_7)} + \beta_5 \frac{h(617)\,(P_6P_{14}P\rho)}{(156)\,(P_6P_{14}P_7)} + \beta_6 \frac{h(157)\,(P_{14}P_5P\rho)}{(156)\,(P_{14}P_5P_7)} \tag{135}$$

This vector has $\beta_1$ for a double axis, with the tangent plane to the cones (3) that of $\beta_1$ and $\beta_4$. We may now cause $\beta_7$ to approach $\beta_1$ as a limit by writing, instead of $\beta_7$, the vector $\beta_1 + x\beta_4 + cx^2\beta_7$, and at the same time putting $hx$ for $h$. The constant $c$ is arbitrary except as noted below. When $x$ approaches zero, the vector (135) approaches the limit

$$\beta_1 \frac{h(P_5P_6P\rho)}{(P_5P_6P_{14})} + \beta_5 \frac{h(614)\,(P_6P_{14}P\rho)}{(156)\,(P_6P_{14}[P_4 + cP_{17}])} + \beta_6 \frac{h(154)\,(P_{14}P_5P\rho)}{(156)\,(P_{14}P_5[P_4+cP_{17}]}, \tag{136}$$

which may be taken as the required normal form. With regard to the denominators of the second and third terms, the cone

$$(P\rho P_{14}P_4) + c(P\rho P_{14}P_{17}) = 0 \tag{137}$$

passes through the axes $\beta_1$, $\beta_2$, and $\beta_3$, with tangent plane at $\beta_1$ the plane of $\beta_1$ and $\beta_4$, and with principal curvature determined by the constant $c$. If the left of (137) be abbreviated $D\rho$, the denominators of the terms in question become, respectively, (156) $D_6$ and $-(156)D_5$. These denominators do not vanish, therefore, so long as the cone (137) does not contain as elements either $\beta_5$ or $\beta_6$. But this cone is an osculating cone to the cones (3) at the element $\beta_1$, and will therefore not contain $\beta_5$ or $\beta_6$ so long as the quadratic vector $F\rho$ is not reducible.

As a verification, the test for a triple axis may be applied to the vector (136) by the rule of Art. 18. The polar vector of $V\rho F\rho$ is

$$V\rho\beta_1 \frac{(P_5P_6P\rho\rho')}{(P_5P_6P_{14})} + V\rho\beta_5 \frac{(614)\,(P_6P_{14}P\rho\rho')}{(156)D_6} + V\rho\beta_6 \frac{(154)\,(P_{14}P_5P\rho\rho')}{-(156)D_5} + V\rho' F\rho,$$

where $F\rho$ has the form (136). Writing $\beta_1$ for $\rho$, the first and fourth terms vanish. To evaluate the other terms note that, by (86),

$$VP_{14}P_1\rho = i(123)^3\,(41\rho), \tag{138}$$

whence, by (23),

$$\begin{aligned}(P_6P_{14}P_1\rho) &= \phantom{-}(316)\,(126)\,(123)^3\,(41\rho),\\ (P_{14}P_5P_1\rho) &= -(315)\,(125)\,(123)^3\,(41\rho).\end{aligned} \tag{139}$$

The vector $\pi$, that is the vector coefficient of $\beta_1$, is thus

$$\beta_5 \frac{(614)\,(316)\,(126)}{(156)D_6} - \beta_6 \frac{(154)\,(315)\,(125)}{(156)D_5}, \tag{140}$$

common factors of the two terms being dropped. Writing $\alpha$ for $\rho$ and $\beta_1$ for $\rho'$ in the polar vector, the result is

$$V\beta_4\beta_1 + V\beta_1\beta_5 \frac{(614)\,(P_6P_{14}P_4)}{(156)D_6} + V\beta_1\beta_6 \frac{(154)\,(P_{14}P_5P_4)}{(156)D_5}, \tag{141}$$

which is $V\beta_1\tau$. The scalar product of (141) and (140) must vanish. This verifies at once, by actual multiplication, the denominators being transformed as in (138), so that

$$\begin{aligned}D_5 &= (P_5P_{14}P_4) + c(315)\,(125)\,(123)^3\,(417),\\ D_6 &= (P_6P_{14}P_4) + c(316)\,(126)\,(123)^3\,(417).\end{aligned} \tag{142}$$

The determinant $S\pi\beta_1\tau$ then vanishes identically, and the test is completed.

Any irreducible quadratic vector, having $\beta_1$ a triple axis, and four other distinct axes, may be thrown into the form (136). For the only limitation is that neither determinant (123) nor (156) shall vanish. We can evidently number the four single axes in pairs so that this restriction is not violated, for we may not have four distinct axes in one plane.

**22.** If, besides having $\beta_1$ as a triple axis, $F\rho$ has also a double axis, the number of possible ways of writing the vector is very large, whether we apply the method of limits, or introduce the vector $Q\rho$ as in the previous discussion of double axes. As the chief question considered in the present paper is the existence of the various types such that $F\rho$ is not reducible, it will be sufficient to note that we may write, for one triple and one double axis,

$$\beta_1 \frac{h(P_{25}P_6P\rho)}{(P_{25}P_6P_{14})} + \beta_2 \frac{h(614)\,(P_6P_{14}P\rho)}{(126)D_6} + \beta_6 \frac{h(124)\,(P_{14}P_{25}P\rho)}{(126)\,(P_{14}P_{25}[P_4+cP_{17}])} \tag{143}$$

obtained from (136) by putting $m\beta_2 + n\beta_5$ instead of $\beta_5$ and letting $n$ approach zero. Similarly, for a vector with a triple axis and two double axes, we may take,

$$\beta_1 \frac{h(P_{25}P_{36}P\rho)}{(P_{25}P_{36}P_{14})} + \beta_2 \frac{h(314)\,(P_{36}P_{14}P\rho)}{(123)D_{36}} + \beta_2 \frac{h(124)\,(P_{14}P_{25}P\rho)}{(123)D_{25}}, \tag{144}$$

obtained from (143) by writing $m\beta_3 + n\beta_6$ instead of $\beta_6$ and letting $n$ approach zero. In keeping with the notation already employed, we take $D_5\rho = 0$ as the polar plane with respect to the cone (137) of the vector $\beta_5$, i. e.

$$D_5\rho = (P_5\rho P_{14}P_4) + c[(315)\,(12\rho) + (31\rho)\,(125)]\,(123)^3 \tag{417}$$

with a similar meaning for $D_6\rho$. Whence we obtain $D_{25}$ and $D_{36}$ by writing $\beta_2$ and $\beta_3$, respectively, for $\rho$.

From their method of derivation, (143) and (144) are the most general vectors of their types. Moreover, (144) is always possible, because the only restriction is that the determinant (123) shall not vanish. But it was shown in Art. 15 that three multiple axes cannot lie in the same plane, i. e. (123) cannot vanish so long as $F\rho$ is irreducible.

The same cannot be said with regard to the above vector (143) because the determinants (123) and (126) must both be different from zero. It may well happen that the triple axis, the double axis, and one of the single axes are in the same plane.

This special case, while it cannot be written in the form (143), is easily studied by the methods of Art. (19). Assume $\beta_1$ for the triple axis, $\beta_2$ for the double axis, $\beta_3$ and $\beta_4$ for the single axes. We cannot have both $\beta_3$ and $\beta_4$ coplanar with $\beta_1$ and $\beta_2$. Suppose $\beta_4$ coplanar with the two multiple axes. Then (123) cannot vanish if $F\rho$ is irreducible. Let the vector be thrown into the form (61). The conditions that $\beta_4$ be coplanar with $\beta_1$ and $\beta_2$ but distinct are given by (93). We have therefore case (c) of Art. 19. Moreover the condition that $\beta_2$ be a double axis becomes $A_{31} = 0$. This, together with (93) and (133) yield, as necessary and sufficient for the present case

$$\left.\begin{array}{l} A_{31} = A_{32} = A_{33} = A_{23}A_{12} - A_{13}A_{22} = 0;\ A_{11} \text{ and } A_{21} \text{ not} \\ \text{both zero; with } A_{22}, A_{23}, A_{12}, \text{ and } A_{13} \text{ all different from zero.} \end{array}\right\} \quad (146)$$

We thus have $\beta_4 = A_{13}\beta_1 + A_{23}\beta_2$, and may write

$$x_2x_3(A_{11}\beta_1 + A_{21}\beta_2) + (\beta_1 + b\beta_2)\,(A_{12}x_3x_1 + A_{13}x_1x_2), \quad (147)$$

(where $b$ is a constant different from zero), as a normal form for a quadratic vector having a triple, a double, and a single axis in the same plane, and one other axis. As perhaps the simplest example of this case we may put $\beta_1 = i$, $\beta_2 = j$, $\beta_3 = k$, $A_{11} = 0$, and the remaining constants equal to unity. The equation (147) then becomes

$$i(zx + xy) + j(yz + zx + xy) \quad (148)$$

The cones (3) become

$$\left.\begin{array}{lr} \text{(a)} & zx(y + z) = 0, \\ \text{(b)} & z(yz + zx + xy) = 0, \\ \text{(c)} & x(zx + xy - y^2) = 0. \end{array}\right\} \quad (149)$$

The axes are (1, 0, 0), (0, 1, 0), (0, 0, 1) and (1, 1, 0). The quadrics $(yz + zx + xy) = 0$ and $(zx + xy - y^2) = 0$ pass through the vector (1, 0, 0) and have the same tangent plane there, viz. $y + z = 0$. Therefore (c) meets both (a) and (b) three times in the element $i$. This element is a double line for both (a) and (b). Hence it is a triple axis. At $j = \beta_2$, or (0, 1, 0), we have also a double line for (a) and for (b); giving a double axis. The axes $i$, $j$, and $i + j$ are coplanar. The left members of (149) have no common factor. Thus all the conditions are satisfied for the case in question.

**23.** The case of a vector having two triple axes is of particular interest as affording the first example of a quadratic vector which cannot always be written in the form (61). For the triple axes and the single axis may be coplanar. To build up this type, we may write, as in (89),

$$Q(\rho) = i(12\rho)^2 + j(12\rho)\,(23\rho) + k(23\rho)\,(31\rho), \tag{150}$$

giving $\beta_2$ a double axis, the tangent to (3) at $\beta_2$ being $(23\rho) = 0$. Put $Q$ for $P$ in (88), with $a = \beta_4$. This gives, dropping the constant $k_7$,

$$\beta_4\frac{(567)(Q_5Q_6Q\rho)}{(456)(Q_5Q_6Q_7)} + \frac{\beta_5(567)(Q_5Q_6Q_{45})(Q_6Q_7Q\rho) + (647)(Q_5Q_6Q_7)(Q_5Q_6Q\rho)}{(456)(Q_5Q_6Q_7)^2} + \beta_6\frac{(457)(Q_{45}Q_5Q\rho)}{(456)\,(Q_{45}Q_5Q_7)}, \tag{151}$$

which is of a type previously considered, viz. it is an example of the most general quadratic vector having two double axes. The tangent to (3) at $\beta_5$ is $(45\rho) = 0$. The single axes are $\beta_1$, $\beta_6$ and $\beta_7$. Let $\beta_6$ be replaced by $\beta_2 + t\beta_3 + at^2\beta_6$, and let $t$ approach zero. $Q_6$ takes the form $Q_2 + tQ_{23} + t^2(Q_3 + aQ_{26})$ plus terms containing higher powers of $t$. But $Q_2$ and $Q_{23}$ vanish identically. Also

$$Q_3 = i(123)^2, \qquad Q_{26} = k(123)\,(236), \tag{152}$$

so that the determinants, (or scalar products), $(Q_5Q_3Q\rho)$ and $(Q_5Q_{26}Q\rho)$ give on expanding

$$(Q_5Q_3Q\rho) = (123)^2(235)\,(23\rho)\,(15\rho), \tag{153}$$

$$(Q_5Q_{26}Q\rho) = (123)^2(236)\,(125)\,(12\rho)\,(52\rho) \tag{154}$$

We have now merely to write $Q_3 + aQ_{26}$ instead of $Q_6$ in (151). The first term of (151) becomes

$$\beta_4\frac{(527)[(123)\,(235)\,(23\rho)\,(15\rho) + a(236)\,(125)\,(12\rho)\,(52\rho)]}{(452)[(123)\,(235)\,(237)\,(157) + a(236)\,(125)\,(127)\,(527)]}$$

which by a notation in keeping with that already used may be abbreviated

$$\beta_4 \frac{(527)\, E^5\rho}{(452)\, E^5{}_7}$$

where $E^5\rho$ denotes a quadratic form whose vanishing defines the cone through $\beta_1$ and $\beta_5$, having its curvature at $\beta_2$ determined by the constant $a$. We now have for the limiting value of (151),

$$\beta_4 \frac{(527)E^5\rho}{(452)\, E^5{}_7} + \beta_5 \frac{-(527)E^5{}_{45}E^7\rho + (247)E^5{}_7E^5\rho}{(452)\,(E^5{}_7)^2} + \beta_2 \frac{(457)(Q_{45}Q_5Q\rho)}{(452)\,(Q_{45}Q_5Q_7)}, \tag{155}$$

where, in the second term, $E^7\rho$ contains $\beta_7$ instead of $\beta_5$. This vector is also of a type previously examined, viz. it has one triple and one double axis. If the determinant $(527) = 0$ it becomes a binomial, in agreement with the vector (147). We have now to write $(\beta_5 + t\beta_4 + t^2b\beta_1)$ instead of $\beta_1$ and let $t$ approach zero. When $t$ approaches zero we have

$$Lim\, \frac{E\rho}{E_7} = \frac{(235)^2(23\rho)\,(45\rho) + a(236)\,(425)\,(52\rho)^2}{(235)^2(237)\,(457) + a(236)\,(425)\,(527)^2}, \tag{156}$$

the numerator being a quadric which vanishes on a cone through $\beta_5$ and $\beta_2$ with tangent planes at those elements respectively $(45\rho) = 0$ and $(23\rho) = 0$, and having the constant $a$ arbitrary.

Again, $E^7\rho$ does not vanish in the limit, but becomes

$$(523)\,(237)\,(23\rho)\,(57\rho) + a(236)\,(527)\,(52\rho)\,(72\rho),$$

obtained by writing 7 for 5 and 5 for 1 in $E^5\rho$.

Now $E_{45}$ may be expanded as

$$E_{45} = (123)\,(235)^2(154) + a(236)\,(125)^2(524),$$

and on writing for $\beta_1$ its new value we have terms containing the square and higher powers of $t$. We thus find

$$Lim\, \frac{E^5{}_{45}}{(E_7{}^5)} = \frac{b(523)\,(235)^2(154) + a(236)\,(425)\,(524)}{[(235)^2(237)\,(457) + a(236)\,(425)\,(527)^2]^2}$$

Considering finally the third term of (155), if we expand and simplify $(Q_{45}Q_5Q\rho)$ we have

$$(Q_{45}Q_5Q\rho) = (123)^2[(21\rho)\,(25\rho)\,(235)\,(154) - (23\rho)\,(15\rho)\,(125)\,(245)], \tag{157}$$

whence, writing for $\beta_1$ its value and taking the limit,

$$Lim\,\frac{(Q_{45}Q_5Q\rho)}{(Q_{45}Q_5Q_7)} = \frac{(23\rho)\,(45\rho)\,(245)^2 + b(25\rho)^2(235)\,(145)}{(237)\,(457)\,(245)^2 + b(257)^2(235)\,(145)}. \tag{158}$$

This result is, in form, like (156), but the constants $a$ and $b$ have different meaning; being, respectively, parameters whose vanishing causes the cones (3) to have inflectional elements at $\beta_2$ and $\beta_5$.

Collecting results, we have as the limiting form of (155),

$$\begin{aligned}&\frac{\{\beta_4(527) + \beta_5(247)\}\,\{(235)^2(23\rho)\,45\rho) + a(236)\,(425)\,(52\rho)^2\}}{(235)^2(237)\,(457) + a(236)\,(425)\,(527)^2}\\ &\quad + \frac{\beta_2\{(23\rho)\,(45\rho)\,(245)^2 + b(25\rho)^2(235)\,(145)\}(457)}{(237)\,(457)\,(245)^2 + b(257)^3(235)\,(145)}\\ &- \beta_5\,\frac{(527)\,\{b(523)\,(235)^2(154) + a(236)\,(425)^2(524)\}\,\{(523)\,(237)\,(23\rho)\,(57\rho) + a(236)\,(527)\,(52\rho)\,(72\rho)\}}{\{(235)^2(237)(457) + a(236)\,(425)\,(527)^2\}^2}\end{aligned} \tag{159}$$

which is the most general form of quadratic vector with two triple axes. It is always possible; for the only limitation on the seven vectors is that the determinant (452) shall not vanish, it being always assumed that the vector is irreducible. Now we may choose either triple axis as $\beta_5$, with the tangent plane to (3) given by $(45\rho) = 0$. If this should pass through $\beta_2$ we have merely to choose $\beta_2$ instead of $\beta_5$; for the tangent at $\beta_5$ cannot also be tangent at $\beta_2$, the vector being irreducible.

If the three axes are coplanar, the determinant (257) vanishes, and the vector as above written becomes a binomial.

As a simple special case, let $i = \beta_2$, $j = \beta_5$, and $i + j = \beta_7$.

Take $k = \beta_3 = \beta_4$, making the tangent planes to (3) at $i$ and $j$ respectively $y = 0$ and $x = 0$. We cannot have both $a$ and $b$ zero, as this would give inflectional elements at both $i$ and $j$ and the vector would become reducible. Take $a = 0$, giving the cones (3) three common points in the plane $y = 0$ at $i$. Then $\beta_6$ disappears from the formula. Leaving $b$ arbitrary, with $\beta_1 = i$, and noting that all the three-row determinants are either $+1$ or $-1$, we easily find

$$jxy - i\,(xy + bz^2), \tag{160}$$

as the value of (159). The cones (3) become

$$\left.\begin{aligned} &(a)\ z(xy + bz^2) = 0,\\ &(b)\qquad xyz = 0,\\ &(c)\ y(x^2 - xy - bz^2) = 0. \end{aligned}\right\} \tag{161}$$

The tangent plane to the quadric cone whose equation is $xy + bz^2 = 0$ is, at the element $i$, the coördinate plane $y = 0$. The cones (a) and (b) both have $i$ for a double element. Thus the cone (c), which at the element $i$ consists of the plane sheet $y = 0$, has three coincident elements in common with the other cones (b) and (c). That is, $i$ is a triple axis of (160).

At the element $j$, the quadric cones

$$xy + bz^2 = 0 \text{ and } x^2 - xy - bz^2 = 0$$

have the common tangent plane $x = 0$. The cones (a) and (b) have $j$ for a double element. Thus the cone (c), which at the element $j$ consists of the quadric sheet $x^2 - xy - bz^2 = 0$, touches one sheet of each of the other cones, and cuts one sheet, giving triple intersection, but the three consecutive elements are not coplanar. By change of coördinate planes, we can, if we wish, obtain non-degenerate cubics having the same order of contact at these elements. The remaining axis is the intersection of the plane $z = 0$ with the quadric cone $x^2 - xy - bz^2 = 0$, i. e. we have $x = y$.

As an example of greater generality, we may take $a$ and $b$ both arbitrary, with the axis $\beta_7$ not coplanar with either of the triple axes $\beta_2$ or $\beta_5$. Suppose $i = \beta_2$, and $j = \beta_5$ as before, and put $k = \beta_7$. Let the tangent planes to the cones (3) meet in the axis $\beta_7$, that is, $k = \beta_3 = \beta_4$. We may take $\beta_1$ any vector not coplanar with $\beta_4$ and $\beta_6$, (for we assume now $b$ different from zero), and may put $\beta_1 = i$. We may take $\beta_6$ any vector not coplanar with $\beta_2$ and $\beta_3$, (for we assume $a$ not zero), and may put $\beta_6 = j$. All the three-row determinants become $+1$, $-1$, or 0. The vector (159) becomes, aside from a constant factor,

$$j(a - b)yz + k(xy - az^2) \tag{162}$$

The cones (3) become

$$\left.\begin{aligned} y[(2a - b)z^2 - xy] &= 0,\\ x(xy - az^2) &= 0,\\ xyz &= 0. \end{aligned}\right\} \tag{163}$$

The existence of triple axes at $i$ and at $j$ may be verified as in the preceding example.

**24.** We may bring the discussion of triple axes into close relation with the theory of quadratic point-transformations by writing

$$R(\rho) = ix_2^2 + j(x_1x_2 - ax_3^2) + kx_2x_3 \tag{164}$$

That $R(\rho)$ has $\beta_1$ for a triple axis independently of the value of the constant $a$ may be verified by either of these three methods: by the rule of Art. 18; by forming the cones (3); or by noting that the scalar components of $R(\rho)$, as quadric cones, meet three times in $\beta_1$, so that $R$ has $\beta_1$ for a triple zero. If, therefore, we write $R$ for $P$ in the general normal form (31) we shall have a general form for a quadratic vector with a triple axis and four other distinct axes. Similarly, by writing $R$ for $P$ in (88), we shall have one triple and one double axis. Various other forms with $R$ in combination with preceding methods are evidently possible.

The vector $R(\rho)$ becomes reducible when $\beta_1$ is an element of inflection for the cones (3), as may be easily shown by applying the rule of Art. 18; a result we might anticipate, because three proper quadric cones cannot have three coincident elements in a plane. If the vector $F(\rho)$ has $\beta_1$ a triple axis, with $\beta_1$ an element of inflection for (3), we may use the preceding general methods.

**25.** As a comprehensive set of normal forms for irreducible quadratic vectors with a triple axis, we may select the four following,—

(a) If there is no double axis, (136) is always possible. For the only restrictions, other than irreducibility, are that the determinants (123) and (156) shall be different from zero. But the axes $\beta_2$, $\beta_3$, $\beta_5$, and $\beta_6$ are the four single axes. Since no four distinct axes are coplanar we may evidently so choose the numbers that these restrictions hold.

(b) If there is one double axis, use (143) provided neither single axis, $\beta_3$ nor $\beta_6$, is coplanar with the triple axis $\beta_1$, and the double axis $\beta_2$. If so, use (147). The generality of these forms has already been proved.

(c) If there are two double axes, use (144).

(d) If there are two triple axes, use (159).

**26.** When an axis is triple, but not of higher multiplicity, it has already been pointed out that at least one of the cones (3) has a uniquely determined tangent plane at the triple axis. If, on the other hand, the axis is quadruple, we may distinguish two cases,—

1°. The polar vector of $V\rho F\rho$, viz. $V\rho' F\rho + V\rho\Phi(\rho, \rho')$, vanishes identically when the axis, (as $\beta_1$), is written for $\rho$; this is equivalent to the condition that all cones (3) have the axis for a double element,—which therefore counts for four intersections.

2°. The polar vector of $V\rho F\rho$ does not vanish identically when the axis is written for $\rho$.

**27.** Case 1° is easily disposed of, for the four elements of the determinant (63) vanish. Choose for $\beta_2$ and $\beta_3$ any two of the three single axes such that (123) does not vanish. The vector $F\rho$ may now be written, (after reducing $\beta_1$, $\beta_2$, and $\beta_3$ to zeros by a term $\rho S\delta\rho$),

$$\beta_1(A_{11}x_2x_3 + A_{12}x_3x_1 + A_{13}x_1x_2) + (\beta_2A_{21} + \beta_3A_{31})x_2x_3 \qquad (165)$$

The remaining axis is

$$\beta_1A_{11}A_{21}A_{31} - (A_{12}A_{31} + A_{13}A_{21} - A_{21}A_{31})(\beta_2A_{21} + \beta_3A_{31}), \qquad (166)$$

as may be directly verified. Conditions for coplanarity of this axis with a pair of the other axes are respectively, $A_{11} = 0$; $A_{21} = 0$; and $A_{31} = 0$. In these cases the fourth axis becomes, respectively, $\beta_2A_{21} + \beta_3A_{31}$; $\beta_3$; and $\beta_2$. Thus we cannot have two of the three single axes, (distinct), coplanar with the multiple axis, but the three may themselves be coplanar. In fact the conditions $A_{21} = 0$ or $A_{31} = 0$ agree with those already found that $\beta_2$ or $\beta_3$, respectively, may be double axes, for we may not have $A_{12} = 0$ nor yet $A_{13} = 0$ if the vector (165) is to be irreducible.

Without assuming the existence of three diplanar axes, we may throw the given vector into the form (92). If $\beta_2$ is a double axis we have as above $A_{31} = 0$, for, by inspection of the polar vector (96), this condition is independent of the terms in $x_3^2$. The vector now takes the form

$$\beta_1(A_{11}x_2x_3 + A_{12}x_3x_1 + A_{13}x_1x_2 + B_1x_3^2) + \beta_2(A_{21}x_2x_3 + B_2x_3^2) + \beta_3B_3x_3^2 \qquad (167)$$

Let us now suppose $\beta_3$, till this time arbitrary, to lie in the tangent plane to the cones (3) at $\beta_2$. This is the same as requiring the tangent plane to be $x_1 = 0$. By (96) this gives, (on putting $\rho = \beta_2$, $x_3 = x_1 = 0$),

$$A_{11} = 0.$$

If we next subtract from the vector the term

$$A_{21}x_3(\beta_1x_1 + \beta_2x_2 + \beta_3x_3)$$

which is of the form $\rho S\delta\rho$ and does not alter the axes, we shall remove the remaining term in $x_2x_3$. If the vector coefficient of $c_3^2$ be called $\zeta$, the vector may be written

$$\beta_1[A_{13}x_1x_2 + (A_{12} - A_{21})x_3x_1] + \zeta x_3^2 \tag{168}$$

The polar vector of $V\rho F\rho$ now becomes

$$V\rho'\{\beta_1[A_{13}x_1x_2 + (A_{12} - A_{21})x_3x_1] + \zeta x_3^2\} + V\rho\beta_1A_{13}(x_1'x_2 + x_1x_2') \\ + V\rho\beta_1(A_{12} - A_{21})\,(x_3'x_1 + x_3x_1') + 2V\rho\zeta x_3'x_3. \tag{169}$$

If the rule for a triple axis be applied to $\beta_2$, we shall now put $\beta_2$ for $\rho'$ and $\beta_3$ for $\rho$, that is $x_1' = x_3' = 0$ and $x_1 = x_2 = 0$. This gives $\zeta$ parallel to the $\tau$ of the rule of Art. 18. Again, $\beta_1$ is parallel to the $\pi$ of the same rule. Hence if $\beta_2$ is a triple axis, we must have $S\beta_2\beta_1\zeta = 0$. We may therefore take

$$\zeta = a_1\beta_1 + a_2\beta_2$$

and the vector becomes

$$\beta_1[A_{13}x_1x_2 + (A_{12} - A_{21})x_3x_1 + a_1x_3^2] + \beta_2a_2x_3^2, \tag{170}$$

which has only two distinct axes, $\beta_1$ and $\beta_2$, quadruple and triple. If $a_1$ is zero, $\beta_2$ is an inflectional element of the cones (3). We cannot have $a_2 = 0$ or $A_{13} = 0$ if the vector is to be irreducible.

It appears therefore that all possible quadratic vectors which are irreducible, and have $\beta_1$ for a quadruple axis of the sort where all cones (3) have double elements at $\beta_1$, are included under (165) and (170). It is so far assumed that $\beta_1$ is not of higher order than four.

**28.** For Case 2° we may follow a similar method. Take $\beta_1$ an axis, supposed at first to be at least triple. Take $\beta_3$ in the tangent plane to (3) at $\beta_1$. If $\beta_1$ is rendered a zero the vector may be written, $\beta_2$ being at present any vector such that (123) does not vanish,

$$\pi x_1x_2 + \tau x_3^2 + \zeta x_2x_3 + \mu x_2^2, \tag{171}$$

where the terms in $x_3x_1$ have been removed as in the preceding article. By the rule of Art. 18, the condition that $\beta_1$ shall be a triple axis is $S\beta_1\pi\tau = 0$, the vectors $\pi$ and $\tau$ agreeing precisely with those of that article.

The condition that $\beta_1$ shall be a quadruple axis will appear as a relation between the vector coefficients in (171). As in Art. 17 we may write

$$\chi\epsilon = g\epsilon \tag{172}$$

If we take the space-derivative of both sides in the direction $\epsilon$ and then operate by $S\epsilon$, we have, remembering that any derivative of the unit-vector $\epsilon$ is at right angles to $\epsilon$,

$$S\epsilon \frac{d}{dh_\epsilon}\chi\epsilon = -\frac{dg}{dh_\epsilon} \tag{173}$$

where the operation $\frac{d}{dh_\epsilon}$ means the same as $-S\epsilon\nabla$.

The condition for a quadruple axis is that the right member of (173), on writing $\beta_1$ for $\rho$ after the differentiation, shall be independent of $\lambda$.

To obtain the condition in convenient form we have to expand the left side in terms of differential operations performed directly on the vector $V\rho F\rho$. We have, since $\chi\epsilon = \frac{d\nu}{dh_\epsilon}$,

$$S\epsilon\frac{d^2\nu}{dh_\epsilon^2} = S\epsilon\frac{d}{dh_\epsilon}\frac{d}{dh_\epsilon}\cdot\frac{\sigma}{T\sigma}, \text{ where } \nu = U\sigma = U\nabla S\lambda\rho F\rho, \text{ as in Art. 17,}$$

$$= S\epsilon\frac{d}{dh_\epsilon}\left\{\frac{1}{T\sigma}\cdot\frac{d\sigma}{dh_\epsilon} - \frac{\sigma}{T^2\sigma}\cdot\frac{dT\sigma}{dh_\epsilon}\right\}$$

$$= S\epsilon\left\{\frac{1}{T\sigma}\cdot\frac{d^2\sigma}{dh_\epsilon} - \frac{2}{T^2\sigma}\cdot\frac{d\sigma}{dh_\epsilon}\cdot\frac{dT\sigma}{dh_\epsilon} + \text{terms in } \sigma\right\},$$

by the ordinary rules for differentiation. If we now write $d\sigma = \phi d\rho$, so that $\frac{d\sigma}{dh_\epsilon} = \phi\epsilon$, and, as in the investigation of triple axes, write $a$

for a unit vector which is constant and coincides with $\epsilon$ when $\beta_1$ is written for $\rho$, the right member becomes, because $S\epsilon\sigma = 0$,

$$\frac{1}{T\sigma} \cdot \frac{d}{dh_\epsilon} \cdot S\alpha\phi\epsilon - \frac{2}{T^2\sigma} \cdot \frac{dT\sigma}{dh_\epsilon} \cdot S\alpha\phi\alpha \tag{174}$$

In the first of these two terms, we have to distribute the operator $\frac{d}{dh_\epsilon}$, differentiating first as if $\epsilon$ were constant, that is, equal to $\alpha$, and second as if $\epsilon$ alone were variable. The two results are

$$\frac{1}{T\sigma} \cdot \frac{d}{dh_\alpha} \cdot S\alpha\phi\alpha + \frac{1}{T\sigma} \cdot S\alpha\phi \frac{d\epsilon}{dh_\epsilon}. \tag{175}$$

Since $S\alpha\phi\alpha$ is the same as the second derivative of $S\lambda\rho F\rho$ in the constant direction $\alpha$, we have, when $\beta_1$ is put for $\rho$ after the differentiation

$$\frac{1}{T\sigma} \cdot \frac{d}{dh_\alpha} S\alpha\phi\alpha = \frac{1}{T\sigma} \frac{d^3}{dh_\alpha{}^3} S\lambda\rho F\rho = \frac{1}{T\sigma} S\lambda \frac{d^3}{dh_\alpha{}^3} \rho F\rho \tag{176}$$

which is in the most convenient form for differentiation.

Taking next the vector $\frac{d\epsilon}{dh_\epsilon}$, occurring in the second term of (175), we note that the derivative of a unit vector is always perpendicular to the unit vector, hence

$$\frac{d\epsilon}{dh_\epsilon} = u\rho + v\nu, \tag{177}$$

where $u$ and $v$ are scalar coefficients; for $\epsilon$, $\nu$, and $\rho$ form a rectangular system. Because $\sigma$ is a homogeneous quadratic vector, $\phi\rho = 2\sigma$; and $S\alpha\sigma = 0$, hence the term in $u$ disappears. The scalar $v$ equals $-g$, for

$$\begin{aligned} \frac{d}{dh\epsilon} S\nu\epsilon &= 0 \\ &= S\frac{\epsilon d\nu}{dh_\epsilon} + S\nu\frac{d\epsilon}{dh_\epsilon}, \text{ by distributing,} \\ &= S\epsilon\chi\epsilon - v, \text{ by (177),} \\ &= -g - v, \text{ because } \chi\epsilon = g\epsilon, \text{ and } \epsilon^2 = -1. \end{aligned}$$

This result gives, for the second term of (175), the value

$$-\frac{gS\alpha\phi\nu}{T\sigma}. \tag{178}$$

The remaining term of (174) is similar to (178). For

$$g = -\frac{S\alpha\phi\alpha}{T\sigma}, \text{ identically, because } S\alpha\phi\alpha = S\lambda\frac{d^2}{dh_\alpha{}^2}\rho F\rho,$$

as in Art. 18. Again,

$$2T\sigma dT\sigma = -d(\sigma^2) = -2S\sigma d\sigma = -2S\sigma\phi d\rho = -2T\sigma S\nu\phi d\rho;$$

whence $\dfrac{dT\sigma}{dh_\epsilon} = -S\nu\phi\epsilon = -S\nu\phi\alpha = -S\alpha\phi\nu$, because $\phi$ is self-conjugate. Thus the two terms of (174) are together equal to

$$\frac{S\cdot\lambda}{T\sigma}\left\{\frac{d^3}{dh_\alpha{}^3} - 3g\frac{d^2}{dh_\nu dh_\alpha}\right\}\rho F\rho, \tag{179}$$

which is in convenient form for differentiation. If $\beta_1$ is a triple axis, $g$ is independent of $\lambda$. If $\beta_1$ is to be a quadruple axis, it is necessary and sufficient that (179) shall be independent of $\lambda$. The reasoning by which this condition has been obtained is independent of the degree of the given vector $F\rho$, and applies, therefore, to vectors of any degree. The denominator $T\sigma$, as already shown, takes the form $S\lambda\beta\pi$, when $\beta_1$ is written for $\rho$. It can easily be shown that, for all homogeneous vectors, the expression (179) takes the form

$$\frac{S\lambda\beta\tau'}{S\lambda\beta\pi}, \tag{180}$$

where $\tau'$ is a new vector.[17] The condition that this fraction shall be independent of $\lambda$ is $S\beta\pi\tau' = 0$.

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[17] To prove this in general, we may write $F\rho = \alpha P + \beta Q + \gamma R$, where $\alpha$, $\beta$, and $\gamma$ form a rectangular unit system as in the text, and $P$, $Q$, $R$, are homogeneous scalar polynomials of degree $n$ in $x$, $y$, $z$. If we write $d$ and $\delta$ for two independent symbols of differentiation; and put $dP = P(\delta\rho)$, $\delta P = P(\delta\rho)$, and $d\delta P = \delta dP = P(d\rho, \delta\rho)$, it being understood that $\beta$ is always put for $\rho$ after the differentiation; with a similar notation for the first and second differentials of $Q$ and of $R$; we must have $P\beta = Q\beta = R\beta = 0$, because $\beta$ is a zero of the

**29.** Applying this condition for a quadruple axis to (171), the third derivative of $\rho F\rho$ in the direction $\alpha$ is $3\alpha F\alpha$, because $\rho F\rho$ is homogeneous of degree 3. The other differential operation is the same as $S\nu\nabla\cdot S\alpha\nabla$. We may write $\beta_3$ for $\alpha$ and $\beta_3\beta_1$ for $\nu$, and may assume these vectors to form a rectangular unit system, although the homogeneity of the conditions makes this assumption unnecessary. The operation $S\beta_3\beta_1\nabla\cdot S\beta_3\nabla\cdot$ on $V\rho F\rho$, where $F\rho$ has the form (171), yields, (forming the second differential vector and writing $\beta_3$ for $d\rho$ and $\beta_3\beta_1$ for $\delta\rho$ and $\beta_1$ for $\rho$),

$$V\beta_3\pi + V\beta_1\zeta,$$

and $V\alpha F\alpha$ gives $3V\beta_3\tau$. By writing $\tau = w\beta_i + g\pi$, and substituting results in (179), we find as the condition that $\beta_1$ shall be a quadruple axis,

$$S\beta_1\pi(w\beta_3 + g\zeta) = 0. \tag{181}$$

**30.** It will now be most convenient to distinguish two subcases, according as $g$ is, or is not, zero. If not, we may add to the vector $F\rho$ a term $\dfrac{wx_2\rho}{g}$ which will remove from the vector $\pi$ its $\beta_1$ component, leaving it parallel to $\tau$. We then have $F\rho$ as

$$\pi(x_1x_2 + gx_3^2) + \zeta x_2x_3 + \mu x_2^2 \tag{182}$$

The condition for a quadruple axis then appears as

$$S\beta_1\pi\zeta = 0. \tag{183}$$

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given homogeneous vector $F\rho$. Also, because $\gamma$ is the normal to the cones (3), $P(\alpha) = R(\alpha) = 0$. Applying the rule of Art. 18 we find

$$\pi = \alpha P(\gamma) + \gamma R(\gamma), \quad \text{and} \quad \tau = \alpha P(\alpha, \alpha) + R(\alpha, \alpha) - 2\alpha Q(\alpha).$$

From the relations $S\beta\pi\tau = 0$ and $g = \dfrac{S\lambda\beta\tau}{S\lambda\beta\pi}$, we have $\tau = u\beta + g\pi$, where $u$ is a scalar and $g$ has the same meaning as in the text. Operating by $S\cdot\alpha\beta$ gives

$$g = \frac{S\alpha\beta\tau}{S\alpha\beta\pi} = \frac{R(\alpha, \alpha)}{R(\gamma)}$$

for all homgeneous vector-polynomials, $\beta$ being a triple axis. If we write $F(\alpha, \alpha, \alpha)$ for the third derivative of $F$ in the direction $\alpha$, and perform the indicated operations, we find all terms not of the required form cancel, and

$$\tau' = F(\alpha, \alpha, \alpha) - 3\alpha Q(\alpha, \alpha) - 3g\{\alpha P(\gamma, \alpha) + \gamma R(\gamma, \alpha) - \alpha Q(\gamma)' - \gamma Q(\alpha)\}.$$

We may therefore write

$$\zeta = u\beta_1 + g_1\pi, \tag{184}$$

and $F\rho$ becomes

$$\pi(x_1x_2 + gx_3^2 + g_1x_2x_3) + u\beta_1x_2x_3 + \mu x_2^2, \tag{185}$$

the most general quadratic vector having $\beta_1$ a quadruple axis, a determinate tangent plane to the cones (3) at $\beta_1$, and $\beta_1$ not an inflectional element of these cones. It is evident that if $\pi$ lies in the tangent plane it is an axis and, conversely, if an axis other than $\beta_1$ exists in the tangent plane, this axis coincides with $\pi$. Let us suppose, as a very special case, that $\pi$ is a double or triple axis, which we may take as $\beta_3$, (since $\beta_3$ is any vector in the tangent plane). The tangent plane to (3) at $\beta_3$ must be distinct from the tangent plane at $\beta_1$ since cubics have no double tangent, and $F\rho$ is assumed not reducible. Applying these conditions according to the methods already exemplified, we have, as the polar vector of $V\rho F\rho$ at $\beta_3$,

$$V\beta_3\{u\beta_1x'_2 - gx_1'\beta_1 - gx_2'\beta_2\}$$

which obviously cannot determine, by its vanishing, a unique tangent plane at $\beta_3$. Thus the supposed case is impossible; and $\pi$ cannot be a double axis.

This possibility disposed of, the vector (185) must always have at least two axes not in the plane $(31\rho) = 0$. Let $\beta_2$ be one of these. The vector becomes

$$\pi(x_1x_2 + gx_3^2 + g_1x_2x_3) + u\beta_1x_2x_3 + a\beta_2x_2^2, \tag{186}$$

where $a$ is a scalar constant. This form, therefore, is equally general with (185), $\beta_1$ being of order not higher than four.

If we consider the vector $\pi$, and the scalars $u$ and $a$, as determined by assigned axes $\beta_4$ and $\beta_5$, I shall now show that the determination is uniquely possible, aside, obviously, from a scalar factor. If we write

$$x_1x_2 + gx_3^2 + g_1x_2x_3 = C\rho$$

the conditions that $F\beta_4$ and $F\beta_5$ shall be parallel, respectively, to $\beta_4$, and $\beta_5$ may be written

$$\begin{aligned} V\beta_4[C_4\pi + u\beta_1(314)\,(124) + a\beta_2(314)^2] &= 0 \\ V\beta_5[C_5\pi + u\beta_1(315)\,(125) + a\beta_2(315)^2] &= 0, \end{aligned} \tag{187}$$

two vector equations equivalent, in general, to four scalar equations. Multiplying the first by $\beta_5$ and the second by $\beta_4$ and taking scalars we obtain two equations,

$$\begin{aligned}(45\pi)C_4 + (451)\,(314)\,(124)u + (452)\,(314)^2a &= 0\\ (45\pi)C_5 + (451)\,(315)\,(125)u + (452)\,(315)^2a &= 0,\end{aligned} \qquad (188)$$

homogeneous in the three unknowns $(45\pi)$, $u$, and $a$. The two-row determinants from the coefficients cannot all vanish if $F\rho$ is not reducible. For the determinant of the second and third columns is

$$(451)\,(452)\,(314)\,(315)\,\{\,(124)\,315) - (125)\,(314)\,\}$$

which by a transformation already used becomes the product of determinants

$$-\,(451)^2(452)\,(314)\,(315)\,(123). \qquad (189)$$

Considering these factors in order, if $(451) = 0$, the axes $\beta_1$, $\beta_4$, and $\beta_5$ are coplanar, and we may put

$$\beta_5 = m\beta_1 + n\beta_5.$$

The matrix of the coefficients becomes

$$\begin{matrix} C_4, & 0, & m(412)\,(314)^2, \\ mn(231)\,(314) + n^2C_4, & 0, & mn^2\,(412)\,(314)^2, \end{matrix}$$

The axes $\beta_4$ and $\beta_5$ being assumed distinct, neither $m$ nor $n$ is zero. We cannot have $(412) = 0$ for the four axes $\beta_1$, $\beta_2$, $\beta_4$, $\beta_5$, would be coplanar and $F\rho$ would be reducible. We cannot have $(314) = 0$ for we cannot have two distinct axes in the tangent plane to (3), viz. $(31\rho) = 0$. And we have $(123)$ different from zero by hypothesis. Hence this matrix cannot have its rank reduced to one.

Taking the second factor of (189), if $(452) = 0$, the three single axes are coplanar. We may put

$$\beta_5 = m\beta_2 + n\beta_4.$$

The matrix of the coefficients becomes

$$\begin{matrix} C_4, & m(421)(314)\,(124), & 0, \\ mn(312)\{(234) + g_1\,(124)\} + n^2C_4, & m(421)\{m(312) + n(314)\}n(214), & 0, \end{matrix}$$

and the only non-vanishing determinant is seen to equal

$$-\,gm^2n(124)^4(123)$$

no factor of which can vanish under the hypotheses.

Taking the third factor of (189), if $(314) = 0$, $\beta_4$ lies in the tangent plane. The two non-vanishing determinants of the matrix reduce to

$$g(124)^2(452)\,(315)^2 \qquad \text{and} \qquad g(124)^2(451)\,(315)\,(125).$$

By hypothesis, $g$ is not zero. We cannot have $(124) = 0$ since $\beta_2$ was taken without the tangent plane. We cannot have $(315) = 0$, since there cannot be two axes in the tangent plane, other than $\beta_1$. Similarly, we cannot have $(451) = 0$. Hence if both determinants vanish, we have simultaneously $(452) = 0$ and $(125) = 0$, the four axes $\beta_1$, $\beta_2$, $\beta_4$, and $\beta_5$, *coplanar, which is impossible under the hypotheses.*

Taking the fourth factor of (189), the same reasoning holds. The last factor differs from zero by hypothesis. Hence the solution of (188) is unique, aside from a factor of proportionality. This factor aside, we find for the constant $a$ the value, by an easy computation

$$a = -\,(123)\,(145)\,\{(314)\,(315)\,(245) + g(124)\,(125)(145)\}, \qquad (190)$$

the result being independent of $g_1$. For the constant $u$ we find

$$u = (123)\,(245)\,[-\,(314)\,(315)\,(345) + g(145)\,\{(314)\,(125) + (315)\,(124)\} + g_1(145)\,(314)\,(315)]. \qquad (191)$$

In determining the vector $\pi$ we shall not fail to remark that neither $C_4$ nor $C_5$ can be zero. This can be shown from the fact that, if $C_4$ is zero, (for example), $F\rho$ is a limiting case of a reducible vector, $C\rho$ defining by its vanishing a quadric having four elements in common with (3) at $\beta_1$ and passing through $\beta_2$, so that if it passes through another axis we have six on a quadric. Better, if $C_4 = 0$, it is evident from (186) that $\beta_4$ lies in the plane of $\beta_1$ and $\beta_2$. But $x_3$ vanishes when $\beta_4$ is put for $\rho$. Hence $\beta_4$ must coincide with $\beta_2$ in direction, contrary to hypothesis. Similarly, $C_5$ cannot vanish.

The most natural way to determine $\pi$, provided (145) does not vanish, is by means of the identity

$$\pi(145) = \beta_1(45\pi) + \beta_4(51\pi) + \beta_5(14\pi) \qquad (192)$$

The component $(45\pi)$ is given by (189). Multiplying both equations of (187) by $\beta_1$ and taking scalars we have

$$\begin{aligned} C_4(14\pi) + a(142)\,(314)^2 &= 0, \\ C_5(51\pi) + a(512)\,(315)^2 &= 0. \end{aligned} \qquad (193)$$

These equations complete the solution. Similarly, if (245) does not vanish we can express $\pi$ in terms of $\beta_2$, $\beta_4$, and $\beta_5$.

**31.** The remaining subcase supposes $\beta_1$ an inflectional element of the cones (3), so that $g = 0$. This is equivalent to saying that the vector $\tau$ of (171) shall be parallel to $\beta_1$. We then have, as the condition that $\beta_1$ shall be a quadruple axis, by (181),

$$S\beta_1\pi\beta_3 = 0, \tag{194}$$

for the scalar $u$ cannot vanish if $F\rho$ is not reducible. (If so, we should have $x_2$ a factor of (171)). The condition (194) requires that $\pi$ shall lie in the tangent plane $x_2 = 0$ to the cones (3) at $\beta_1$. Since $\beta_3$ is so far any vector in that plane, we may take $a\beta_3 = \pi$, and (171) becomes

$$a\beta_3x_1x_2 + u\beta_1x_3^2 + \zeta x_2x_3 + ux_2^2 \tag{195}$$

Since we cannot now have any axis in the tangent plane distinct from $\beta_1$, (for if so we should have four axes in a plane), we may assume $\beta_2$ an axis, and, by a properly chosen term in $\rho$, remove the term in $x_2^2$ from the vector $F\rho$. We then have

$$a\beta_3x_1x_2 + u\beta_1x_3^2 + \zeta x_2x_3, \tag{196}$$

as the most general form for this sub-type. (The constants $a$ and $u$ are, of course, altered by addition of a term in $\rho$): We have the vector $\zeta$ and the scalars $a$ and $u$ at our disposal to determine two more axes $\beta_4$ and $\beta_5$ as in the former subcase. This, however, presents no new difficulty.

**32.** The methods already exemplified are amply sufficient to impose on the two vectors (186) or (196) conditions that $\beta_2$ shall be a double or a triple axis. As they are the most general possible vectors of their types, as was shown, they contain all further special cases, having $\beta_1$ for a quadruple axis.

As an example, let $\beta_1 = i$, $\beta_3 = j$, $\beta_3 = \pi = k$, and let a third single axis be $j + k$, so that the three single axes are coplanar, and one of them is in the tangent plane, ($y = 0$), to (3) at $i$. Let $g = 1$ and $g_1 = 0$. We find from (186)

$$k(xy + z^2) + jy^2$$

as the value of $F\rho$, and the cones (3) become

$$y(xy + z^2 - yz) = 0, \quad xy^2 = 0, \quad x(xy + z^2) = 0.$$

Any linear function of the left members gives a uniquely determined tangent plane $y = 0$ at the element $i$, provided that the coefficient of the first is not zero. That $j$, $k$, and $j + k$ are axes verifies by direct substitution, as also the quadruple character of the axis $i$.

As a second example, let $i$ be a quadruple, and $j$ a triple axis. Let the tangent plane at $i$ be $y = 0$ and at $j$ be $z = 0$. Let $g = 1$ and $g_1 = 0$. The vector

$$j(xy + z^2) + iyz$$

satisfies these conditions, the cones (3) becoming

$$z(xy + z^2) = 0, \; yz^2 = 0, \text{ and } x^2y + xz^2 - y^2z = 0.$$

That there are no axes except $i$ and $j$ is obvious by inspection, and the quadruple character of $i$ verifies easily. This is a special case of (186).

The constant $g_1$ is closely related to the aberrancy [18] of any cubic curve obtained from the cones (3) by plane sections through $\beta_1$, and vanishes if the aberrancy of such a cubic vanishes at $\beta_1$.

Conditions that a quadratic vector may have an axis of order higher than four may be obtained by similar methods, but such conditions are not needed for the complete determination of the axes of a quadratic vector. For suppose a vector to have an axis of the fifth order. It is evident that there can be, at most, two other axes if the vector is irreducible. An examination of these is sufficient proof of the quintuple character of the multiple axis. Or again, suppose a vector to have only two axes. If these be tested, by the rules above given, one will be found to be of the fourth order, at least. It will then be of the fifth, or of the sixth, order, according as the remaining axis is of order two or one.

If a quadratic vector has only one axis, that must be of the seventh order. As a simple example,

$$i(xy + z^2 + y^2) + jy^2$$

has $i$ for its only axis, the cones (3) becoming

$$y^2z = 0, \quad z(xy + y^2 + z^2) = 0, \quad y(y^2 + z^2) = 0.$$

The tangent plane at $i$ is not determinate, all combinations of these equations having a double element at $i$. This is, therefore, not a special case of (186).

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[18] Salmon, Higher Plane Curves, 3rd. Ed. Art. 407.

## PART FOUR.

**33.** Our analysis of quadratic vectors is now complete, in the sense that the nature of any given vector can be completely determined by the foregoing methods. Furthermore, normal forms, or model vectors, have been given including all possible vectors. I shall apply what precedes to the study of a general, and very simple, normal form, by means of which the properties of a quadratic vector are made to depend on those of a pair of vectors of the first degree; namely

$$V\phi\rho\theta\rho + \rho S\delta\rho, \tag{197}$$

where $\phi$ and $\theta$ are linear in $\rho$, and $\delta$ is a constant vector.[19] The first term of this vector, the vector product $V\phi\rho\theta\rho$, has three zeros, for there exist three directions, in general distinct, which are altered in the same manner by the operations $\phi$ and $\theta$. These directions[20] are the axes of the linear vector function $\phi^{-1}\theta$. Let three vectors along these directions be $\beta_1$, $\beta_2$, and $\beta_3$. Let them be converted by $\phi$ into $\lambda_1$, $\lambda_2$, $\lambda_3$, respectively. We may then write

$$\phi\dot{\rho}\cdot(123) = \lambda_1(23\rho) + \lambda_2(31\rho) + \lambda_3(12\rho),$$

or with the notation already adopted,

$$\phi\rho = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \tag{198}$$

and also

$$\theta\rho = g_1\lambda_1 x_1 + g_2\lambda_2 x_2 + g_3\lambda_3 x_3, \tag{199}$$

where $g_1$, $g_2$, and $g_3$ are the roots of the cubic in $\phi^{-1}\theta$, that is they satisfy three relations of the form $\phi^{-1}\theta\beta = g\beta$. Whence it follows that $\theta\beta = g\phi\beta$ and $V\phi\beta\theta\beta = 0.$, for $\beta_1$, $\beta_2$, or $\beta_3$. If we now multiply together the corresponding members of (198) and (199), introducing the notation

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[19] In a former paper, (Phil. Mag. Jan. 1909, page 124) I gave this form (without proof), in connection with differential operators of the second order, the symbol $\nabla$ being written for $\rho$).

[20] Cf. the appendix by the late Prof. C. J. Joly to Hamilton's 'Elements of Quaternions,' 2nd. Ed. p. 363.

$$(g_3 - g_2)V\beta_2\beta_3 = \alpha_1,\ (g_1 - g_3)V\beta_3\beta_1 = \alpha_2,\ (g_2 - g_1)V\beta_1\beta_2 = \alpha_3, \tag{200}$$

we find identically

$$V\phi\rho\theta\rho = \alpha_1 x_2 x_3 + \alpha_2 x_3 x_1 + \alpha_3 x_1 x_2. \tag{201}$$

It has already been shown that any quadratic vector of the general type, or any having three distinct diplanar axes, can be thrown into the form of the right member, by means of the addition of a properly chosen term $\rho S\delta\rho$. We have now to examine the converse of the process by which (201) was just obtained, viz. to convert the right member into the left, or to factor, vectorially, into the linear vectors $\phi\rho$ and $\theta\rho$. In the most general form of quadratic vector this will be possible. For we may write

$$\begin{aligned}\phi\rho &= h_1 x_1 V\alpha_2\alpha_3 + h_2 x_2 V\alpha_3\alpha_1 + h_3 x_3 V\alpha_1\alpha_2,\\ \theta\rho &= c_1 x_1 V\alpha_2\alpha_3 + c_2 x_2 V\alpha_3\alpha_1 + c_3 x_3 V\alpha_1\alpha_2,\end{aligned}$$

where the six constants $h$ and $c$ are undetermined. If we take the vector product of corresponding members, utilizing the identity, proved in all works on vectorial algebra,

$$V \cdot V\alpha_1\alpha_2 V\alpha_3\alpha_1 = \alpha_1 S\alpha_1\alpha_2\alpha_3$$

with two others of like form, and compare with (201), we find these three relations to determine the six constants,

$$h_3 c_2 - h_2 c_3 = h_1 c_3 - h_3 c_1 = h_2 c_1 - h_1 c_2 = \frac{1}{S\alpha_1\alpha_2\alpha_3} \tag{202}$$

whence, evidently, there are an infinite number of ways to write down $\phi\rho$ and $\theta\rho$. For example, a simple, although unsymmetrical, solution, is

$$h_1 = 1,\quad h_2 = -1,\quad h_3 = 0,\quad c_1 S(\alpha_1\alpha_2\alpha_3) = -1,\quad c_2 = 0,\quad c_3 S(\alpha_1\alpha_2\alpha_3) = +1.$$

If we let the resulting values of $\phi$ and $\theta$, or any two we may construct satisfying the conditions, be called $\phi_0$ and $\theta_0$, then the new pair,

$$\phi = u\phi_0 + v\theta_0,\quad \theta = u_1\phi_0 + v_1\theta_0,$$

will also satisfy them, provided $uv_1 - u_1 v = 1$. It thus appears that we have $V\phi\rho\theta\rho$ determinable by fifteen scalars; each linear vector in general involving nine scalars, but in the vector product we have the four parameters $u$, $v$, $u_1$ $v_1$, with the restriction as given.

This process assumes that $V\alpha_2\alpha_3$, $V\alpha_3\alpha_1$, and $V\alpha_1\alpha_2$ are diplanar, since $\phi\rho$ and $\theta\rho$ were assumed in terms of them. This, in turn, implies that $\alpha_1$, $\alpha_2$, and $\alpha_3$ are diplanar, in general true, but not necessary, even in what I have called a vector of the first, or general type. For it was shown that if $F\rho$ has three zeros, and if, of the four remaining axes, three are coplanar, the last or seventh axis will also be a zero, (cf. Art. 6), and the vector itself will be always in one plane, i. e. $\alpha_1$, $\alpha_2$, $\alpha_3$ will be coplanar.

The possibility of the normal form (197) therefore depends, when sets of coplanar axes exist, upon so selecting the zeros that, of the four other axes, no three shall be coplanar, while the three zeros themselves cannot be coplanar. Each relation of coplanarity between sets of three vectors diminishes the number of possibilities. Thus if no three axes are coplanar, three can be chosen in thirty-five ways, each leading to a separate form like the right member of (201). If three axes are coplanar, the number of possibilities reduces to thirty-one, since we must now exclude any choice of three zeros from the four non-coplanar axes, and three can be chosen from four in four ways.

That the choice of three diplanar axes, so that of the four that remain no three shall be coplanar, will always be possible may be proved by exhaustion as follows. Let $\beta_1$, $\beta_2$, and $\beta_3$ be chosen as any three diplanar axes. Let $\beta_4$, $\beta_5$, and $\beta_6$ be any remaining three which are not coplanar. If $\beta_7$ is coplanar with two of this latter three, suppose $(567) = 0$. If of the remaining four vectors $\beta_1$, $\beta_2$, $\beta_3$, $\beta_4$, no three are coplanar, the problem is solved, since (456) is not zero.

Of these four, (123) does not vanish by hypothesis. If $\beta_7$ is coplanar with two out of the three $\beta_1$, $\beta_2$; $\beta_3$, suppose $(237) = 0$. Then (127) is not zero, since no four axes are coplanar.

Choose $\beta_1$, $\beta_2$, and $\beta_7$ to be zeros. If of the four others $\beta_3$, $\beta_4$, $\beta_5$, $\beta_6$, no three are coplanar, the problem is solved. Of the four, (456) is not zero by hypothesis, and (356) is not zero for $(567) = 0$ and no four axes are coplanar. If $\beta_3$ and $\beta_4$ are coplanar with either $\beta_5$ or $\beta_6$ (so far treated alike), suppose $(345) = 0$.

Choose $\beta_3$, $\beta_5$, and $\beta_6$ to be zeros. If of the four others $\beta_1$, $\beta_2$, $\beta_4$, $\beta_7$, no three are coplanar, the problem is solved. Of these four, (127) and (247) can neither be zero, since $(237) = 0$. Now (124) and (147) cannot both be zero. Two cases are to be distinguished,—

Case 1. $(124) = 0$. Choose $\beta_1$, $\beta_4$, and $\beta_7$ to be zeros. Of the four others, $\beta_2$, $\beta_3$, $\beta_5$, $\beta_6$, (235) and (236) are not zero since $(237) = 0$. And (256) and (356) are not zero, since $(567) = 0$. The problem is therefore solved. We may still have $(163) = 0$, when the seven axes will lie in sets of three on five planes, no three planes having a common axis, except those through $\beta_3$.

Case 2. (147) = 0. If $\beta_1$, $\beta_3$, and $\beta_6$ are diplanar, choose them to be zeros. Of the four others, $\beta_2$, $\beta_4$, $\beta_5$, $\beta_7$, (247) and (257) cannot be zero, since (237) = 0. (457) cannot be zero since (147) = 0. (245) cannot be zero since (345) = 0. The problem is then solved. But if (136) = 0 choose $\beta_2$, $\beta_4$ and $\beta_6$ to be zeros, if diplanar. Then of the four others $\beta_1$, $\beta_3$, $\beta_5$, $\beta_7$, we cannot have (135) nor (137) zero because (136) = 0. We cannot have (157) nor (357) zero since (567) = 0. The problem is then solved. But if (246) = 0, choose $\beta_1$, $\beta_2$, and $\beta_5$ to be zeros. Of the four others, $\beta_3$, $\beta_4$, $\beta_6$, $\beta_7$, we cannot have (346) nor (467) zero, since (246) = 0. We cannot have (347) = 0 nor (367) = 0 since (237) = 0. The problem is then solved if (125) does not vanish. But the three vectors $\beta_1$, $\beta_2$, $\beta_5$ are the intersections of planes through the other four axes taken in pairs. It has just been shown that no three of these four can be coplanar. Therefore (125) cannot vanish, and the proof is complete. The axes now lie in sets of three on six planes, the four not chosen as zeros being at the intersection of three planes each.

The form (197) is then always possible for a quadratic vector of the first or general class, having just seven distinct axes.

**34.** If an irreducible quadratic vector possesses one or more double axes, but no triple axis, it may still be thrown into the form (197), but the manner of obtaining a proper $\phi$ and $\theta$ is somewhat different. Let, at first, each double axis be replaced by two single axes, one coinciding with the original double axis, the other lying in the original tangent plane to the cones (3) at the double axis. Then let the selection of three axes proceed as in Art. 33, these three to be diplanar, and of the four others no three to be coplanar. Then let the axes in the respective tangent planes approach their original positions. We now distinguish two cases.

Case 1. If the three axes selected remain single axes, let them be reduced to zeros, the vector then taking the form of the right member of (201). The vectors $\alpha_1$, $\alpha_2$, $\alpha_3$, cannot be coplanar, and the factorization into the form (197) proceeds as before.

Proof. Suppose $\alpha_1$, $\alpha_2$, and $\alpha_3$ to be coplanar. Put

$$\alpha_3 = m\alpha_1 + n\alpha_2.$$

The vector $F\rho$ takes the form

$$\alpha_1(x_2x_3 + mx_1x_2) + \alpha_2(x_3x_1 + nx_1x_2).$$

Four of the seven axes are zeros, viz. the four intersections of

$$x_2x_3 + mx_1x_2 = 0. \quad x_3x_1 + nx_1x_2 = 0.$$

There can be no other axis without the plane of $\alpha_1$ and $\alpha_2$. For the above pair of simultaneous equations cannot have five solutions since the number is finite. No axis not given by a solution of these equations can be a zero. That is, the remaining axes are not zeros. Hence they are in the plane of $\alpha_1$ and $\alpha_2$. Since, by hypothesis for this case, the three selected zeros are single axes, the fourth zero is single. Of the remaining non-zero axes, one is therefore a double axis. In virtue of the method by which the first three zeros were selected, the last non-zero axis cannot lie in the tangent plane at the double axis, which, therefore, cannot be the plane of $\alpha_1$ and $\alpha_2$.

Consider the normal form (88); which is precisely the present case,—three single axes being zeros, the double axis being $\beta_5$, and the tangent plane at $\beta_5$ passing through $\alpha$, which by hypothesis, is distinct from $\beta_5$. If (88) lies in a constant plane, that plane is therefore the tangent plane, contrary to the above result. Hence the vector cannot lie in a constant plane; that is, $\alpha_1$, $\alpha_2$, $\alpha_3$, cannot be coplanar.

Case 2. If the three vectors selected do not remain single in the limiting vector, let the double axis be $\beta_1$ and the other be $\beta_2$ and let the vector be thrown into the form (171), and $\beta_2$ be rendered a zero. We then have

$$F\rho = \pi x_1x_2 + \tau x_3^2 + \zeta x_2x_3. \qquad (203)$$

The vectors $\pi$, $\tau$, and $\zeta$ will not be coplanar.

Proof. Suppose $\pi$, $\tau$, $\zeta$, to be coplanar. Put

$$\zeta = m\pi + n\tau.$$

The vector $F\rho$ takes the form

$$\pi(x_1x_2 + mx_2x_3) + \tau(x_3^2 + nx_2x_3).$$

By the same reasoning as in Case 1, the non-zero axes must lie in the plane of $\pi$ and $\tau$. By virtue of the method of selecting the first three zeros, if the non-zero axes are three single axes, they are not coplanar. Hence they cannot be all single. One non-zero axis is therefore double. By the same reasoning as in Case 1, the tangent plane at this latter double axis cannot be that of $\pi$, and $\tau$.

Consider the model form obtained by writing $Q$ for $P$ in (88), which is precisely the present case. As above, the vector, if in a constant plane, is in the tangent plane at $\beta_5$, in contradiction with the result just obtained. Hence $\pi$, $\tau$, and $\zeta$ are not coplanar.

I shall now show that the right member of (203) can always be thrown into the form $V\phi\rho\theta\rho$ when $\zeta$, $\pi$, and $\tau$ are not coplanar. Write

$$\phi\rho = x_1 V\pi\tau + x_2 V\zeta\pi + x_3\mu, \tag{204}$$

where $\mu$ is to be determined. Also

$$\theta\rho = g_1x_1V\pi\tau + g_2x_2V\zeta\pi + (g_1\mu + g_3V\pi\tau)x_3, \tag{205}$$

these simultaneous forms of $\phi$ and $\theta$ corresponding to coincidence of two axes of the linear vector $\phi^{-1}\theta\rho$. If we now take the vector product $V\phi\rho\theta\rho$ and equate to (203) we find, comparing vector coefficients of $x_1x_2$, $x_3^2$, and $x_2x_3$,

$$(g_2 - g_1)S\zeta\pi\tau = 1, \quad g_3S\pi\mu = 1, \quad S\tau\mu = 0,$$
$$\zeta = (g_1 - g_2)VV\zeta\pi\cdot\mu - g_3\pi S\zeta\pi\tau.$$

From these equations we obtain the solution, for $g_3$ and $\mu$,

$$g_3S\zeta\pi\tau = -1, \qquad \mu = V\zeta\tau + V\tau\pi \tag{206}$$

whence $\phi$ and $\theta$ are known, $g_1$ and $g_2$ being arbitrary provided their difference be constant.

**35.** Taking next the case of a quadratic vector having a triple axis, but no axis of higher order, a number of situations may arise. Suppose first that we have one triple and four single axes. Let the triple axis be $\beta_1$. It is always possible so to choose $\beta_2$ and $\beta_3$ that of the two other axes neither shall lie in the tangent plane to (3) at $\beta_1$, (because only one single axis can lie in that plane), and, at the same time, so that these two remaining axes shall not be coplanar with $\beta_1$, (because four axes cannot be coplanar). Let $\beta_1$, $\beta_2$, and $\beta_3$ be rendered zeros. We then have the form of the right member of (201). The three vectors $\alpha_1$, $\alpha_2$, and $\alpha_3$ must be diplanar, and we can factor into $V\phi\rho\theta\rho$ as before.

Proof. Suppose $(\alpha_1\alpha_2\alpha_3) = 0$. These three vectors will not all be parallel, for $F\rho$ is not reducible. Let one of them be expressed in terms of the other two, (non-parallel). The scalar coefficients of these two define, by their vanishing, two quadrics. Two cases may arise.

Case 1. These quadrics are tangent at $\beta_1$. Neither of the single axes not first chosen to be zeros can be now a zero. But all non-zero axes are in the plane of the vectors $\alpha_1, \alpha_2, \alpha_3$. But $\beta_1$ must lie in that plane by reason of its triple character. This is contrary to the hypothesis that $\beta_1$ and the two axes are diplanar.

Case 2. The quadrics are not tangent at $\beta_1$. One of the two single axes not selected to be zeros must be a zero, since the quadrics have four intersections. But the plane of the vectors $\alpha$ contains three axes, is therefore the tangent plane to (3) at $\beta_1$, and passes through the non-zero axis, contrary to hypothesis.

A similar proof can be obtained analytically by considering the condition of Art. 19 that $\beta_1$ shall be a triple axis.

Under case 2 use was made of the fact that, if the plane of a quadratic vector is constant, and if two of the three axes which it must then possess in that plane are coincident, the plane is tangent to the cones (3) at the double axis,— assuming this tangent plane to be uniquely determined. An analytical proof of this is desirable. Suppose first that the double axis is a zero, as in the case of the triple axis just considered. Let coördinate axes be taken so that $i$ is the zero, $j$ also in the constant plane, so that the vector becomes

$$i(bxy + cy^2 + \text{terms in } z) + j(b_1xy + c_1y^2 + \text{terms in } z).$$

The equation to determine axes in the plane $z = 0$ then becomes

$$x(b_1xy + c_1y^2) = y(bxy + cy^2).$$

That there may be only one axis other than $y = 0$ we must have $b_1 = 0$. But the polar vector of $V\rho F\rho$ at $i$ takes the form

$$k(b_1y' + \text{a term in } z').$$

Hence the condition that the tangent plane to (3) shall be the plane $z = 0$ is also that we have $b_1 = 0$.

If the double axis is not a zero, we shall have the vector in the form

$$i(ax^2 + bxy + cy^2 + \text{terms in } z) + j(b_1xy + c_1y^2 + \text{terms in } z),$$

where $i$ is taken as an axis. The equation to determine axes in the plane $z = 0$ becomes

$$x(b_1xy + c_1y^2) = y(ax^2 + bxy + cy^2)$$

giving $b_1 = a$ if there is to be but one axis other than $i$. But the polar vector becomes at $i$,

$$V(ix' + jy' + kz')ia + Vi(jb_1y' + \text{a term in } z').$$

The only terms other than those in $z'$ are

$$-kay' + kb_1y'$$

Hence $b_1 = a$ gives $z = 0$ as the tangent plane to (3) at $i$.

The reasoning of the preceding article may now be extended to cover the case of a quadratic vector with one triple axis and either one or two double axes. We proceed as in Art. 34. If the axes originally selected to be zeros along with $\beta_1$ remain single axes, we still obtain the form of (201). The resulting vectors $\alpha$ cannot be coplanar and we may factor as before. If, on the other hand, the axes selected to be zeros approach each other, we obtain the form of (203). The reasoning already employed shows that the plane of the resulting vector cannot be constant, and we may factor as in Art. 34.

**36.** If the given quadratic vector has two triple axes and one single axis, the preceding reasoning fails. A normal form for this case is (159). It will be simplest to consider two subcases, according as the three distinct axes are, or are not, diplanar.

Case 1. The three axes $\beta_2$, $\beta_5$, and $\beta_7$ are diplanar. Let these axes be made zeros by adding to the normal form (159) a proper term of the form $\rho S\delta\rho$. The vector then takes the form of the right member of (201). The constant vectors $\alpha$ will not be in the same plane, and we factor into $V\phi\rho\theta\rho$ as in Art. 33.

Proof. Suppose the vectors $\alpha$ to be coplanar. Let one of them be expressed in terms of the other two. The scalar coefficients of these two will then define, by their vanishing, two quadrics. The four intersections of these quadrics can only be at $\beta_2$, $\beta_5$ and $\beta_7$, since there are no other axes. Hence one of the axes $\beta_2$ or $\beta_5$ must be a double intersection for the two quadrics. Suppose the axes to be numbered so that $\beta_5$ is a double intersection for the two quadrics. The four intersections of these quadrics count as four of the seven axes of the quadratic vector. The plane of the vectors $\alpha$ must contain the other three, which, however, can only be at $\beta_2$ and $\beta_5$ since there are no other axes in that plane. It was proved in Art. 35 that if a quadratic vector of constant plane has one axis in that plane a zero, and only one other axis in that plane, the plane itself is the tangent plane to the cones (3) at the zero. That is, the tangent plane to (3) at $\beta_2$ passes through $\beta_5$.

Consider the normal form (159). The tangent plane to the cones (3) at $\beta_2$ is the plane $(23\rho) = 0$. Hence we must have $(235) = 0$.

The form (159) now becomes, by putting (235) = 0,

$$\frac{\{\beta_4(527)+\beta_5(247)\}(52\rho)^2}{(527)^2}+\frac{\beta_2(23\rho)\,(45\rho)}{(237)}-\frac{\beta_5(524)\,(52\rho)\,(72\rho)}{(527)^2} \qquad (207)$$

We note that the constant $a$ cannot be zero, since this would give $\beta_2$ an inflectional element of the cones (3) and $\beta_5$ could not then lie in the tangent plane if $F\rho$ is irreducible. Also, (457) cannot vanish, since the planes $(45\rho) = 0$ and $(25\rho) = 0$ taken together constitute a quadric cone which would then contain six axes, as the limiting form of a reducible vector.

Let now the vector (207) be thrown into the form of the right member of (201) by addition of the term

$$\frac{-\rho(52\rho)\,(452)}{(527)}$$

which renders $\beta_7$ a zero without destroying the zero character of $\beta_2$ or $\beta_5$. By use of the identity

$$\rho(452) = \beta_4(52\rho) + \beta_5(24\rho) + \beta_2(45\rho)$$

we find the vector takes the form

$$\frac{\beta_5(52\rho)\{(247)\,(52\rho)-(524)\,(72\rho)-(24\rho)\,(527)\}}{(527)} + \beta_2\left\{\frac{(23\rho)\,(45\rho)}{(237)}-\frac{(45\rho)\,(52\rho)}{(527)}\right\}$$

the terms in $\beta_4$ destroying each other. But we have identically

$$(247)\,(52\rho) - (527)\,(24\rho) = (452)\,(72\rho)$$

whence the coefficient of $\beta_5$ vanishes. The vector is thus reducible, contrary to hypothesis. Hence its plane cannot be constant.

Case 2. The three axes $\beta_2$, $\beta_5$, and $\beta_7$ are coplanar. The form (159) becomes, putting (257) = 0,

$$\frac{\{\beta_5(247)+\beta_2(457)\}\,(23\rho)\,(45\rho)}{(237)\,(457)} + (a_1\beta_5 + b_1\beta_2)\,(52\rho)^2, \qquad (208)$$

where $a_1$ and $b_1$ are constants, which, like the original constants $a$ and $b$, cannot vanish in this case, $F\rho$ being irreducible. It is not possible to throw this vector into the form (201), because we have not three diplanar axes. The form (203) is still possible. By the identity

$$\beta_7(452) = \beta_5(247) + \beta_2(457)$$

the vector (208) may be written, (neglecting a multiplicative constant),

$$\beta_7(23\rho)\,(45\rho) + (a_1\beta_5 + b_1\beta_2)\,(52\rho)^2,$$

the constants $a_1$ and $b_1$ being arbitrary but not zero. By subtraction of the vector term $\rho$ (237) $(45\rho)$ the axis $\beta_7$ is made a zero. We may also write, identically,

$$\rho(237) = \beta_2(37\rho) + \beta_3(72\rho) + \beta_7(23\rho),$$

which gives us our quadratic vector as

$$-\,\beta_2(37\rho)\,(45\rho) - \beta^3(72\rho)\,(45\rho) + (a_1\beta_5 + b_1\beta_2)\,(52\rho)^2 \qquad (209)$$

The axes $\beta_2$, $\beta_5$, and $\beta_7$ are coplanar, so that we may write

$$\beta_5 = m\beta_2 + n\beta_7.$$

We therefore have $(52\rho) = n(72\rho)$.

If we now put $x_1 = (37\rho)$, $z_2 = (45\rho)$, and $x_3 = (72\rho)$, we shall have thrown (209) into the form (203). It is evident that the vector coefficients are not coplanar and the factorization proceeds as in Art. 34.

All quadratic vectors, therefore, having a triple axis, but no axis of higher order, can be thrown into the form $V\phi\rho\theta\rho + \rho S\delta\rho$.

**37.** Taking, finally, vectors having axes of order higher than the third, it has already been shown that these differ in their properties according as the cones (3) have, all of them, a double element at the multiple axis; or have a uniquely determined tangent plane there.

If $\beta_1$ is an axis of the fourth or higher order, and if we are dealing with the case of vanishing polar vector, so that the tangent plane is not unique or determinate; and if, also, there exist two other axes diplanar with $\beta_1$, the form (201) is not possible. For we can throw at once into the form (165), obviously a vector of constant plane, incapable, therefore, of being factored as $V\phi\rho\theta\rho$.

If, however, we subtract from (165) the term $\rho A_{12}x_3$, the resulting quadratic vector may be written

$$A_{13}\beta_1x_1x_2 - A_{12}\beta_3x_3^2 + (A_{11}\beta_1 + A_{21}\beta_2 - A_{12}\beta_2 + A_{31}\beta_3)x_2x_3, \qquad (210)$$

which is in the form (203) and can therefore be factored into $V\phi\rho\theta\rho$, as in Art. 34, provided the vector coefficients are not in the same plane. It was shown in Art. 27 that neither $A_{13}$ nor $A_{12}$ can be zero. The condition that the vector coefficients be coplanar is therefore

$$A_{21} - A_{12} = 0. \qquad (211)$$

If this condition is satisfied, (210) cannot be factored into $V\phi\rho\theta\rho$. If, however, we subtract from (165) the term $\rho A_{13}x_2$, the resulting quadratic vector may be written

$$A_{12}\beta_1x_3x_1 - A_{13}\beta_2x_2^2 + (A_{11}\beta_1 + A_{21}\beta_2 + A_{31}\beta_3 - A_{13}\beta_3)x_2x_3, \qquad (212)$$

which may be factored like (203) if the vector coefficients are diplanar, that is unless

$$A_{31} - A_{13} = 0. \qquad (213)$$

Suppose (211) and (213) both satisfied. Subtract from (165) the term $\rho(A_{12}x_3 + A_{13}x_2)$. The resulting quadratic vector may be written

$$- A_{13}\beta_2x_2^2 - A_{12}\beta_3x_3^2 + A_{11}\beta_1x_2x_3. \qquad (214)$$

Neither $A_{12}$ nor $A_{13}$ can be zero, the vector being irreducible by hypothesis. If $A_{11}$ is not zero we may put

$$\left.\begin{aligned} \phi\rho &= A_{11}x_2V\beta_1\beta_2 + A_{12}x_3V\beta_2\beta_3 \\ \theta\rho &= \frac{A_{13}x_2}{A_{11}S\beta_1\beta_2\beta_3}V\beta_2\beta_3 + \frac{x_3}{S\beta_1\beta_2\beta_3}V\beta_3\beta_1 \end{aligned}\right\} \qquad (215)$$

and we then have (214) identically equal to $V\phi\rho\theta\rho$. If, on the other hand, $A_{11} = 0$, (214) cannot be factored into $V\phi\rho\theta\rho$. The resulting simple vector

$$A_{13}\beta_2x_2^2 + A_{12}\beta_3x_3^2 \qquad (216)$$

may, however, be factored by first adding a term $\rho S\delta\rho$, where $\delta$ may be chosen in an infinite number of ways. It can be shown that any term

$$-\rho(bx_2 + cx_3)$$

will suffice, if $b$ and $c$ satisfy

$$A_{12}A_{13} - bA_{12} - cA_{13} = 0.$$

A simple solution is found by multiplying (216) by 2, and writing $b = A_{13}$, and $c = A_{12}$. The resulting quadratic vector is

$$2(A_{13}\beta_2x_2^2 + A_{12}\beta_3x_3^2) - \rho(A_{13}x_2 + A_{12}x_3)$$

which by taking $\rho = x_1\beta_1 + x_2\beta_2 + x_3\beta_3$ becomes

$$\beta_1(-A_{13}x_1x_2-A_{12}x_1x_3) + \beta_2(A_{13}x_2^2-A_{12}x_2x_3) + \beta_3(A_{12}x_3^2-A_{13}x_2x_3), \tag{217}$$

which, by Art. 4, has the same axes as (216). If we take

$$\left.\begin{aligned}\phi\rho &= x_2V\beta_1\beta_2 + x_3V\beta_3\beta_1,\\ \theta\rho &= (A_{12}x_3-A_{13}x_2)V\beta_2\beta_3+x_1(A_{12}V\beta_1\beta_2-A_{13}V\beta_3\beta_1),\end{aligned}\right\} \tag{218}$$

we shall have (217) identically equal to $V\phi\rho\theta\rho$, aside from the scalar factor $S\beta_1\beta_2\beta_3$.

This completes the factorization of (165) for all possible cases.

It was shown in Art. (27) that when the quadratic vector has a triple axis, (besides the quadruple axis with vanishing polar vector), it can be thrown into the form (170), evidently of constant plane. If, however, we subtract the term

$$\rho(A_{12} - A_{21})x_3 + \rho A_{13}x_2$$

the resulting quadratic vector may be written

$$-x_2^2A_{13}\beta_2-x_2x_3(A_{12}\beta_2-A_{21}\beta_2+A_{13}\beta_3) + x_3^2(a_1\beta_1+a_2\beta_2-A_{12}\beta_3 +A_{21}\beta_3) \tag{219}$$

which is the same form as (214) and can be factored as in (215) provided the vector coefficients are diplanar. We cannot have $A_{13} = 0$. The condition of coplanarity for the three coefficients is therefore $a_1 = 0$; that is, the element $\beta_2$ is an inflectional element of the cones (3). If such is the case, we may, instead, subtract from (170) the term $\rho(A_{12} - A_{21})x_3$. The resulting quadratic vector may be written

$$x_1x_2A_{13}\beta_1+x_3^2(a_1\beta_1+a_2\beta_2-A_{12}\beta_3+A_{21}\beta_3)-x_2x_3(A_{12}-A_{21})\beta_2 \tag{220}$$

which is in the form (203) and can be factored into $V\phi\rho\theta\rho$ if the vector coefficients are diplanar; that is, unless

$$A_{12} - A_{21} = 0.$$

If such is the case, and if, also, $a_1 = 0$, the vector (170) takes the simple form

$$\beta_1 A_{13} x_1 x_2 + \beta_2 a_2 x_3^2 \tag{221}$$

If we subtract the term $\rho x_3 \sqrt{a_2 A_{13}}$, the resulting quadratic vector may be written

$$\beta_1(A_{13} x_1 x_2 - b x_2 x_3) + \beta_2(a_2 x_3^2 - b x_2 x_3) - \beta_3 b x_3^2, \tag{222}$$

where $b = \sqrt{a_2 A_{13}}$. If we take

$$\begin{aligned} \phi\rho &= \sqrt{a_2}\, x_3 V\beta_2\beta_3 + \sqrt{A_{13}}\, x_1 V\beta_3\beta_1, \\ \theta\rho &= (\sqrt{a_2}\, x_3 - \sqrt{A_{13}}\, x_2) V\beta_1\beta_2 + \sqrt{A_{13}}\, x_3 V\beta_3\beta_1, \end{aligned} \tag{223}$$

we shall have (222) identically equal to $V\phi\rho\theta\rho$ aside from the scalar factor $S\beta_1\beta_2\beta_3$.

We have thus considered all possibilities for an axis $\beta_1$ when the cones (3) all have a double element at $\beta_1$, and $\beta_1$ is not of higher than the fourth order. If $\beta_1$ is of fifth or higher order we may assume $\beta_2$ and $\beta_3$ any two vectors not coplanar with $\beta_1$. By virtue of the vanishing polar vector we may then throw the quadratic vector into the form

$$\alpha_1 x_2^2 + \alpha_2 x_2 x_3 + \alpha_3 x_3^2, \tag{224}$$

and if the vectors $\alpha$ are diplanar we may factor as in (214) and (215). If the $\alpha$'s are coplanar, $\beta_1$ must lie in their plane, for if not we shall have three other axes in that plane, (not necessarily distinct), and $\beta_1$ will be of the fourth order only. We may suppose $\beta_2$, as yet arbitrary, to be some other vector in this plane. (224) may then be written

$$\beta_1(B_{11} x_2^2 + B_{12} x_2 x_3 + B_{13} x_3^2) + \beta_2(B_{21} x_2^2 + B_{22} x_2 x_3 + B_{23} x_3^2), \tag{225}$$

where the $B$'s are constant scalars. The plane of $\beta_1$ and $\beta_2$ is the same as the plane $x_3 = 0$. It is evident, therefore, that the only axis, if any, which (225) possesses distinct from $\beta_1$ is

$$\beta_1 B_{11} + \beta_2 B_{21}. \tag{226}$$

If $B_{21}$ is not zero we may suppose $\beta_2$, as yet arbitrary in the plane $x_3 = 0$, to be an axis; when $B_{11}$ will disappear. By subtracting the term $\rho B_{21} x_2$, the resulting quadratic vector may be written

$$-x_1x_2B_{21}\beta_1+x_2x_3(B_{12}\beta_1+B_{22}\beta_2-B_{21}\beta_3)+x_3^2(B_{13}\beta_1+B_{23}\beta_2), \qquad (227)$$

the axis $\beta_2$ being made a zero, and, consequently, the term in $x_2^2$ disappearing. (227) is in the form (203) and can be factored into $V\phi\rho\theta\rho$ if the vector coefficients are diplanar. We may not have $B_{21} = 0$, for if so the vector is reducible. Thus the condition for coplanarity is that $B_{23}$ shall vanish. If so, the quadratic vector (225) becomes

$$\beta_1(B_{12}x_2x_3 + B_{13}x_3^2) + \beta_2(B_{21}x_2^2 + B_{22}x_2x_3); \qquad (228)$$

and if $\beta_{12}$ is not zero we may subtract the vector term

$$\rho\left\{ B_{21}x_2 + \frac{B_{21}B_{13}}{B_{12}}x_3 \right\} \qquad (229)$$

writing the resulting quadratic vector as

$$\beta_1(B_{12}x_2x_3+B_{13}x_3^2-B_{21}x_1x_2-bx_1x_3)+\beta_2(B_{22}-b)x_2x_3-\beta_3(B_{21}x_2x_3 +bx_3^2) \qquad (230)$$

$$\left.\begin{array}{l}\text{where } b \text{ stands for the coefficient of } x_3 \text{ in (229). Taking}\\ \phi\rho = x_2V\beta_1\beta_2(B_{21}B_{13}-B_{22}B_{12})-V\beta_3\beta_1(x_3B_{21}B_{13}+x_2B_{12}B_{21})\end{array}\right\} \qquad (231)$$

$$\theta\rho = B_{21}x_3V\beta_2\beta_3+(B_{12}x_3-B_{21}x_1)V\beta_1\beta_2,$$

we have (230) identically equal to $V\phi\rho\theta\rho$ aside from the scalar factor $B_{21}B_{12}\cdot S\beta_1\beta_2\beta_3\cdot$

If $B_{12} = 0$ (225) becomes

$$\beta_1B_{13}x_3^2 + \beta_2(B_{21}x_2^2 + B_{22}x_2x_3); \qquad (232)$$

if we add the term $-c\rho(x_2B_{21} + x_3B_{22})$, where $c$ is a scalar constant neither zero nor unity, the resulting quadratic vector may be written

$$\beta_1(B_{13}x_3^2-cB_{21}x_1x_2-cB_{22}x_1x_3)+\beta_2(1-c)(B_{21}x_2^2+B_{22}x_2x_3)-c\beta_3(B_{21}x_2x_3 +B_{22}x_3^2) \qquad (233)$$

If we take

$$\phi\rho = \frac{c^2B_{21}x_1}{1-c}V\beta_3\beta_1+c((x_2B_{21}+x_3B_{22})V\beta_2\beta_3+(x_3B_{13}-cx_1B_{22})V\beta_1\beta_2,$$

$$\theta\rho = \frac{1-c}{c}x_2V\beta_1\beta_2 + x_3V\beta_3\beta_1, \qquad (234)$$

we shall have (233) identically equal to $V\phi\rho\theta\rho$ aside from the scalar factor $S\beta_1\beta_2\beta_3$.

Returning to (225), if $B_{21} = 0$, the vector can have no axis except $\beta_1$. If we take

$$y = -\frac{B_{22}}{c},\ z = -\frac{B_{23}}{c}, \tag{235}$$

where $c$ is any non-vanishing scalar, and subtract from (225) the term $\rho(yx_2 + zx_3)$, the resulting quadratic vector may be written

$$\beta_1(B_{11}x_2^2+B_{12}x_2x_3+B_{13}x_3^2-yx_1x_2-zx_1x_3)+\beta_2(B_{22}x_2x_3-zx_2x_3-yx_2^2) \\ -\beta_3(yx_2x_3+zx_3^2). \tag{236}$$

If we now take

$$\left.\begin{aligned} \phi\rho &= (x_2B_{22}+x_3B_{23})V\beta_2\beta_3-(x_1B_{22}+x_2B_{11}c)V\beta_3\beta_1 \\ &\qquad +[x_1(cB_{22}-B_{23})+x_2(c^2B_{11}-cB_{12})-cB_{13}x_3]V\beta_1\beta_2, \\ \theta\rho &= -x_3V\beta_3\beta_1+(x_2+cx_3)V\beta_1\beta_2 \end{aligned}\right\} \tag{237}$$

we shall have (236) identically equal to $V\phi\rho\theta\rho$ aside from the scalar factor $cS\beta_1\beta_2\beta_3$.

This completes the factorization into $V\phi\rho\theta\rho$ for all possible cases where all cones (3) have $\beta_1$ a double element.

If the cones (3) have $\beta_1$ an axis of the fourth or higher order, with a uniquely determined tangent plane, it has been shown that we may throw into one of the two forms (185) or (195), according as $\beta_1$ is an ordinary, or an inflectional, element.

The sub-case (185) may be at once factored if the vectors $\pi$, $\mu$, and $\beta_1u$ are diplanar. For if we take

$$\left.\begin{aligned} \phi\rho &= x_2V\mu\pi - ux_3V\pi\beta_1, \\ \theta\rho &= x_1V\pi\beta_1 - x_2V\beta_1\mu + x_3\left(\frac{g}{u}V\mu\pi + g_1V\pi\beta_1\right) \end{aligned}\right\} \tag{238}$$

we shall have (185) identically equal to $V\phi\rho\theta\rho$ aside from the factor $S\pi\mu\beta_1$.

If $u = 0$, this method fails. If there exists an axis not in the plane $x_2 = 0$ we may suppose, as already shown, that $\mu = a\beta_2$. Then the constant $a$ cannot be zero. If we now subtract the term $a\rho x_2$, the resulting quadratic vector may be written

$$x_1x_2(\pi - a\beta_1) + x_2x_3(g_1\pi - a\beta_3) + x_3^2g\pi, \tag{239}$$

which factors like (203) if the vector coefficients are not in the same plane, that is unless $S\beta_3\beta_1\pi = 0$. If so, we must either have $\pi$ parallel to $\beta_1$, or may take it parallel to $\beta_3$, which is as yet any vector in the tangent plane. If $\pi$ is parallel to $\beta_3$ we may subtract the term $\rho(ax_2 + gx_3)$ from (186). The resulting quadratic vector may be written

$$x_2x_3\{(g_1 - a)\beta_3 - g\beta_2\} - x_3x_1g\beta_1 + x_1x_2\{c\beta_3 - a\beta_1\}, \tag{240}$$

where $c$ is a constant which cannot vanish. This is in the form (201) and can be factored, since neither $c$ nor $g$ are zero for this case. If $\pi$ is parallel to $\beta_1$, the tangent plane to (3) ceases to be uniquely determined, contrary to hypothesis.

If there exists no axis without the tangent plane, $u$ being still zero, we must have $S\beta_1\pi\mu = 0$; for if not we should have three axis, (not necessarily distinct), aside from $\beta_1$. These cannot be distinct and in the tangent plane, nor, as was shown, can they be coincident in that plane. Hence $S\beta_1\pi\mu = 0$. But a quadratic vector of constant plane, having a zero in that plane, and only one other axis in that plane, must have that plane tangent to the cones (3) at the zero.[21] Hence $\pi$ lies in the plane $x_2 = 0$, and may be taken to be parallel to $\beta_3$. Since $\mu$ is in the same plane we may take $\mu = b_1\beta_1 + b_3\beta_3$. The vector (185) now takes the form

$$\beta_3(ax_1x_2 + gx_3^2 + g_1x_2x_3) + (b_1\beta_1 + b_3\beta_3)x_2^2, \tag{241}$$

where $a$ is a non-vanishing constant.

To see whether a term $\rho S\delta\rho$ can be added to this vector so that it can be factored into $V\phi\rho\theta\rho$, the most elegant method would be to consider, after Hamilton, the pure and the rotational parts of strains defined by $\phi$ and $\theta$. As I have not in the present paper introduced these ideas, I shall employ the more cumbrous method of undetermined coefficients; and shall thereby avoid a digression upon simultaneous forms of $\phi$ and $\theta$. Since the vectors $\beta_1$, $\beta_2$, and $\beta_3$ are diplanar, a general form of $\phi$ and $\theta$ with undetermined constants $p$ and $q$ may be taken as

$$\begin{aligned}\phi\rho &= x_1(pV\beta_2\beta_3+qV\beta_3\beta_1+rV\beta_1\beta_2)+x_2(p'V\beta_2\beta_3+q_1V\beta_3\beta_1+r'V\beta_1\beta_2)\\ &\qquad +x_3(p''V\beta_2\beta_3+q''V\beta_3\beta_1+r''V\beta_1\beta_2),\\ \theta\rho &= x_1(p_1V\beta_2\beta_3+q_1V\beta_2\beta_1+r_1V\beta_1\beta_2)+x_2(p'_1V\beta_2\beta_3+q'_1V\beta_3\beta_1\\ &\qquad +r'_1V\beta_1\beta_2)+x_3(p''_1V\beta_2\beta_3+q''_1V\beta_3\beta_1+r''_1V\beta_1\beta_2).\end{aligned}$$

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[21] Proved in Art. 35.

On the one hand we have the vector product of these two expressions. On the other, we have to add to (241) a term which we may write

$$-(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3)(x x_1 + y x_2 + z x_3),$$

where $x$, $y$, and $z$ are undetermined, and are the components of the vector $\delta$. Multiplying out, and equating coefficients of like terms in the variables $x_1$, $x_2$, $x_3$, we have a system of eighteen equations. It is clear at the start that we may take $x = 0$, for this is the same as saying that $\beta_1$ must be a zero; which follows from the facts: that $\beta_1$ is a sextuple axis; and $V\varphi\rho\theta\rho$ has 3 zeros. Also, we may take $p_1 = q_1 = r_1 = 0$. For this is the same as making the coefficient of $x_1^2$ zero, a necessary condition that $\beta_1$ be a zero. We then have the system of equations, fifteen in number,

Coef. of $x_2^2$.

1. $q'_1 r' - q' r'_1 = b_1$
2. $p' r'_1 - p'_1 r' = -y$
3. $p'_1 q' - p' q'_1 = b_3$

Coef. of $x_3^2$.

4. $p''_1 q'' - p'' q''_1 = g - z$
5. $p'' r''_1 - p''_1 r'' = 0$
6. $q''_1 r'' - q'' r''_1 = 0$

Coef. of $x_1 x_2$

7. $q'_1 r - q r'_1 = -y$
8. $p r'_1 - p'_1 r = 0$
9. $p'_1 q - p q'_1 = a$

Coef. of $x_2 x_3$

10. $q'_1 r'' - q' r''_1 + q''_1 r' - q'' r'_1 = 0$
11. $p' r''_1 - p'_1 r'' + p'' r'_1 - p''_1 r' = -z$
12. $p'_1 q'' - p' q''_1 + p''_1 q' - p'' q'_1 = g_1 - y$

Coef. of $x_3 x_1$

13. $q''_1 r - q r''_1 = -z$
14. $p r''_1 - p''_1 r = 0$
15. $p''_1 q - p q''_1 = 0.$

Either $\phi\rho$ or $\theta\rho$ may be divided by a scalar which is also multiplied into the other, leaving $V\phi\rho\theta\rho$ unchanged. We may therefore without loss of generality assume that some one letter, as $r'_1$, is either zero or unity.

Case 1. Let $r'_1 = 1$. I shall show first that $p$ must be zero. For, if not, we have, (by 8), $p = p'_1 r$. Substituting in 14 we have, (since $r$ cannot be zero if $p$ is not zero), $p''_1 = p'_1 r''_1$. Then writing for $p$ and for $p''_1$ their values in 15 we have, since $p'_1$ cannot vanish if $p$ does not,

$$r''_1 q - q''_1 r = 0. \tag{A}$$

Comparing with 13, this gives $z = 0$. Hence $r''_1$ is not zero, for if so, by 13 and 14, (since $r$ is not zero), $p''_1 = 0$ and $q''_1 = 0$, making $\theta\rho$ a monomial and $V\phi\rho\theta\rho$ reducible. We may then write $r''_1 = cr$, where $c$ is not zero. By (A), $q''_1 = cq$ and by 14 $p''_1 = cp$. From 5 and 6 we have

$$p'' r - p r'' = 0, \quad q'' r - q r'' = 0,$$

and since $r$ is not zero it follows that $p''q - pq'' = 0$. But from 4, writing for $p''_1$ and $q''_1$ their values, $cp$ and $cq$,

$$-c(p''q - pq'') = g - z.$$

Therefore $g = z = 0$, contrary to fact, since if $g$ vanishes the quadratic vector is reducible. Thus if $r'_1 = 1$, $p = 0$, which is equivalent to saying that either $p$ or $r'_1$ must vanish.

Case 2. Let $p = 0$. We need not now assume $r'_1 = 1$. By 8, either $p'_1$ or $r$ must vanish. Suppose $p'_1 = 0$. By 9, we have $a = 0$, But if $a = 0$ the vector (241) has, at $\beta_1$, a quadruple axis of the first kind, contrary to hypothesis.

Suppose $p = r = 0$. Then, by 9, neither $p'_1$ nor $q$ can vanish. If we take $q = 1$, we have $p'_1 = a$, (by 9), and $r'_1 = y$, (by 7). Also $r''_1 = z$, (by 13), and $p''_1 = 0$, (by 15). Now $r''_1$ cannot vanish, for if so we have, by 6, either $q''_1 = 0$, or $r'' = 0$; but if $q''_1 = 0$, $g = 0$, by 4; and if $r'' = 0$, $p'' = 0$ by 11, (we cannot have $r'_1 = 0$, since not both $y$ and $z$ vanish), and again $g = 0$ (by 4). Hence $r''_1$ is not zero.

We therefore have $p'' = 0$, (by 5), and $z = g$, (by 4). Equation 6 now becomes

$$q_1''r'' - gq'' = 0, \tag{A}$$

which will serve to determine $q''$. There remain the six equations 1, 2, 3, 10, 11, 12. Substituting values, these become,

1. $q'_1r' - q'y = b_1$
2. $p'y \quad - ar' = -y$
3. $aq' \quad - p'q'_1 = b_3$
10. $q'_1r'' - q'g + q''_1r' - q''y = 0$
11. $p'g - ar'' = -g$
12. $aq'' - p'q''_1 = g_1 - y.$

By elimination of $q''$ and $p'$ from 11, 12, and (A), we have

$$q''_1 = g_1 - y. \tag{B}$$

By elimination of $q'$ and $r'$ from 1, 2, and 3,

$$y(q'_1 - b_3) = ab_1. \tag{C}$$

Since neither $a$ nor $b_1$ can be zero, $y$ cannot. Hence this equation gives a value for $q'_1$. By 11, we have $r''$ in terms of $p'$. It is then an easy calculation to substitute, in 10, all other unknowns in terms of $p'$ and $y$. We find all coefficients of powers of $y$ cancel out, giving $b_1g = 0$, which is impossible since the vector is irreducible. Hence we cannot have $p = 0$.

Case 3. Let $r'_1 = 0$. We cannot also have $p'_1 = 0$; for if so, we have $y = 0$, (by 2), and $r = 0$, (by 7, since $q'_1$ is not zero by 9); then $r''_1 = 0$, (by 14), and $z = 0$, (by 13), but $z$ and $y$ are not both zero. We then must have, by 8, $r'_1 = r = 0$. By 7, $y = 0$. By 2, $r' = 0$. By 1, $b_1 = 0$, which is not true. Hence $r'_1$ cannot be zero.

Thus the vector (241) cannot be written in the form

$$V\phi\rho\theta\rho + \rho S\delta\rho.$$

This completes the examination of (185) when $u = 0$. The factorization (238) also fails when $S\pi\mu\beta_1 = 0$. If there exists an axis not in the plane $x_2 = 0$, we may, as before, subtract the term $a\rho x_2$, with $\mu = a\beta_2$. If $a$ does not vanish we may factor as in (203). We cannot now conclude that $a$ is not zero, since we have not $u = 0$. If $a = 0$, and $S\pi\beta_1\beta_2 = 0$, the vector (185) becomes

$$(c_1\beta_1 + c_2\beta_2)\,(x_1x_2 + gx_3^2 + g'x_2x_3) + u\beta_1x_2x_3 \tag{242}$$

where $c_1$ and $c_2$ are constants, and $c_2$, at least, is not zero since $\pi$ is not parallel to $\beta_1$. If $c_1$ is not zero, we may add the term

$$-\rho\left(\frac{c_2u}{c_1}\right)$$

when the quadratic vector may be written

$$\begin{aligned}\beta_1\{c_1x_1x_2 + c_1gx_3^2 + (c_1g_1 + u)x_2x_3 - zx_1x_3\}\\ + \beta_2\{c_2x_1x_2 + c_2gx_3^2 + c_2g_1x_2x_3 - zx_2x_3\} - \beta_3zx_3^2,\end{aligned} \tag{243}$$

where $z$ has been written for $\dfrac{c_2u}{c_1}$. If we then take

$$\begin{aligned}\phi\rho &= c_2x_3V\beta_2\beta_3 - c_1x_3V\beta_3\beta_1 - c_2x_1V\beta_1\beta_2,\\ \theta\rho &= V\beta_2\beta_3 + V\beta_3\beta_1\left\{\frac{ux_3}{c_1} - \frac{c_1x_2}{c_2}\right\} + V\beta_1\beta_2\left\{gx_3 + \left(g_1 + \frac{u}{c_1}\right)x_2\right\},\end{aligned} \tag{244}$$

we find (243) identically equal to $V\phi\rho\theta\rho$, aside from the factor $S\beta_1\beta_2\beta_3$.

This method fails if $c_1 = 0$. If so, the vector (242) becomes, aside from a scalar factor,

$$\beta_2(ax_1x_2 + gx_3^2 + g_1x_2x_3) + u\beta_1x_2x_3, \tag{245}$$

where $a$ is a constant which is not zero. This quadratic vector has $\beta_1$

for a quadruple axis and $\beta_2$ for a triple axis. If we add the vector term

$$-\rho(ax_1 + g_1x_3)$$

the resulting quadratic vector may be written

$$\beta_1(ux_2x_3 - ax_1^2) + (g\beta_2 - g_1\beta_3)x_3^2 + (-g_1\beta_1 - a\beta_3)x_3x_1, \qquad (246)$$

and if we then take

$$\begin{aligned} \phi\rho &= -ax_1(g_1V\beta_3\beta_1 + gV\beta_1\beta_2) - ax_2uV\beta_3\beta_1 \\ &\qquad + x_3(ga\,V\beta_2\beta_3 - g_1^2V\beta_3\beta_1 - gg_1V\beta_1\beta_2), \\ \theta\rho &= ax_1V\beta_3\beta_1 + x_3(g_1V\beta_3\beta_1 + gV\beta_1\beta_2), \qquad (247) \end{aligned}$$

we find (246) identically equal to $V\phi\rho\theta\rho$, aside from a scalar factor $gaS\beta_1\beta_2\beta_3$.

If no axis exists except in the plane $x_2 = 0$, (having now by hypothesis $S\pi\mu\beta_1 = 0$ and $u$ not zero), we must, as before, have $\pi$ an axis which we may take as $\beta_3$. The quadratic vector may then be written

$$\beta_3(ax_1x_2 + gx_3^2 + g_1x_2x_3) + u\beta_1x_2x_3 + (b_1\beta_1 + b_3\beta_3)x_2^2, \qquad (248)$$

where the constants $a$, $b_1$, and $b_3$, have the same meaning as in (241).

To see whether a term $\rho S\delta\rho$ can be found which shall render this vector factorizable, we may set up equations as for (241), with the difference that the right member of 10 will now be $u$ instead of zero. In the equations under (241), for Cases 1 and 3, equation 10 was not used. Hence the reasoning still holds. For case 2, the reasoning is as before, up to the substitution in equation 10. The result gives $y = \dfrac{b_1g}{u}$ as a unique solution. The values of $\phi\rho$ and of $\theta\rho$ which follow are, on letting $p'$, (which, from the equations, is arbitrary), have the value $-1$, and clearing of fractions,

$$\begin{aligned} \phi\rho &= -gx_2V\beta_2\beta_3 + (gx_1 - ux_2)V\beta_3\beta_1, \\ \theta\rho &= agux_2V\beta_2\beta_3 + \{(au^2 + b_3gu)x_2 + (gg_1u - b_1g^2)x_3\}V\beta_3\beta_1 \\ &\qquad + \{b_1g^2x_2 + g^2ux_3\}V\beta_1\beta_2; \end{aligned}$$

while (248), after subtracting the term $\rho(yx_2 + gx_3)$, becomes

$$\begin{aligned} \beta_1(ux_2x_3 + b_1x_2^2 - yx_1x_2 - gx_1x_2) + \beta_2(-yx_2^2 - gx_2x_3) + \beta_3(ax_1x_2 + g_1x_2x_3 \\ + b_3x_2^2 - yx_2x_3). \qquad (250) \end{aligned}$$

We then find that (250) is identically equal to $V\phi\rho\theta\rho$, aside from the scalar factor $g^2 u S\beta_1\beta_2\beta_3$.

This completes all possible special forms of the vector (185). The only form which cannot be made factorizable by a term $\rho S\delta\rho$ is the vector (241).

The only remaining irreducible vector to be considered is (195). If there is at least one axis other than $\beta_1$, we can write as (196), which factors like (203) if the vector coefficients are diplanar, that is, unless $S\beta_3\beta_1\zeta = 0$. If so, we may write (196) as

$$\beta_1(b_1x_2x_3 + ux_3^2) + \beta_3(b_3x_2x_3 + ax_1x_2), \tag{251}$$

where $\zeta = b_1\beta_1 + b_3\beta_3$. To see whether a term $\rho S\delta\rho$ can be added to this vector so that it can be factored into $V\phi\rho\theta\rho$, we note that it has $\beta_1$ for a quintuple axis, and $\beta_2$ for a single axis, and the other axis is

$$\beta_1 b_3 b_1 + \beta_2 a u - \beta_3 a b_1.$$

Therefore $V\phi\rho\theta\rho$, if it exists, must have $\beta_1$ for a zero, since three of its axes must be zeros, (not necessarily distinct). We may then set up equations, fifteen in number, as for the vector (241) and the left members will be identical with those for (241). The right members will be the same for equations 2, 5, 7, 8, 9, 11, 13, 14, and 15. The remaining equations become

1. $q'_1r' - q'r'_1 = 0$
3. $p'_1q' - p'q'_1 = 0$
4. $p''_1q'' - p''q''_1 = -z$
6. $q''_1r'' - q''r''_1 = u$
10. $q'_1r'' - q'r''_1 + q''_1r' - q''r'_1 = b_1$
12. $p'_1q'' - p'q''_1 + p''_1q' - p''q'_1 = b_3 - y$

The reasoning under case 1 is precisely as before, up to the obtaining of the relations $q''_1 = cq$ and $p''_1 = cp$, since none of the new equations are used so far. Then by 4 and 5 we have

$$p''r - pr'' = 0, \quad p''q - pq'' = 0,$$

and since $p$ is not zero it follows that $q''r - qr'' = 0$. But, (from 6), putting for $q''_1$ and $r''_1$ their values $cq$ and $cr$, we have

$$-c(q''r - qr'') = u$$

Therefore $u = 0$, which makes (251) reducible contrary to hypothesis. Hence if $r'_1 = 1$ we cannot have $p$ different from zero. Hence either $r'_1$ or $p$ must be zero.

Case 2. Let $p = 0$. As before, $p'_1$ cannot be zero. Hence $r = 0$. And as before, if $q = 1$ $p'_1 = a$, $r'_1 = y$, $r''_1 = z$, and $p''_1 = 0$.

We cannot have $r''_1 = 0$. For if so we cannot also have $q''_1 = 0$, for $\theta\rho$ would be a monomial. Then $p'' = 0$, (by 4), and $r'' = 0$, (by 11). This makes $u = 0$, (by 6), which is contrary to hypothesis.

We must then have $p'' = 0$,(by 5) and again $z = 0$, (by 4), or $r''_1 = 0$, just proved impossible. Hence this case is impossible.

Case 3. Let $r'_1 = 0$. We cannot have $p'_1 = 0$, by the same reasoning as before. We must then have $r = 0$, (by 8), $y = 0$, (by 7), and $r' = 0$, (by 2), as before. Then $r''_1 = 0$, (by 14), and $z = 0$, (by 13). But not both $y$ and $z$ are zero for this leaves (251) unchanged. Hence this case is impossible. Hence no term $\rho S \delta \rho$ can be found.

It remains to consider the possibility that (195) shall not possess an axis other than $\beta_1$. If we write $\zeta = b_1\beta_1 + b_2\beta_2 + b_3\beta_3$, and $u = c_1\beta_1 + c_2\beta_2 + c_3\beta_3$, the vector (195) becomes

$$\beta_1(b_1x_2x_3+c_1x_2^2+ux_3^2)+\beta_2(b_2x_2x_3+c_2x_2^2)+\beta_3(b_3x_2x_3+c_3x_2^2+ax_1x_2), \tag{252}$$

which we may denote as usual by $F\rho$. If $\rho = \beta_1x_1 + \beta_2x_2 + \beta_3x_3$, the vector equation $V\rho F\rho = 0$ defines the axes of $F\rho$, and, by multiplying out, is equivalent to the three scalar equations

$$\begin{aligned} x_2(b_3x_2x_3 + c_3x_2^2 + ax_1x_2) - x_3(b_2x_2x_3 + cx_2^2) &= 0, \\ x_3(b_1x_2x_3 + c_1x_2^2 + ux_3^2) - x_1(b_3x_2x_3 + c_3x_2^2 + ax_1x_2) &= 0, \\ x_1(b_2x_2x_3 + c_2x_2^2) - x_2(b_1x_2x_3 + c_1x_2^2 + ux_3^2) &= 0. \end{aligned}$$

If there is no axis except $\beta_1$, these equations have no solution when $x_2$ is not zero. Furthermore, if $x_2$ is not zero, any solution of the first and third equations simultaneously must be a solution of the second. By elimination of $x_1$ from the first and third equations we have the cubic

$$\begin{aligned} b_2^2x_3^3+(2b_2c_2-b_2b_3-au)x_2x_3^2+(c_2^2-b_3c_2-b_2c_3-ab_1)x_2^2x_3 \\ -(ac_1+c_2c_3)x_2^3=0. \end{aligned} \tag{253}$$

If this equation can be solved for $x_3$, we can find $x_1$ from the first of the above cubics, and so have an axis other than $\beta_1$. That no such axis exist it is necessary that $b_2 = 0$, and hence also that $au = 0$ contrary to hypothesis. Hence such an axis must exist. This completes the study of (195), and hence of irreducible quadratic vectors.

## SUMMARY OF RESULTS.

**38.** All irreducible quadratic vectors may be thrown into the form $V\phi\rho\theta\rho + \rho S\delta\rho$, with exception of two special cases.

First exception.

$$\beta_3(ax_1x_2 + gx_3^2 + g_1x_2x_3) + (b_1\beta_1 + b_3\beta_3)x_2^2. \tag{241}$$

This vector may be detected, when its components $X$, $Y$, $Z$, are given in any form, by the following properties; it has a sextuple axis, and a single axis; the tangent plane to the cones (3) is unique and determinate for at least one of the cones; the single axis lies in this tangent plane; on taking $\beta_1$ for the multiple axis, $\beta_3$ for the single axis, (so that $x_2 = 0$ gives the tangent plane), and $\beta_2$ any vector without the tangent plane, we may remove the terms in $x_1^2$, $x_1x_2$, and $x_1x_3$ from the coefficient of $\beta_1$, by adding a properly chosen term $\rho S\delta\rho$; and the vector then takes the form (241).

Second exception.

$$\beta_1(b_1x_2x_3 + ux_3) + \beta_3(b_3x_2x_3 + ax_1x_2). \tag{251}$$

The properties of this vector are: it has a quintuple axis; the tangent plane to at least one of the cones (3) is unique and determinate at this axis, which is an inflectional element of the cone; on taking $\beta_1$ for the quintuple axis, $\beta_2$ for another axis, $\beta_3$ any vector in the tangent plane except $\beta_1$, we may remove the terms $x_1^2$, $x_1x_2$, and $x_1x_3$ from the coefficient of $\beta_1$; the vector then takes the form (251).

If the quadratic vector does not come under either exceptional case, the method of throwing it into the form $V\phi\rho\theta\rho + \rho S\delta\rho$ depends upon the configuration of the axes. The form of the constant vector $\delta$ is not, usually, unique, but may have thirty-five possibilities or fewer. The values obtained in this paper, are, therefore, not the only ones that could be given for the majority of cases. They have been selected so that $\delta$ should be rational, and as simple as possible in terms of known axes.

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# *THE MOUNTAIN ANTS OF WESTERN NORTH AMERICA.*

By William Morton Wheeler.

# THE MOUNTAIN ANTS OF WESTERN NORTH AMERICA.[1]

By William Morton Wheeler.

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The study of several collections of ants received from Professors J. C. Bradley, C. F. Baker, T. D. A. Cockerell, C. C. Adams, S. J. Hunter, Dr. W. M. Mann, Dr. R. V. Chamberlin, Mr. E. J. Oslar and others and of my own collections made during several seasons in Colorado, New Mexico, Texas, Arizona and Southern California, and especially during the summer of 1915 in the Yosemite Valley and at Lake Tahoe, California and in the Canadian Rockies, enables me to give a much more consistent and comprehensive account of the distribution of the Formicidae of Western North America than was possible heretofore. These collections represent two distinct faunas, one of which belongs to Merriam's Lower and Upper Sonoran Zones and comprises species of several neotropical and tropicopolitan genera and subgenera, while the other, occurring at higher elevations belongs to Merriam's Transition and Canadian Zones and is represented by species of the genera *Monomorium, Solenopsis, Myrmecina, Myrmica, Leptothorax, Aphaenogaster, Stenamma, Liometopum, Tapinoma, Prenolepis s. str., Lasius, Formica, Polyergus* and a few subgenera of *Camponotus* (*Camponotus s. str. and Myrmoturba*). There is some overlapping of the Sonoran and mountain faunas due to the ascent of such forms as *Pogonomyrmex occidentalis, Myrmecocystus mexicanus* and a few species of *Crematogaster, Pheidole* and *Solenopsis* into the Transition Zone and the descent of a few species of *Camponotus s. str., Myrmica* and *Formica* into the Sonoran Zones. In the following pages I have listed the known forms belonging to the Transition and Canadian Zones of Western North America and have added descriptions of 32 new forms (three species, twelve subspecies and seventeen varieties) which I have been able to recognize among the recently collected material. I have not included any of our Ponerinae in the list, because the distribution of the species of *Ponera, Stigmatomma; Proceratium* and *Sysphincta*, with the exception of *Ponera coarctata* subsp. *pennsylvanica*, is imperfectly known, and because I have no new data for publication. The ants of the genera *Proceratium* and *Sysphincta* are very rare and seem to belong to the Upper and Lower Austral Zones, but they will probably be discovered in the Western States. I have, in fact, seen a male specimen which seems to belong to one of these genera, from California. *Ponera pennsylvanica* is confined to the Eastern and Central States, Ontario and Nova Scotia. The genus is represented in the Western and Southern States by at least two closely allied species (*P. trigona* var. *opacior* and *P. opaciceps*), whose precise distribution is still unknown.

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[1] Contributions from the Entomological Laboratory of the Bussey Institution, Harvard University, No. 118.

The great importance of the ants in the study of geographical distribution has not been overlooked by students of this fascinating subject. These insects are, indeed, specially fitted for the mapping of geographical areas, for several reasons. They are not, like many other groups of insects, absolutely dependent on specific food-plants, their colonies are stable and stationary entities, chained to the soil or to certain general plant associations, and they are exceedingly sensitive to climatic and other environmental influences as shown by the extraordinary development of geographical races (subspecies) and varieties in practically all the species of extensive range. A few authors have attempted to minimize these peculiarities on the ground that the marriage-flight of male and female ants must permit of a wide dissemination of the species. It is true that many species of ants have a very wide range, e. g. *Formica fusca*, which is circumpolar and *Camponotus maculatus* which is cosmopolitan, but this is, in all probability, the result of great geologic age, and while we must admit that the nuptial flight of the female ant is practically the only means of rapidly disseminating the species, it is easy to exaggerate its importance. It is natural to suppose that small flying insects, like many female ants, must be carried long distances by air-currents, and these females, when fecundated, are, of course, so many potential colonies. But such observations as can be made in the field do not support this supposition. Most female ants are heavy-bodied and have feeble powers of flight. Moreover, the time during which they can use their wings, especially after fecundation, is limited to a few hours at most. The wing muscles very soon begin to degenerate and compel the insects to descend, abandon their organs of flight and become as completely terrestrial as the workers. During marriage flights female ants are therefore usually observed to return in great numbers to the ground at no great distance from their parental nests.

The height to which ants are able to ascend on their nuptial flights will be ascertained only when some of our young myrmecologists become aviators. We know that winged ants are often carried to high mountain peaks. Forel (1874) records the occurrence of males and females of *Formica rufa* and *pratensis* on the perpetual snows of Alpine glaciers, and Mrs. Slosson sent me several male and female ants of different genera from the summit of Mt. Washington, N. H. (Wheeler 1905). I have myself taken similar specimens on the summits of other peaks in the White Mountains. But, as Forel has shown, female ants never succeed in establishing colonies at these altitudes. They are merely transported to the summits by the air-currents which are known to ascend mountain slopes during the day-time and to carry up great numbers of insects of all orders. Unless, therefore, such females were able to descend on the opposite slopes,— and this is probably of very rare occurrence — high mountain ranges must constitute barriers as effective as are considerable bodies of water or deserts to the distribution of most ants. I am convinced that the Sierra Nevada in California is such a barrier to many forms common on the Pacific Coast and in Europe the Alps certainly act as a similar barrier to many species common in Italy and Central Europe.

For the purpose of bringing before the reader as clearly as possible the results of my study of the ants of the Transition and Canadian Zones, I have cited the various species, subspecies and varieties from the Coast Range of California, the Sierra-Cascade Ranges, the Rocky Mountains and the portion of North America east of these ranges in four columns in the accompanying Tables I to IX (pp. 464 to 481). As might be expected, the great ranges of the Rocky Mountains, from British America to Mexico, show the greatest number and diversity of forms. The Eastern portion of North America has, with the exception of a certain number of holarctic and neotropical species, a fauna peculiar to itself, and the Sierras and Californian Coast each possesses peculiar elements, though also possessing many forms in common with the Rocky Mountains. One is struck in the tables by the meagerness of the two Californian mountain faunas. This might be attributed to their much smaller territory, but such can hardly be a complete explanation, for ant-colonies in California, even those of the more dominant species, are much less numerous than they are in the Rocky Mountains and Eastern States. I believe that the difference is due to the peculiar annual distribution of temperature and moisture in California. The ants of the Transition and Boreal Zones require a considerable amount of humidity and warmth during their breeding season. These conditions are not realized simultaneously in California, where the rainy season comes during the winter and the summer is rainless except in the high Sierras. The more perfect adaptation of the species of the Sonoran zones to a smaller amount of moisture and to winter temperatures not sufficiently low to inhibit completely the activities of the worker ants, probably accounts for the greater number of species and colonies at lower altitudes in Southern California, where the conditions are much like those of Arizona. Even moderately low temperatures, when coupled with considerable humidity, a condition which prevails in California during the winter months, is very unfavorable to ants, and when such conditions are most accentuated, the ant-fauna is reduced to a mere remnant, although the vegetation, if the temperature is not too low, may be luxuriant. This is the case in New Zealand where I sometimes searched in vain for an ant-colony in forests whose luxuriance rivalled those of the tropics. But we have a striking example of the depressing effects of cold and moisture on ant-life much nearer home. The cool Selkirk Mts. of British Columbia have an abundant supply of moisture and an unusually rich flora, but their ant-fauna is reduced to a few boreal species. The adjacent Canadian Rockies, however, though in the same latitude, are less humid and have a poorer flora, but their ant-fauna is decidedly richer in species and colonies.

In mountain regions slope exposure in its relation to insolation is a very important factor in the local distribution of ants, but it is impossible at present to give more than a general statement in regard to this matter. Northern slopes in the northern hemisphere are usually, for very obvious reasons, almost or quite destitute of ants. In regard to the other slopes my observations in the Alps of Switzerland and the mountains of the United States, British America, Mexico and Central America confirm those of Forel in the Alps and the mountains of North Carolina. He finds that ants prefer the eastern and southern slopes as these are the situations in which they have the longest day for their activities during the breeding season, since they are early awakened by a sufficiently high temperature of the soil and air from the lethargy induced by the chill night hours, and even though the slope may be in shade during the afternoon the warmth is sufficient to sustain their activities till sun-set. On western slopes, however, the morning hours are too cool and are therefore practically lost to the ants, whereas the afternoon hours are too warm.

This preference of our northern ants for eastern and southern slopes is further confirmed by the shape of the nest and the position of the nest-entrance of certain species. This matter was considered in my ant-book (1910, p. 205) in the following passage: "I have already called attention to the constant position of the nest opening at the base of the southern or eastern slope of the mounds of *Pogonomyrmex occidentalis*. Huber says that the yellow ants (*Lasius flavus*) of Switzerland "serve as compasses to the mountaineers when they are enveloped in dense fogs or have lost their way at night; for the reason that the nests, which in the mountains are much more numerous and higher than elsewhere, take on an elongated, almost regular form. Their direction is constantly from east to west. Their summits and more precipitous slopes are turned towards the winter sunrise, their longer slopes in the opposite direction." These remarks of Huber have been recently confirmed by Tissot (Wasmann 1907) and Linder (1908). The latter has shown that the elongate shape of the mounds is due to the fact that the ants keep extending them in an easterly direction in such a manner that only the extreme easterly, highest and most precipitous portions are inhabited by the insects. I have observed a similar and equally striking orientation of the mounds of *Formica argentata* [*fusca* var. *argentea*] in the subalpine meadows of Colorado." In the southern hemisphere, as we should expect, the ants prefer the northern and eastern slopes of the mountains. I found many striking instances of this preference while collecting in the mountains of New Zealand, New South Wales and Queensland.

Merriam and his collaborators in their studies of the floras and faunas of the mountains of western North America have published interesting observations which deserve consideration since they have a bearing on the distribution of the Formicidae though they show that these insects would hardly suffice to determine the boundaries of the various life-zones on mountain slopes. In his work on Mt. Shasta, Merrian (1899) says: "The influence of slope exposure on the faunas and floras of mountain regions is profound. Measured by a scale of altitudes it amounts on ordinary slopes to nearly a thousand feet and on steep slopes is still more marked. Thus on mountains it is usual for plants and animals of particular species to occur on warm southwesterly slopes at elevations 800 to 1000 feet higher than on cool northeasterly slopes — similarly on north and south ridges, the fauna and floras of the warm west slopes often belong to lower zones than those of equal elevations on the cool east slopes." Merriam had previously shown the existence of very similar conditions in a very different region, the San Francisco Mountains of Arizona (1890). There he found the normal difference in altitude of the same zone on the southwest and northeast slopes to be about 900 feet. After giving numerous examples of this altitudinal distribution on Mt. Shasta, he calls attention to other factors, besides those of insolation, which influence the range of plants and animals: "It is well-known that in ordinary calm weather the air-currents on mountain sides and in deep canyons ascend by day and descend by night. The ascending currents are warm, the descending currents cold. The night current, being in the main free from local influences that affect its temperature, must exert an essentially equal affect on all sides of a mountain; but the temperature of the ascending day current, being constantly exposed to and in fact created by the influence of the sun, must vary enormously on different slopes. The activity and effectiveness of this current increase with the steepness of the slope and the directness of its exposure to the afternoon sun. Hence the hottest normal slopes—those that face the sun at nearly a right angle during the hottest part of the day — are rendered still more potent by increased steepness, the direct exposure of the sun keeping up the supply of heat while the steepness of the slope accelerates the rate of movement of the diurnal ascending current, carrying the heated air upward a very great distance before it has time to be cooled to the general temperature of the stratum it penetrates. Thus it is that species characteristic of the Transition zone on Shasta — species which on normal southwesternly slopes attain their upper limits at an altitude of 5500 to 5700 feet — are in favorable places enabled to live at elevations of 7900 or even 8000 feet, considerably more than 2000 feet above their normal limits."

Every observer in the field must have been impressed with the fact that steepness of slope is an important factor in the local distribution of mountain ants. These insects always greatly prefer the more gradual slopes and alpine meadows, probably because the soil of such places retains a more abundant and more equable supply of moisture and because their surfaces are much less exposed to rapid evaporation both from direct insolation and from air-currents. All of these ecological factors demand much more careful study.

It is, of course, well known that the delimitation of the various life-zones in mountain regions depends not only on slope-exposure but also on latitude. That the upper limit of the zones descends in more northern and ascends in more southern latitudes even within the confines of a single one of our western states is well shown in the following table from Cary's "Biological Survey of Colorado" (1911):

| Zone | North Colorado | | Southern Colorado | |
|---|---|---|---|---|
| | Northeast exposure | Southwest exposure | Northeast exposure | Southwest exposure |
| | Feet | Feet | Feet | |
| Upper Soronan | ——— to 5600 | ——— to 6500 | ——— to 6500 | ——— to 7800 |
| Transition | 5600 to 7500 | 6500 to 8200 | 6500 to 8000 | 7800 to 9000 |
| Canadian | 7500 to 10000 | 8200 to 10400 | 8000 to 10500 | 9000 to 11000 |
| Hudsonian | 10000 to 10900 | 10400 to 11600 | 10500 to 11200 | 11000 to 12000 |
| Arctic-Alpine | 10900 to ——— | 11600 to ——— | 11200 to ——— | 12000 to ——— |

In his "Life Zones and Crop Zones of New Mexico, Bailey (1913) gives the upper boundary of the Upper Sonoran as 5000–7000 or even 8000 ft., the boundaries of the Transition as extending from 7000 to 8500 ft. on northeastern and 8000 to 9500 on southwestern slopes, of the Canadian as from 8500 to 11,000 and on warm slopes from 9500 to 12,000, of the Hudsonian from 11,000 to 12,000 on northeastern and 12,000 to 13,000 on southwestern slopes, the Arctic-Alpine on the Sangre de Cristo Range as all above 12,000 ft. on the coldest slopes, and on especially steep slopes as all above 11,500 ft.; on the warmest slopes as all above 13,000 ft. or on very gradual slopes all above 12,500 ft. In Arizona the boundaries of the life-zones ascend somewhat higher, as indicated by the following altitudes from Merriam's work (1890) on the San Francisco Mountains (southwest slopes): Lower Sonoran 4000–6000 ft., Upper Sonoran 6000–7000 ft., Transition 7000–8200 ft., Canadian 8200–9200 ft., Hudsonian 9200–10,500 ft., Arctic-Alpine 10,500–11,500 ft. In the Chisos, Davis and Guadeloupe Mountains of Western Texas, according to Bailey (1905) the Transition Zone extends from about 6000 ft. on northeast slopes to the top of the ranges (8000–9500 ft.). In Mexico the upper boundary of the Transition must be even higher. North of Colorado the zonal boundaries descend rapidly till in the latitude of Vancouver and Maine the Canadian zone, which extends across the continent, is at sea-level, so that we find at this level such forms as *Camponotus whymperi*, *modoc*, and *laevigatus*, the two latter of which do not descend below 4000 to 6000 ft. in the Sierras, while *whymperi* and *laevigatus* are not known from elevations under 7000 to 8000 ft. in Colorado. On the other

TABLE

| A.<br>Pacific Coast<br>Transition | B.<br>Sierra-Cascade<br>Transition and Boreal |
|---|---|
| **Monomorium** | **Monomorium** |
|  | minimum |
| minimum subsp. *ergatogyna* |  |
| **Solenopsis** | **Solenopsis** |
| molesta var. validiuscula |  |
| **Myrmecina** | **Myrmecina** |
| **Myrmica** | **Myrmica** |
|  | brevinodis var. sulcinodoides |
|  | var. subalpina |
|  | scabrinodis subsp. schencki var. *tahoēnsis* |
|  | *bradleyi* |

I.

| C. Rocky Mountain Transition and Boreal | D. Eastern Transition and Boreal |
|---|---|
| **Monomorium** | **Monomorium** |
| minimum | minimum |
| subsp. *compressum* | |
| subsp. *cyaneum* | |
| **Solenopsis** | **Solenopsis** |
| molesta | molesta |
| var. validiuscula | |
| var. *castanea* | |
| **Myrmecina** | **Myrmecina** |
| graminicola subsp. americana | graminicola subsp. americana |
| var. brevispinosa | var. brevispinosa |
| subsp. *texana* | |
| **Myrmica** | **Myrmica** |
| brevinodis | |
| var. sulcinodoides | brevinodis var. *canadensis* |
| var. *decedens* | |
| var. *brevispinosa* | |
| var. subalpina | |
| var. *frigida* | |
| scabrinodis subsp. *lobicornis* var. *glacialis* | scabrinodes var. *sabuleti* |
| | var. *fracticornis* |
| | var. *detritinodis* |
| scabrinodis subsp. schencki var. *monticola* | scabrinodis subsp. schencki var. *emeryana* |
| *mexicana* | *punctiventris* |
| | var. *pinetorum* |
| *mutica* | *rubra* subsp. *laevinodis* var. *bruesi* |
| *aldrichi* | subsp. *neolaevinodis* |
| *hunteri* | subsp. *champlaini* |

TABLE

| A.<br>Pacific Coast<br>Transition | B.<br>Sierra-Cascade<br>Transition and Boreal |
|---|---|
| **Leptothorax** | **Leptothorax** |
| *andrei* | |
| *eldoradensis* | |
| nitens var. *heathi* | nitens |
| var. *mariposa* | |
| var. *occidentalis* | |
| | *nevadensis* |
| | subsp. *rudis* |
| | rugatulus var. *mediorufus* |
| | acervorum subsp. canadensis var. *calderoni* |
| **Symmyrmica** | **Symmyrmica** |
| **Harpagoxenus** | **Harpagoxenus** |

II.

| C.<br>Rocky Mountain<br>Transition and Boreal | D.<br>Eastern<br>Transition and Boreal |
|---|---|
| **Leptothorax** | **Leptothorax** |
| *schmitti* | |
| *neomexicanus* | |
| nitens | |
| *tricarinatus* | |
| *melanderi* | |
| *furunculus* | *longispinosus* |
| *mexicanus* | *fortinodis* |
| *terrigena* | *schaumi* |
| *texanus* | texanus var. *davisi* |
| *obturator* | |
| *rugatulus* | *curvispinosus* |
| var. *cockerelli* | subsp. *ambiguus* |
| subsp. *annectens* | |
| subsp. *brunnescens* | |
| muscorum var. *sordidus* | |
| var. *septentrionalis* | |
| acervorum subsp. canadensis | acervorum subsp. canadensis |
| var. convivialis | var. convivialis |
| var. *yankee* | |
| subsp. *crassipilis* | |
| emersoni subsp. *glacialis* | *emersoni* |
| subsp. *hirtipilis* | |
| hirticornis subsp. *formidolosus* | *hirticornis* |
| *provancheri* | |
| **Symmyrmica** | **Symmyrmica** |
| *chamberlini* | |
| **Harpagoxenus** | **Harpagoxenus** |
| | *americanus* |

TABLE

| A.<br>Pacific Coast<br>Transition | B.<br>Sierra-Cascade<br>Transition and Boreal |
|---|---|
| **Stenamma** | **Stenamma** |
| brevicorne subsp. *heathi* | |
| subsp. *sequoiarum* | |
| *nearcticum* | |
| **Aphaenogaster** | **Aphaenogaster** |
| subterranea subsp. occidentalis | subterranea subsp. occidentalis |
| | subsp. valida var. *manni* |
| *patruelis* | |
| var. *bakeri* | |
| var. *carbonaria* | |
| mutica | |
| **Liometopum** | **Liometopum** |
| *occidentale* | apiculatum |
| apiculatum subsp. luctuosum | |

III.

| C.<br>Rocky Mountain<br>Transition and Boreal | D.<br>Eastern<br>Transition and Boreal |
|---|---|
| **Stenamma** | **Stenamma** |
|  | *brevicorne* |
| brevicorne subsp. diecki | subsp. diecki |
|  | var. *impressum* |
|  | subsp. *impar* |
|  | subsp. *schmitti* |
| *manni* |  |
| **Aphaenogaster** | **Aphaenogaster** |
| subterranea subsp. occidentalis | *fulva* |
| subsp. *valida* | subsp. *aquia* |
| subsp. *borealis* | var. *picea* |
| fulva subsp. aquia var. rudis | var. rudis |
| var. *azteca* |  |
| *uinta* |  |
| *texana* |  |
| var. *furvescens* | texana var. *carolinensis* |
|  | *mariae* |
| mutica | *tennesseensis* |
|  | var. *ecalcarata* |
|  | *treatae* |
|  | subsp. *wheeleri* |
|  | *lamellidens* |
| **Liometopum** | **Liometopum** |
| apiculatum |  |
| subsp. luctuosum |  |

TABLE

| A.<br>Pacific Coast<br>Transition | B.<br>Sierra-Cascade<br>Transition and Boreal |
|---|---|
| **Tapinoma** | **Tapinoma** |
| sessile | sessile |
| **Prenolepis** | **Prenolepis** |
| imparis | imparis |
| **Lasius** | **Lasius** |
| niger var. sitkaënsis | niger var. sitkaënsis |
| var. neoniger | var. neoniger |
|  | subsp. alienus var. americanus |
|  | brevicornis subsp. *microps* |
|  | flavus subsp. claripennis |
|  | umbratus subsp. subumbratus |
|  | *humilis* |
|  | latipes |
| interjectus subsp. *californicus* |  |

IV.

| C.<br>Rocky Mountain<br>Transition and Boreal | D.<br>Eastern<br>Transition and Boreal |
|---|---|
| **Tapinoma** | **Tapinoma** |
| sessile | sessile |
| **Prenolepis** | **Prenolepis** |
| imparis | imparis |
| | var. *testacea* |
| | var. *parva* |
| **Lasius** | **Lasius** |
| niger var. sitkaënsis | niger var. sitkaënsis |
| var. neoniger | var. neoniger |
| subsp. alienus var. americanus | subsp. alienus var. americanus |
| brevicornis | brevicornis |
| flavus subsp. nearcticus | flavus subsp. nearcticus |
| subsp. claripennis | |
| umbratus subsp. subumbratus | umbratus subsp. subumbratus |
| subsp. mixtus var. aphidicola | subsp. mixtus var. aphidicola |
| subsp. *vestitus* | subsp. *speculiventris* |
| | subsp. *minutus* |
| latipes | latipes |
| *occidentalis* | |
| murphyi | murphyi |
| interjectus | interjectus |
| subsp. *coloradensis* | |
| subsp. *arizonicus* | |
| subsp. *mexicanus* | |
| claviger | claviger |
| | subsp. *subglaber* |

TABLE

| A. Pacific Coast Transition | B. Sierra-Cascade Transition and Boreal |
|---|---|
| **Formica** | **Formica** |
| sanguinea subsp. subnuda | sanguinea subsp. subnuda |
| | manni |
| | rufa subsp. obscuripes |
| | var. melanotica |
| truncicola subsp. integroides | truncicola subsp. integroides |
| var. *subfasciata* | var. *tahoënsis* |
| | var. *propinqua* |
| | var. haemorrhoidalis |
| subsp. integra var. subcaviceps | subsp. integra var. subcaviceps |
| | oreas var. comptula |

V.

| C.<br>Rocky Mountain<br>Transition and Boreal | D.<br>Eastern<br>Transition and Boreal |
|---|---|
| **Formica** | **Formica** |
| sanguinea subsp. subnuda | sanguinea subsp. subnuda |
| subsp. puberula | subsp. puberula |
| | subsp. *subintegra* |
| | var. *gilvescens* |
| subsp. *obtusopilosa* | subsp. *rubicunda* |
| | var. *sublucida* |
| *munda* | subsp. *aserva* |
| var. *alticola* | *pergandei* |
| manni | |
| *perpilosa* | |
| *bradleyi* | |
| rufa subsp. obscuripes | |
| var. melanotica | rufa subsp. obscuripes var. melanotica |
| truncicola subsp. integroides var. *coloradensis* | |
| var. *ravida* | |
| var. haemorrhoidalis | |
| subsp. *mucescens* | |
| subsp. obscuriventris var. *aggerans* | truncicola subsp. *obscuriventris* |
| | var. *gymnomma* |
| subsp. integra | subsp. integra |
| *foreliana* | *ferocula* |
| *ciliata* | |
| *comata* | |
| *criniventris* | |
| *oreas* | |
| var. comptula | |

TABLE

| A.<br>Pacific Coast<br>Transition | B.<br>Sierra-Cascade<br>Transition and Boreal |
|---|---|
| **Formica** | **Formica** |
| | microgyna subsp. rasilis |
| | var. *pinetorum* |
| | subsp. *californica* |
| | var. *hybrida* |
| | *nevadensis* |
| fusca | fusca |
| | var. marcida |
| | var. subaenescens |
| var. argentea | var. argentea |
| var. neorufibarbis | var. neorufibarbis |
| | var. *blanda* |
| | subsp. pruinosa var. *lutescens* |
| rufibarbis var. *occidua* | |

VI.

| C.<br>Rocky Mountain<br>Transition and Boreal | D.<br>Eastern<br>Transition and Boreal |
|---|---|
| **Formica** | **Formica** |
| *dakotensis* | |
| var. *montigena* | dakotensis var. *speculiventris* |
| var. *saturata* | |
| *microgyna* | microgyna subsp. *scitula* |
| var. *recidiva* | *difficilis* |
| subsp. rasilis | var. *consocians* |
| var. *spicata* | *impexa* |
| var. *pullula* | *nepticula* |
| var. *nahua* | *morsei* |
| *whymperi* | |
| var. *alpina* | whymperi var. *adamsi* |
| exsectoides var. *hesperia* | *exsectoides* |
| subsp. *opaciventris* | var. *davisi* |
| ulkei | ulkei |
| | var. *hebescens* |
| fusca | fusca |
| var. marcida | |
| var. subsericea | var. subsericea |
| var. subaenescens | var. subaenescens |
| var. argentea | var. argentea |
| var. *gelida* | var. *algida* |
| var. neorufibarbis | |
| var. *neoclara* | |
| subsp. *pruinosa* | |
| rufibarbis var. *gnava* | |

TABLE

| A.<br>Pacific Coast<br>Transition | B.<br>Sierra-Cascade<br>Transition and Boreal |
|---|---|
| **Formica** | **Formica** |
| cinerea var. neocinerea | |
| var. *lepida* | |
| subsp. *pilicornis* | |
| | *sibylla* |
| *subpolita* | |
| var. camponoticeps | subpolita var. camponoticeps |
| | neogagates |
| neogagates subsp. lasioides var. vetula | subsp. lasioides var. vetula |
| **Polyergus** | **Polyergus** |
| rufescens subsp. breviceps | rufescens subsp. breviceps |
| var. *umbratus* | |
| subsp. *laeviceps* | |

VII.

| C.<br>Rocky Mountain<br>Transition and Boreal | D.<br>Eastern<br>Transition and Boreal |
|---|---|
| **Formica** | **Formica** |
| cinerea var. neocinerea | cinerea var. neocinerea |
| var. *altipetens* | var. *rutilans* |
| var. *canadensis* | |
| *montana* | |
| *hewitti* | |
| *subcyanea* | |
| subpolita var. *ficticia* | |
| neogagates | neogagates |
| var. *morbida* | var. *vinculans* |
| subsp. lasioides | subsp. lasioides |
| var. vetula | var. vetula |
| *limata* | |
| | *pallidefulva* |
| | var. *succinea* |
| | subsp. *schaufussi* |
| | var. *dolosa* |
| pallidefulva subsp. schaufussi var. incerta | var. incerta |
| subsp. nitidiventris | subsp. nitidiventris |
| var. fuscata | var. fuscata |
| *moki* | |
| **Polyergus** | **Polyergus** |
| rufescens subsp. breviceps | rufescens subsp. breviceps |
| var. *montezuma* | |
| var. *fusciventris* | |
| subsp. *mexicanus* | |
| subsp. bicolor | subsp. bicolor |
| lucidus subsp. *montivagus* | *lucidus* |

TABLE

| A. Pacific Coast Transition | B. Sierra-Cascade Transition and Boreal |
|---|---|
| **Camponotus** | **Camponotus** |
| laevigatus | laevigatus |
| | herculeanus var. modoc |
| | subsp. ligniperda var. noveboracensis |
| *hyatti* | |
| var. *bakeri* | |
| *anthrax* | |
| fallax var. nearcticus | fallax var. nearcticus |
| | var. minutus |
| subsp. subbarbatus | |
| subsp. discolor var. clarithorax | subsp. discolor var. clarithorax |

VIII.

| C.<br>Rocky Mountain<br>Transition and Boreal | D.<br>Eastern<br>Transition and Boreal |
|---|---|
| **Camponotus** | **Camponotus** |
| laevigatus | |
| herculeanus, var. modoc | |
| var. whymperi | herculeanus var. whymperi |
| | subsp. *pennsylvanicus* |
| | var. *ferrugineus* |
| | var. *mahican* |
| subsp. ligniperda var. noveboracensis | subsp. ligniperda var. noveboracensis |
| | var. *rubens* |
| *schaefferi* | *castaneus* |
| *texanus* | subsp. *americanus* |
| *sayi* | |
| fallax var. nearcticus | fallax var. nearcticus |
| var. minutus | var. minutus |
| var. decipiens | var. decipiens |
| | var. *tanquaryi* |
| | var. *pardus* |
| subsp. *rasilis* | |
| var. *pavidus* | |
| | subsp. subbarbatus |
| | var. *paucipilis* |
| subsp. discolor | subsp. discolor |
| var. clarithorax | var. clarithorax |
| | var. *cnemidatus* |

TABLE

| A.<br>Pacific Coast<br>Transition | B.<br>Sierra-Cascade<br>Transition and Boreal |
|---|---|
| **Camponotus** | **Camponotus** |
| maculatus subsp. vicinus | maculatus subsp. vicinus |
| var. plorabilis | |
| var. luteangulus | |
| var. semitestaceus | var. semitestaceus |
| var. nitidiventris | |
| var. *maritimus* | |
| var. infernalis | var. infernalis |
| subsp. *dumetorum* | |
| | |
| | |
| | |
| subsp. *maccooki* | |
| fumidus var. *fragilis* | |
| | |
| | |
| | |
| | |
| *ocreatus* | |
| *mina* | |
| | |
| *yogi* | |

IX.

| C. Rocky Mountain Transition and Boreal | D. Eastern Transition and Boreal |
|---|---|
| **Camponotus** | **Camponotus** |
| maculatus subsp. vicinus | |
| var. plorabilis | |
| var. luteangulus | |
| var. nitidiventris | |
| var. infernalis | |
| subsp. *sansabeanus* | |
| var. *torrefactus* | |
| subsp. *bulimosus* | |
| fumidus var. *festinatus* | |
| var. *spurcus* | |
| *vafer* | |
| *acutirostris* | |
| var. *clarigaster* | |
| ocreatus subsp. *primipilaris* | |
| mina subsp. *zuni* | |
| *bruesi* | |
| *ulcerosus* | |
| *pylartes* | *impressus* |
| var. *hunteri* | |
| abditus var. *etiolatus* | |

hand, in the Huachuca Mountains of Arizona such neotropical forms as species of *Eciton, Odontomachus, Pheidole,* etc. are abundant at altitudes of 4000 to 6000 ft.[2]

Before considering the historical problems suggested by the ant-faunas of the four regions of the tables, it will be advisable to analyze them more closely. The total number of forms recorded for all the regions is 422 distributed as follows:

| A | B | C | D |
|---|---|---|---|
| 58 or 13.7% | 60 or 14.2% | 180 or 42.6% | 124 or 29.4% |

In reality the total number of *different* forms is only 311. If we count only the forms peculiar, endemic, or precinctive to each region (printed in italics in the tables) we have the following:

|  | A | B | C | D | Totals |
|---|---|---|---|---|---|
| Species | 11 | 5 | 47 | 25 | 88 |
| Subspecies | 7 | 3 | 27 | 18 | 55 |
| Varieties | 11 | 10 | 38 | 32 | 91 |
|  | 29 | 18 | 112 | 75 | 234 |

There are therefore 77 forms common to two or more of the regions. This would yield the following percentages for the given groups:

| A | B | C | D | Common |
|---|---|---|---|---|
| 9.3% | 5.8% | 36.0% | 24.1% | 24.8% |

The total number of endemic forms in the western fauna (A + B + C) is 159 or 68%, whereas the 75 eastern forms represent only 32%. The number of forms in common gives a good index of the affinities of the different regions, and may be tabulated as follows:

---

[2] In his interesting paper on the insects of Custer County, Colorado, Cockerell (1893) does not accept Merriam's terminology for the life-zones of that region. He distinguishes three zones, a "subalpine," up to about 6500 ft., a "mid-alpine" between 6500 and 10,000 ft. and a "high-alpine" zone above the latter elevation, and correlates these with Merriam's zones in the statement that "an analysis of the insects of the Colorado Mountains shows that the high-alpine and mid-alpine elements, though sufficiently distinct, are both essentially boreal. If we follow Dr. Merriam's arrangement, it appears that the high-alpine is truly boreal, while the mid-alpine belongs to the transition region, containing a considerable number of strictly American types. The subalpine, on the other hand is southern or Sonoran."

| Faunas of the Four Regions | Species in common | Subspecies in common | Varieties in common | Total number of forms in common | Percentages |
|---|---|---|---|---|---|
| A + B | 4 | 6 | 12 | 22 | 28.7% |
| A + B + C | 4 | 5 | 8 | 17 | 22.1% |
| A + B + C + D | 3 | 2 | 6 | 11 | 14.3% |
| A + C | 5 | 5 | 14 | 24 | 31.2% |
| B + C | 8 | 9 | 19 | 36 | 46.7% |
| B + D | 6 | 4 | 11 | 21 | 27.3% |
| C + D | 12 | 14 | 21 | 47 | 61.0% |
| A + D | 3 | 2 | 7 | 12 | 15.6% |
| B + C + D | 6 | 3 | 11 | 20 | 26.0% |
| A + C + D | 3 | 2 | 7 | 12 | 15.6% |
| A + B + D | 3 | 2 | 6 | 11 | 14.3% |

As would be expected, the affinities between the Pacific Coast and Eastern faunas are least developed, while those between the Sierra-Cascade and Rocky Mts. and especially those between the latter and the Eastern fauna are much greater.

If to the Transition and Boreal forms included in the tables we add the species, subspecies and varieties of the Lower and Upper Sonoran and Lower and Upper Austral Zones, the quantitative and qualitative differences between the western and eastern ant-faunas are even more striking. The same would be true of a comparison of the subgenera and genera of the two regions, for we find that, if we exclude the neotropical elements, no less than eight genera and subgenera are restricted to the western fauna (*Liometopum*, *Messor*, *Deromyrma*, *Neomyrma, Symmyrmica* and *Myrmecocystus*) and seven are peculiarly eastern (*Bothriomyrmex, Hypoclinea, Strumigenys, Epoecus, Dichothorax, Harpagoxenus* and *Brachymyrmex*). Moreover certain genera are almost exclusively western or eastern. *Pogonomyrmex* e. g., represented by numerous species in the Southwestern States has only one species (*P. badius*) in the Southeastern States, and *Proceratium* and *Sysphincta* are at any rate very largely confined to the East.

The great majority of the forms recorded in the tables are undoubtedly peculiar to the Transition Zone. Only the following would seem to belong to the Canadian, or boreal fauna:

*Myrmica brevinodis* vars. *sulcinodoides, canadensis,* and *frigida,*
*M. scabrinodis* subsp. *lobicornis* var. *glacialis,*
*Leptothorax acervorum* subsp. *canadensis* and its vars. and the subsp. *crassipilis,*
*L. muscorum* and its vars.,
*L. provancheri,*
*L. emersoni* and its subspecies,
*Stenamma nearcticum,*
*S. brevicorne,* its subspecies and varieties,
*Lasius niger* var. *sitkaënsis,*
*L. flavus* subsp. *claripennis,*
*L. umbratus* subsp. *subumbratus,*
*Formica bradleyi,*
*F. sanguinea* and subsp. *subnuda* and *aserva,*
*F. rufa obscuripes* and its var. *melanotica,*
*F. truncicola* and its subspecies and varieties,
*F. whymperi* and its varieties,
*F. dakotensis* and its varieties,
*F. ulkei,*
*F. fusca* and its varieties *neorufibarbis, marcida, subaenescens, argentea, gelida* and *algida* and the subsp. *pruinosa,*
*F. hewitti,*
*F. cinerea* var. *altipetens* and *canadensis,*
*F. neogagates,* its subspecies and var. *vetula,*
*Camponotus laevigatus,*
*C. herculeanus* var. *whymperi* and subsp. *ligniperda* var. *noveboracensis.*

Most of these are what the Germans would call "stenotherm kälteliebend" (stenothermal psychrophilous). Some of them, however, and especially those common to the four regions of the tables, are strikingly eurythermal ("eurythermal ubiquists" of Zschokke, 1907, 1908). A list of the latter would include the following:

*Myrmica scabrinodis* and most of its subspecies and varieties,
*Aphaenogaster subterranea* subsp. *occidentalis,*
*Tapinoma sessile,*
*Prenolepis imparis,*
*Lasius niger* subsp. *alienus* var. *americanus,*
*L. brevicornis,*
*L. flavus* subsp. *nearcticus,*
*Formica sanguinea* subsp. *rubicunda, subintegra* and *subnuda,*
*F. fusca* and its varieties *subsericea* and *argentea,*
*F. cinerea* var. *neocinerea,*
*F. neogagates* subsp. *lasioides* var. *vetula,*
*Polyergus rufescens* subsp. *breviceps,*
*Camponotus herculeanus* subsp. *pennsylvanicus,*
*C. fallax* var. *nearcticus,*
*C. maculatus* subsp. *vicinus* and its varieties.

It will be noticed that the bulk of the forms common to all four regions of the tables is made up of some eight of the forms included in this list. The ants of both the preceding lists, owing to their pronounced eurythermy or psychrophilous stenothermy, constitute the great majority of the forms common at higher elevations in the mountains of North America. Incidentally attention may be called to the high degree of melanism of nearly all the forms enumerated in these lists. This is a well-known peculiarity of many arctic-alpine insects (*Cf.* Zschokke, 1908, p. 42).

The ant-fauna of the Nearctic Transition and Boreal Zones as a whole shows very close affinities to the fauna of the corresponding zones of the Palearctic Region, as will be evident from a study of the following list in which the most closely allied forms of the two regions are arranged in parallel columns:—

| *Palearctic* | *Nearctic* |
|---|---|
| Ponera coarctata | P. coarctata subsp. pennsylvanica |
| Monomorium minutum | M. minimum |
| Solenopsis fugax | S. molesta |
| Myrmica sulcinodis | M. brevinodis |
| M. scabrinodis var. sabuleti | M. scabrinodis var. sabuleti |
| M. scabrinodis subsp. lobicornis | M. scabrinodis subsp. lobicornis var. glacialis |

| *Palearctic* | *Nearctic* |
|---|---|
| M. scabrinodis subsp. schencki | M. scabrinodis subsp. schencki var. emeryana. |
| M. levinodis | M. levinodis subsp. neolevinodis |
| M. rubida | M. mutica |
| Leptothorax acervorum | L. acervorum subsp. canadensis |
| L. muscorum | L. muscorum var. sordidus |
| L. flavicornis | L. curvispinosus and rugatulus |
| Harpagoxenus sublevis | Harpagoxenus americanus |
| Formicoxenus nitidulus | Symmyrmica chamberlini |
| Stenamma westwoodi | Stenamma nearcticum |
| Aphaenogaster subterranea | A. subterranea subsp. occidentalis |
| Tapinoma erraticum | T. sessile |
| Bothriomyrmex meridionalis | B. dimmocki |
| Liometopum microcephalum | L. occidentale |
| Hypoclinea 4-punctata | H. plagiata |
| Prenolepis imparis subsp. nitens | P. imparis |
| Lasius niger | L. niger var. sitkaënsis and neoniger |
| L. niger subsp. alienus | L. niger subsp. alienus var. americanus |
| L. flavus | L. flavus subsp. nearcticus |
| L. umbratus subsp. mixtus | L. umbratus subsp. mixtus var. aphidicola |
| Formica sanguinea | F. sanguinea subsp. rubicunda |
| F. rufa subsp. pratensis | F. rufa subsp. obscuripes |
| F. truncicola | F. truncicola subsp. integroides |
| F. exsecta | F. exsectoides |
| F. fusca | F. fusca |
| F. cinerea | F. cinerea var. neocinerea |
| F. rufibarbis | F. rufibarbis var. occidua |
| Polyergus rufescens | P. rufescens subsp. breviceps |
| Camponotus herculeanus var. whymperi | C. herculeanus var. whymperi |
| C. herculeanus subsp. ligniperda | C. herculeanus subsp. ligniperda var. noveboracensis |
| C. herculeanus subsp. pennsylvanicus | C. herculeanus subsp. pennsylvanicus |
| C. fallax | C. fallax var. nearcticus |
| C. maculatus subsp. aethiops | C. maculatus subsp. vicinus |
| C. truncatus | C. impressus |

In this list of 41 Nearctic forms 25 are specifically, 6 subspecifically and two varietally identical with Palearctic forms.

The results of the foregoing study of the Transition and Boreal ant-fauna agree in the main with those derived from other animals and of plants, and suggest the same problems as to the original source of the North American ant-fauna, the meaning of the differences between its western and eastern constituents and of the much greater richness of the former in species, subspecies and varieties. An intensive study of the geographical distribution of any circumscribed group of organisms necessarily involves an appeal to general historical considerations, since no group can be satisfactorily studied as an isolated unit. One is compelled, therefore, to assume an attitude towards certain hypotheses which have been gradually elaborated and are more or less firmly supported by the researches of many workers on many different groups. In assuming such an attitude one is inevitably more or less biased by the particular group or groups with which one is most familiar. Before considering the hypothetical centers of origin and the migrations of the various existing categories of insects and especially of the ants, it seems advisable to determine, if possible, the geological age of these categories. This has been attempted in three different ways: first, by a study of paleontology, second, by a study of present distribution on the supposition that forms with a wide and especially with a wide and discontinuous range are older than forms with a limited, continuous range, and third, by a combination of both of these methods. It is evident that the first method is of great importance, the second by itself of comparatively little value and open to many objections, and that the value of the third method depends largely on the paleontological facts to which it may be able to appeal. It is, however, the most comprehensive method and owing to the incompleteness of the paleontological record, the only one that can be resorted to in the study of many groups of organisms at the present time.

Our knowledge of fossil ants is rather limited but of great significance. The earliest known species are those of the Baltic amber, of Lower Oligocene age. Mayr (1868) and I (1914) have described nearly a hundred of these belonging to no less than 43 genera, 19 of which are extinct and 24 still extant, viz: *Ectatomma* (subgen. *Rhytidoponera*), *Euponera* (subgen. *Trachymesopus*), *Platythyrea*, *Ponera*, *Sima*, *Monomorium*, *Erebomyrma*, *Vollenhovia*, *Stenamma*, *Aphaenogaster*, *Myrmica*, *Leptothorax*, *Dolichoderus* (subgen. *Hypoclinea*), *Iridomyrmex*, *Liometopum*, *Plagiolepis*, *Gesomyrmex*, *Dimorphomyrmex*, *Oecophylla*, *Prenolepis*, *Lasius*, *Formica*, *Pseudolasius* and *Camponotus*.

From the Scilian amber, which is of later, Miocene age, Emery (1891) has described 14 ants representing 13 genera, 11 of which are still extant, viz: *Ectatomma, Ponera, Cataulacus, Podomyrma* (subgen. *Acrostigma*), *Aëromyrma, Crematogaster, Tapinoma, Technomyrmex, Plagiolepis, Gesomyrmex* and *Oecophylla*. Heer (1848, 1849) had previously described a number of ants from the Miocene shales of Oeningen and Radoboj, but owing to the imperfectly developed taxonomic categories of his day, referred them to such generalized genera as *Formica, Ponera* and *Myrmica*. Mayr (1867), however, examined many of Heer's types from Radoboj and was able to recognize among them representatives of the following modern or extant genera: *Aphaenogaster, Leptothorax, Liometopum, Dolichoderus* (subgen. *Hypoclinea*), *Lasius, Formica, Oecophylla* and *Camponotus*. Among several thousand ants from Florissant, Colorado, also of Miocene age, I am now able to recognize specimens belonging to the recent genera *Pheidole, Crematogaster, Aphaenogaster, Liometopum, Dolichoderus* (subgen. *Hypoclinea*), *Lasius, Formica* and *Camponotus*, in addition to a few extinct genera (e. g. *Agroecomyrmex*). It is evident, therefore, that a large number of important ant genera of the present had been developed by early Tertiary times, and as the species representing these genera are quite as highly specialized as their existing congeners, I believe that we must assume that their genera go back at least to the Basic Eocene or even to the Upper Cretaceous. And since these genera clearly represent four of the five subfamilies of living ants, and among them the most highly specialized subfamily, the Camponotinae, we are justified in assuming that the subfamilies of the Formicidae were differentiated during the Mesozoic, probably as early as the Jurassic or Triassic. This assumption is in general accord with the opinions of Emery (1893) and Handlirsch (1913). According to the latter "we know today that by the end of the Cretaceous all the main groups of insects had been completed, that the species living today arose not later than the Pleistocene and the majority of them in the Pleiocene and in certain cases go back even to the Oligocene. The present genera were certainly nearly all completed in the late Tertiary, many of them already in the Oligocene and perhaps some of them in the Upper Cretaceous." He believes (1909) that the Formicidae as a family could scarcely have originated before the Upper Cretaceous. I am inclined to believe that these estimates, at least as far as the ants are concerned, are too conservative. If I understand Emery correctly, his estimates are somewhat closer to my own, for he is inclined to assign the genera of the oldest subfamily, the Ponerinae, and several Myrmicine genera to the Mesozoic, and many Dolichoderinae and Camponotinae to the early Tertiary. Kolbe (1913), however, comes still closer to my point of view in a very suggestive study of the distribution of certain ancient genera of Coleoptera, an order which can scarcely be much older than the Hymenoptera. He calls attention to the fact that if we compare the beetles of Australia and Europe we find that they possess no less than 146 genera in common, and that owing to the fact that Australia was isolated during the Eocene we are justified in regarding all such genera as of Mesozoic age. I believe that the same conclusion is admissible in the case of other insects and especially in regard to the ants and would hold good also of the genera common to Australia and America. In the following list, including all the known genera of Australian ants, the genera printed in large type are represented also in the Neotropical and Nearctic faunas and those preceded by an asterisk occurred in the Tertiary of Europe or are represented in the living fauna of that continent:

SPHINCTOMYRMEX
*CERAPACHYS
*Phyracaces
Myrmecia
Amblyopone
*PLATYTHYREA
ACANTHOPONERA
Onychomyrmex
Paranomopone
Diacamma
*ECTATOMMA
Bothroponera
Odontoponera
*EUPONERA
*PONERA
Dorylozelus
Prodiscothyrea
Prionogenys
LEPTOGENYS
*ANOCHETUS
ODONTOMACHUS
Aenictus
Metapone
*Sima
*Oligomyrmex
Pheidologeton
*CREMATOGASTER
*SOLENOPSIS
*PHEIDOLE
Lordomyrma
*Vollenhovia
*Podomyrma
*MYRMECINA
Machomyrma
Dacryon
*MONOMORIUM
*CARDIOCONDYLA
*APHAENOGASTER
*TETRAMORIUM
Pristomyrmex
Triglyphothrix
Mayriella
ROGERIA
Prodicroaspis
Promeranoplus
Meranoplus

Calyptomyrmex
*STRUMIGENYS
*EPITRITUS
Orectognathus
Epopostruma
Rhopalothrix
*DOLICHODERUS (subgen. HYPOCLINEA)
Leptomyrmex
Frogattella
Turneria
*IRIDOMYRMEX
*BOTHRIOMYRMEX
*TAPINOMA
*Technomyrmex
Acropyga
*Plagiolepis
*Acantholepis (subgen. Stigmacros)
Prolasius
Melophorus
*Pseudolasius
Notoncus
*Oecophylla
Myrmecorhynchus
*PRENOLEPIS
Opisthopsis
Echinopla
Calomyrmex
*CAMPONOTUS
Polyrhachis

Of the 75 genera in this list 31 or 41.3% are known to exist or have existed in Europe and 27 or 36% in America; 37 or 49.3% are unknown in either of these regions, but more than half of them are represented in the Oriental region. As the migration of ants from the latter region into Australia since its isolation has been very much restricted, these genera must also be regarded as of Mesozoic origin. It should also be noted that 21 or 28% of the 75 Australian genera belong to the most ancient and primitive subfamily of the Ponerinae, a group comparable to the Monotremes and Marsupials among mammals and one which reaches no such proportions in any of the other geographical regions. I believe, therefore, that we have underestimated the antiquity of the genera of ants and that the great majority of them are of Pretertiary or at the latest of early Eocene development. The same may be true even of certain species of tropicopolitan or cosmopolitan distribution, e. g. *Solenopsis geminata*, *Odontomachus haematoda* and especially *Camponotus* (*Myrmoturba*) *maculatus*, which is represented by numerous local races and varieties not only on all the continents but also on many islands (*e. g.* Hawaii!). There are good reasons for believing, however, that the great majority of existing species and subspecies are of Postmiocene origin. In North America and Eurasia, at any rate, only subspecies and varieties seem to have developed since the Ice Age. This is indicated by the very small number of varieties common to the Nearctic and Palearctic faunas as compared with the number of common species and subspecies (see pp. 485–486).

Paleographers agree in characterizing the Upper Jurassic as a period of great continental emergence, warm climate and a cosmopolitan flora. During this and the ensuing Cretaceous most of the families and genera of insects, like the flora on which so many of them were vitally dependent, must have assumed their modern facies and have become very widely distributed. According to Osborn (1910, p. 95) the "most memorable fact about the flora is one recently insisted upon by Knowlton (1909), namely that as we pass from the Cretaceous into the Eocene there is no appreciable change in the flora. From this it would appear that there was no secular change of climate; that the temperature was the same." There is nothing to indicate that the insects underwent any profounder change than the plants, so that we are unable to believe that these animals exhibited anything like the catastrophic elimination which occurred in several other groups of organisms both terrestrial and aquatic at the end of the Cretaceous. There is therefore no justification for assuming a close parallel in the course of development of such insects as the ants during the Tertiary with that of the mammals, whose phylogeny during that period was very complicated and greatly accelerated. The repeated migrations of mammals between North America and Eurasia during Cretaceous and Posteocene time were probably paralleled by the ants but we have no precise evidence of such movements. A single land-bridge, the Siberian-Alaskan, which is accepted by all students of geographical distribution, and according to most of them was in existence during the Cretaceous and again from late Miocene to Pleistocene times, is sufficient to account for the present constitution of our North American ant-fauna. Scharff (1907, 1912) and others have adduced considerable evidence in favor of another land-bridge connecting North America with Great Britain and Scandinavia during Preglacial and early Glacial time, but others reject this construction though they have not succeeded in accounting for the fauna and flora of Greenland and Iceland and the distribution of many eastern Nearctic and western European forms on any other hypothesis. That there was a gradual cooling of the climate from late Eocene to the Glacial Epoch is also generally admitted and the resulting development of pronounced zonal climates had a very powerful effect, as we know, on the fauna and flora of the northern hemisphere. The elimination of species thus induced over the area covered by the great ice-sheet both in Europe and North America and the southward migration of surviving species away from its border have been so often discussed that I need not dwell on them here. The ants of the

Baltic amber and of Florissant, like the plants of the same formations, show very clearly the gradual cooling of climate during the early and middle Tertiary. In the latitude of Sweden, where the amber was formed, the climate seems to have been subtropical as early as the Lower Oligocene, since the ants belonging to boreal genera such as *Formica, Lasius, Prenolepis s. str.* etc. constitute a dominant component of the fauna, at least in individuals. During the Miocene the climate of Colorado, as indicated by the Florissant plants, resembled that of the Gulf States at the present time. The ants perhaps indicate a slightly cooler and dryer climate, not unlike that now prevailing at low altitudes in Colorado or New Mexico.

I am inclined to believe, with Scharff, that the extent of the southward migration or displacement of organisms beyond the border of the ice sheet during glacial times has been exaggerated by many authors. Still there must have been some displacement and considerable extinction. It is at any rate clear that owing to the absence of such a complete barrier to southward migration as the Alps and the Mediterranean, our North American fauna suffered much less severely during the Ice Age than that of Europe. Moreover our fauna has been greatly enriched since the Pleistocene by a northward immigration of numerous neotropical species into the Southern United States by way of Mexico and the West Indies. The neotropical immigrants among ants belong to the Doryline genus *Eciton*, to several Ponerine genera (*Neoponera, Pseudoponera, Ectatomma, Leptogenys* and *Odontomachus*), to several Myrmicine genera (*Pseudomyrma, Cryptocerus, Macromischa, Xenomyrmex, Xiphomyrmex,* possibly *Pogonomyrmex* and especially to the fungus-growing tribe Attini (*Atta, Acromyrmex, Trachymyrmex* and *Cyphomyrmex*), to the Dolichoderine genera *Forelius, Dorymyrmex* and *Iridomyrmex* and to the Camponotine genera *Brachymyrmex, Prenolepis* (subgen. *Nylanderia*) and *Camponotus* (subgen. *Myrmothrix, Myrmobrachys* and *Myrmamblys*). Some of these genera (*Pseudoponera, Odontomachus, Leptogenys, Iridomyrmex*) are common to paleotropical regions and at once suggest the question as to whether they originally reached South America during the Cretaceous by way of Antarctica from Australia or came from Asia by way of North America or over other land-connections from other parts of the Old World, and therefore involve a discussion of the hypothetical southern land-bridges, which several recent writers, notably H. von Ihering and Scharff, have been very actively constructing in order to explain certain cases of wide and discontinuous distribution among organisms. So far as the Formicidae are concerned I unreservedly agree with those who repudiate all such connections, with the exception of the Siberian-Alaskan and possibly the North Atlantic bridges. I am quite unable to find anything in the neotropical ant-fauna that makes it necessary to assume former connections of South America with Australia or with Africa. The only genus supposed to be peculiar to Australia and South America is *Melophorus*, which is represented by numerous species in the former region and to which Forel and Emery referred a few Chilian and Patagonian species formerly regarded as belonging to the holarctic genus *Lasius*. But Emery later showed that the Chilian and Patagonian forms really constitute a distinct subgenus, which he called *Lasiophanes*. From a recent study of the Australian species I am convinced that they should be generically separated from the South American species. So far as Africa and South America are concerned, they have no genera in common which have not a much wider distribution in the Palearctic or Oriental regions.

Among the numerous writers on geographical distribution who have recently rejected or ignored the speculations of the bridge-builders, I will consider only Kolbe, Handlirsch and Matthew, as they seem to me to have reached conclusions very similar to those suggested by my study of the Formicidae. These writers recall those who, like Allen (1878), Scribner (1882) and Haacke (1887) long ago pointed to the north polar region as the original center of organic distribution, but differ in placing this center in Central or Eastern Asia. Kolbe (1913) calls attention to the vast extent and great permanence of the Asiatic continent during geologic time as contrasted with Europe and selects the region between the Caspian Sea and Eastern China, and especially Turkestan and Thibet, as the most ancient of the sources and reservoirs of Palearctic animal life. This is indicated by both the mammals and the insects. Of the single beetle genus *Carabus* Central Asia alone possesses 35 endemic subgenera! Europe is merely a zoögeographical appendix of Asia, to which the African, Australian and North American faunas are easily traceable by emigration during the Cretaceous when there existed a broad Siberian-Alaskan land-bridge.

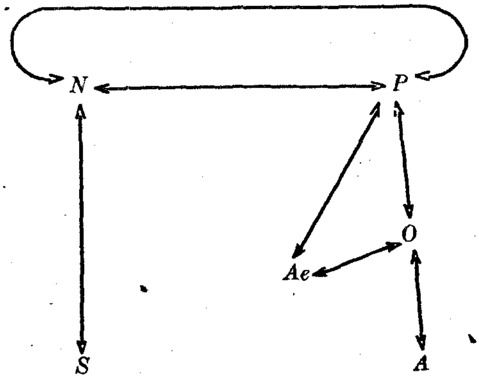
Kolbe does not discuss the origin of the neotropical fauna nor the antarctic and other land-bridges, but Handlirsch (1913) comes to close grips with these constructions in a valuable statistical study of more than 16,000 insect genera, comprising 180,000 species or about one third of the known forms. His results in regard to the distribution of the endemic as contrasted with the more widely distributed genera in the various geographical regions are given in the following interesting table:

| | Total of Genera | a. Endemic Genera | b. More widely distributed Genera | Ratio a to b |
|---|---|---|---|---|
| Neotropical | 5617 | 3437 | 2180 | 1:0.63 |
| Nearctic | 3467 | 797 | 2670 | 1:3.35 |
| Palearctic | 4956 | 1859 | 3097 | 1:1.67 |
| Ethiopian | 3968 | 2249 | 1719 | 1:0.76 |
| Oriental | 4137 | 1641 | 2496 | 1:1.52 |
| Australian | 3101 | 1400 | 1701 | 1:1.21 |

Of Wallace's regions, the Neotropical, owing to its very large proportion of endemic genera, is therefore the most independent and best established, the Nearctic the least. In an examination of 8300 selected genera, Handlirsch finds that only about 4% have a discontinuous distribution and would therefore constitute the chief basis for the contentions of the bridge-builders. But it appears that the generic relations between the Neotropical and Ethiopian regions, for which von Ihering constructed his "Archhelenis" land-bridge, are actually feebler than those between the Neotropical and Oriental, or between the Nearctic and Oriental or between the Palearctic and Australian! Furthermore, the Neotropical is really more closely related to the Palearctic than to the African insect fauna. These facts are clearly brought out in the following numbers of genera found to be common to the different regions, after excluding the cosmopolitan species:

| | |
|---|---|
| Palearctic and Nearctic | 1225 |
| Nearctic and Neotropical | 1159 |
| Palearctic and Oriental | 1083 |
| Oriental and Australian | 754 |
| Oriental and Ethiopian | 701 |
| Palearctic and Ethiopian | 687 |
| Palearctic and Neotropical | 571 |
| Palearctic and Australian | 472 |
| Nearctic and Oriental | 306 |
| Neotropical and Oriental | 259 |
| Ethiopian and Australian | 327 |
| Neotropical and Australian | 228 |
| Neotropical and Ethiopian | 195 |
| Nearctic and Australian | 197 |
| Nearctic and Ethiopian | 159 |

The study of such a circumscribed group as the Formicidae would be even more unfavorable to the views of von Ihering and other bridge-builders than the more comprehensive studies of Handlirsch, because the relationships of the ants of South America to those of Africa or Australia would be represented, as I have stated, only by genera of cosmopolitan range or at any rate common to the Palearctic and Oriental regions, from which they could have found their way to the New World over the Behring Sea and North Atlantic land-bridges. Handlirsch summarizes his views on the migrations of forms between the different regions in the following simple diagram, in which *S* stands for the Neotropical and *Ae* for the Ethiopian region:



Matthew (1915) has reached very similar conclusions from a study of the distribution of Vertebrates and particularly of the mammals. He finds the same fallacies as Handlirsch in the work of the bridge-builders and expresses them in the following paragraphs:

"1. The discontinuous distribution of modern species is again and again taken as proof that the regions now inhabited must have been connected across deep oceanic basins, without considering the possibility that it is a remnant of a wider past distribution, or that it is due to parallel evolution from a more primitive type of intermediate distribution, now extinct. Yet so many instances are known where the geological record has furnished proof that one or the other of these explanations applies to cases of discontinuous distribution, that it would seem that these ought to be the first solutions of the problem to be considered, and that in view of the known imperfection of the geologic record, mere negative evidence is not sufficient to cause them to be set aside.

"2. No account is taken of faunal interchanges often much more extensive, which would presumably have taken place if the land-bridges assumed had existed, but which have not taken place. It may here be urged that this too is negative evidence. But the negative evidence derived from an appeal to the geological record is weak, not *per se*, but because of the demonstrated imperfection of the record. On the other hand, there are many instances where a land-bridge is well proven, and in these cases it is not a few scattered exceptions, but an entire fauna that has migrated, subject only to the restrictions imposed by climatic or topographic barriers of other kinds."

In accounting for the present discontinuous distribution of many ancient and primitive forms Matthew seems to me to have made good use of a principle which seems to have been first suggested by Haacke (1887). This writer called attention to the fact that at the present time the most primitive types of the various groups of animals are mostly confined to the tropics and the southern hemisphere. This can be most readily explained on the supposition that the situations in which such forms now live are not their original habitats but those to which they have been relegated by more recent and more specialized forms evolving and usurping their places in the territory originally occupied by the group. Hence the oldest and most primitive members of a group come to be found today at the periphery of its range and the more recent and specialized forms in or near its center. Clark (1915) has reached the same conclusion from his study of the distribution of the Onychophora. He says: "Any animal type, once evolved, will extend itself immediately in every direction as far as the natural barriers to its dispersal; a more specialized form (a dominant type) of the same animal, better fitted for the conditions under which it lives, will sooner or later be evolved somewhere in the central, or more favorable portion of the territory inhabited by the original type; this new type will at once extend itself as did the original type; but in the meantime there may have arisen certain barriers, which the second type cannot cross and beyond which, therefore, the first type is secure. Up to these barriers — high mountains, deserts, newly formed arms of the sea, or whatever they may be — the second type will gradually supplant the first, as a result of its better economic equipment and more perfect physical resistence, and the advantage which this better equipment and resistance give it in the struggle for existence. Thus we shall eventually find a specialized type beyond the limits of which occurs a more generalized type of the same organism. The subsequent evolution of additional types, which will most frequently occur at or near the so-called center of distribution as a natural result of the greater facility for adaptation due to the greater distance apart of the physico-economic barriers and the consequently greater radius of each type, will result in the gradual formation of a dispersal figure which would be ideally represented by a series of concentric circles, each of the circles representing a barrier, the small central circle enclosing the most perfected type and the peripheral band the most generalized, the intervening areas including intermediate types increasing in specialization toward the center."

The Formicidae show in a very striking manner the relegation of the most primitive forms to the tropics and southern hemisphere and especially to the Neotropical, Oriental and Australian regions. As all of these forms are exquisitely thermophilous and stenothermal, whereas the Palearctic and Nearctic faunas and particularly the forms peculiar to the mountains consist of more specialized, stenothermal and psychrophilous species together with a small number of eurythermal ubiquists, we are led to believe that the development of zonal climates during the Tertiary has been the essential factor in determining the distribution of the present world-wide distribution of ants. The mountain faunas are therefore of comparatively recent origin and this is particularly true of that of the Rocky Mountains, to judge from the large number of subspecies and varieties, most of which have, in all probability, developed since the Pleistocene. The Rocky Mountains as an independent center of formation of new forms contrast markedly with the Alps and Himalayas, for there are relatively few ants peculiar to the latter and especially to the Alps. This may be attributed to geological conditions. Geologists maintain that the Rocky Mountains began to be elevated as early as late Cretaceous time and by the Eocene had attained altitudes of 4000 to 5000 feet. They continued to rise during the Tertiary Period to altitudes of 13,000 to 14,000 feet, with a corresponding elevation of the bases between them and considerable erosion of their summits. The Alps however, did not appear till the close of the Oligocene and only during the Miocene were the Himalayas uplifted. The Alpine area, moreover, was surrounded by water till the Miocene when it became joined by a broad land-connection with Central Asia. Its connection with France by means of another land-connection is said to have occurred at the end of the Miocene. These conditions, together with the later extensive glaciation of the Alps, must have been very potent factors in preventing the development of an indigenous ant fauna. On the other hand the Rocky Mountains occupy a very large area and were much less affected by the unfavorable climate of the Ice Age. They therefore remained as a preserve of Pleiocene species and during more recent times became a center of origin of many races and varieties. For evidence in support of this contention we have only to compare the Nearctic and Palearctic forms of the genera *Myrmica, Leptothorax, Lasius, Formica, Polyergus* and *Camponotus*.

I advanced the opinion that the Rocky Mountains were probably the center of origin of the genus *Formica* in my paper on this group (1913), but I now accept Handlirsch's view that not only the genus *Formica*, but the whole family Formicidæ had its origin during the Mesozoic in Eurasia and believe, with Kolbe, that Central or Eastern Asia is, as seems to be the case with so many other groups of organisms, the most likely spot in which to seek the origin of the ants. The views of Matthew and Clark on the development of the specialized forms in the center of the geographical range of a group and the relegation of the older and more primitive forms to the periphery of that range, appear at first sight to support my former contention, but it now seems to me to be more probable that the Rocky Mountains constitute only a secondary, more recent center of speciation, and that they are much more important as a region of conservation. It may be noted in this connection that Japan is evidently a similar secondary center of speciation for the same genera, since in that country the common holarctic species of *Myrmica, Lasius, Formica, Polyergus* and *Camponotus* have developed several peculiar subspecies and varieties. The same is true of the Eastern United States, which are also characterized by the development of endemic forms of *Myrmica, Formica,* etc.

This brings us, finally, to the problem with which we started, the pronounced difference between the ant faunas of western and eastern North America, a difference very similar to what has been so often noticed in other groups of organisms. It may be readily attributed to a difference in survival after the glaciation of the northern portion of the continent, since the ice-sheet is known to have advanced considerably further south in the eastern than in the western half of the continent, while the Gulf of Mexico formed an impassable barrier to a directly southward emigration of species. Hence we should expect a much more meager survival of species in the Eastern than in the Western United States. The differences of character in the endemic forms of the two regions, however, must be due to other conditions, some of which were undoubtedly preglacial, while others were as clearly postglacial. If we may judge from the Florissant deposits, the North American ant-life of Miocene times was poorer in species than that of Europe, but we must bear in mind that the Florissant locality is a very limited and elevated area and that the preserved ants are nearly all males and females which happened to fall into a small lake during their nuptial flight and after sinking to the bottom became embedded in volcanic sediment. Many of the species became extinct, but some of them, notably those of the genera *Aphaenogaster*, *Lasius* and *Liometopum*, were undoubtedly very closely related to forms of the same genera still living in Colorado. To the descendants of this Miocene fauna there was added during the Pleiocene a number of forms by immigration from Asia over the Siberian-Alaskan land-bridge, probably at the same time and by the same route as the Strepsicerine and Hippotragine antelopes. Probably, too, certain North American ants passed into Asia at the same time, just as seems to have been the case with the camels. Emery has recorded the occurrence of *Camponotus herculeanus* subsp. *pennsylvanicus* in Siberia and Burmah, and the closely related *C. japonicus* was originally described by Mayr as a mere variety of that subspecies. Probably such species as *Aphaenogaster subterranea*, *Myrmica lobicornis*, *Leptothorax muscorum*, some form of the subgenus *Neomyrma* closely related to the Eurasian *rubida* and the ancestor of *N. mutica*, *hunteri* and *aldrichi*, together with several species of *Formica*, notably *cinerea*, *rufibarbis*, *rufa*, *truncicola* and *Polyergus rufescens*, first entered North America during the Pleiocene. Even at the present time few of these species have succeeded in extending their range to the Atlantic States. The origin of the peculiarly eastern forms is more obscure. Probably a number of them are Mesozoic and early Tertiary survivors, notably the Ponerinae, and the species of *Strumigenys*, *Myrmecina* and *Aphaenogaster*. Others may have come from Europe over the North Atlantic land-bridge during the late Tertiary and have given rise to such forms as *Formica ulkei*, *exsectoides*, and *pallidefulva* and *Camponotus castaneus*. Some of these have migrated westward as far as the easternmost ranges of the Rocky Mountains but none has reached the Pacific Coast Eurythermal ubiquists like *Tapinoma sessile*, *Prenolepis imparis* and *Formica fusca* may have existed in North America since the Oligocene or Eocene. *F. fusca* and *P. imparis* are, as I have shown (1914), almost identical with *F. flori* and *P. henschei* of the Baltic amber. That elements derived from such various sources and migrations, probably separated by long periods of time, should have gradually evolved a number of subspecies and varieties in the localities to which they were at first confined, is only what might be expected. And that such varieties and subspecies should be much more numerous in the Rocky Mountains than in the Eastern States is also to be expected, when we consider the much greater variety of physical conditions in that region. The high mountains running north and south through many degrees of latitude, with very high timber and snow lines and often broken into isolated ranges separated by arid basins or "parks," sometimes of great extent, constitute a much more favorable territory for the production of endemic races and varieties than the compact east and west massif of the Alps with their low timber and snow lines and narrow valleys. Some very short mountain ranges, especially in Arizona, New Mexico and Western Texas are, in fact, quite insular, being completely surrounded by the desert from which they rise, and like islands have developed numerous endemic, or precinctive forms. This is very clearly seen in the Huachuca Mountains of Arizona, which are inhabited by several ants and other insects not known to occur in any other localities. The Coast Range and Sierra Cascade Ranges have also each developed a number of endemic forms. Unfortunately we are unable at the present time to appreciate the precise character and extent of this endemicity in our western mountains, because our knowledge of the distribution of any single insect group in any one of the various ranges is very fragmentary. This is equally true of the Appalachian System (White Mountains, Adirondacks, mountains of North Carolina and Georgia). Certainly no more interesting work could be undertaken by our taxonomic entomologists than a detailed and systematic survey of the various groups of insects in all these ranges after the manner of the fine surveys of the distribution of vertebrates and woody plants by Dr. C. H. Merriam and his collaborators.

## List and Descriptions of Western Mountain Ants.

### MYRMICINAE.

1. *Monomorium minimum* Buckley.

The typical form of this species, which is common in Texas and the Atlantic States at lower elevations and south of New England, has been taken by Prof. C. F. Baker in Ormsby County, Nevada, on the eastern shore of Lake Tahoe. I have found it in Arizona and Colorado and Dr. W. M. Mann collected it in the mountains of Hidalgo, Mexico.

2. *Monomorium minimum* subsp. *ergatogyna* Wheeler.

Known only from Catalina Island, Cala., where it was taken several years ago by Prof. C. F. Baker.

3. *Monomorium minimum* subsp. *compressum* Wheeler.

Taken by Dr. W. M. Mann at Guerrero Mill, Hidalgo, Mexico.

4. *Monomorium minimum* subsp. *cyaneum* Wheeler.

A beautiful metallic blue form from the same locality as the preceding.

5. *Solenopsis molesta* Say.

The typical form of this species is abundant throughout the Eastern United States and Southern Ontario. It occurs also, but more sporadically, in the mountains of Colorado and New Mexico at altitudes below 8000 ft.

6. *Solenopsis molesta* var. *validiuscula* Emery.

Originally described from Los Angeles and San Jacinto, California. I have taken it in the Santa Inez Mts. near Santa Barbara in the same state, and Dr. Mann has give me specimens which he collected at Wawawai, Washington.

7. *Solenopsis molesta* var. *castanea* Wheeler.

A dark color variety originally described from Woodland Park, in the Ute Pass, Colorado (Wheeler).

8. *Myrmecina graminicola* Latr. subsp. *americana* Emery.

A rather rare ant, occurring in rich, shady woods in the Eastern States and as far west as Texas and the Grand Canyon, Arizona.

9. *Myrmecina graminicola* subsp. *americana* var. *brevispinosa* Emery.

In distribution this form is similar to the preceding. I have taken it as far west as New Braunfels, Texas, and have seen specimens collected by Mr. E. S. Tucker at Plano in the same state.

10. *Myrmecina graminicola* subsp. *texana* Wheeler.

Known only from Austin, Texas, where I found it many years ago in moist places in the canyons of the Edwards Plateau.

11. *Myrmica brevinodis* Emery.

The typical form of this species is common in the Transition Zone of Colorado, about Colorado Springs, Denver, Boulder, Buena Vista, etc.

12. *Myrmica brevinodis* var. *brevispinosa* Wheeler.

Known from New Mexico and Colorado.

13. *Myrmica brevinodis* var. *frigida* Forel.

The types of this variety were taken by Whymper in the Ice River Valley, British Columbia (5000 ft.).

14. *Myrmica brevinodis* var. *sulcinodoides* Emery.

British Columbia: Hector, Carbonate and Spillimachen R., Selkirk Mts. (C. J. Bradley); Field and Yoho Pass, Emerald Lake (Wheeler).

Alberta: Lake Louise (Wheeler).

Colorado: Rico, 10,000 ft. and Hayden Peak, 10,000 ft. (E. J. Osler); Boulder and Ward (W. W. Robbins); Lost Lake, Eldora, 9500 ft. (D. M. Andrews).

California: Lake Tahoe, 6000 ft. (Wheeler).

This variety is so much like the typical European *M. sulcinodis* Nyl. that one is inclined to regard *brevicornis* as merely a subspecies. The form described by Forel as *brevinodis* var. *whymperi* from Vermilion Pass, Alberta, is, so far as I am able to judge from two cotypes received from Prof. Forel, merely *sulcinodoides*. For a description and further citations of localities of this variety from Utah, Colorado and New Mexico, see my revision of the forms of *brevinodis* in Bull. Wis. Nat. Hist. Soc. 5, 1907, p. 73 *et seq*. The workers from Lake Tahoe have the epinotal spines rather short and perceptibly curved downward at the tips. It does not seem desirable to regard them as representing a distinct variety.

15. *Myrmica brevinodis* var. *decedens* Wheeler.

Colorado: Buena Vista, 7900 ft. (type locality) and Florissant, 8500 ft. (Wheeler).

New Mexico: Pecos Mts., San Miguel County (Mitchell).

The workers from New Mexico are considerably darker than the types and may prove to belong to a distinct variety.

16. *Myrmica brevinodis* var. *alaskensis* var. nov.

*Worker.* Length 3.5 mm.

Resembling the variety *sulcinodoides* but smaller and of a different color. Head black above; thorax, pedicel and gaster castaneous; mandibles, antennae and legs brownish yellow. Antennae slightly thicker at the base than in *sulcinodoides*. Rugae on the clypeus very coarse, much fewer in number (only 8) than in other forms of the species; frontal area distinctly outlined, subopaque and very finely punctate, not longitudinally rugulose. Rugae on the sides of the head coarsely reticulate, not longitudinal, those on the thorax and pedicel much as in *sulcinodoides* but a little finer; surface of head, thorax and pedicel a little more opaque. Epinotal spines somewhat shorter than the base of the epinotum, curved downward at their tips. Summit of petiolar node distinctly more rounded than in *sulcinodoides* and transversely rugose, postpetiolar node less convex behind. Pilosity like that of *sulcinodoides*.

Described from eight workers taken at Seward, Alaska by Mr. F. H. Whitney.

17. *Myrmica brevinodis* var. *subalpina* Wheeler.

British Columbia: Hector, Field and Carbonate (J. C. Bradley); Emerald Lake (Wheeler).

Alberta: Banff (Wheeler); Jasper (C. G. Hewitt).

Washington: Orcus Island (W. M. Mann).

This variety, originally described from Florissant, Colo., forms flourishing colonies under logs and stones in moist, sunny places. I found it very abundant on the southern slope of Tunnel Mt. at Banff. It closely resembles the eastern var. *canadensis* Wheeler, but the wings of the male and female are whitish hyaline throughout and not infuscated at the base. The workers of certain colonies present transitions in color to the var. *sulcinodoides*, but this forms smaller colonies and prefers higher elevations.

18. *Myrmica mexicana* Wheeler.

This species, related to our eastern *M. punctiventris* Roger, was taken by Dr. Mann at Guerrero Mill, Hidalgo, Mexico.

19. *Myrmica scabrinodis* Nyl. subsp. *lobicornis* Nyl. var. *glacialis* Forel.

Alberta: Vermilion Pass, type locality (Whymper); Lake Louise and Moraine Lake, Valley of the Ten Peaks (Wheeler).

British Columbia: Emerald Lake (Wheeler); Carbonate and Prairie Hills, Selkirk Mts. (J. C. Bradley).

Montana: Helena (W. M. Mann).

Utah: Salt Lake County (R. V. Chamberlin).

Colorado: Florissant, Ute Pass, Cheyenne Canyon and Manitou (Wheeler); Creede Co. 8844 ft. (S. J. Hunter); Boulder (P. J. Schmitt); Pikes Peak, 10,000 ft., Willow Creek and West Cliff, 7864 ft. (T. D. A. Cockerell).

New Mexico: Harvey's Ranch, Las Vegas Range, 10,000 ft. (E. L. Hewett); Beulah, 8000 ft. (T. D. A. Cockerell).

Arizona: San Francisco Mts., 13,000 ft. (W. M. Mann); Coconino Forest, Grand Canyon (Wheeler).

Forel described this form from worker specimens as a variety of the typical *scabrinodis*, but has more recently placed it under the subsp. *schencki* Emery. During the summer of 1915 I found the males and females in many nests in British Columbia and Alberta and these phases show unmistakably that *glacialis* must be regarded as a form of *lobicornis*, a subspecies common in the Alps and Northern Europe but not hitherto known to occur in America. The antennal scapes of the male *glacialis* are strongly bent at the base and fully $\frac{2}{5}$ as long as the funiculus. They are therefore only a little shorter than in typical *lobicornis*. The other differences are equally insignificant. The *glacialis* male is a little smaller and has somewhat shorter epinotal teeth, the sculpture of the head and thorax is somewhat feebler so that the surface is more shining. The female is also somewhat smaller than the female of the typical form, its head and thorax are more shining and less coarsely sculptured, and the thorax and pedicel are darker, the angles at the base of the antennal scapes decidedly smaller. The worker specimens from the Grand Canyon, San Francisco Mts. and Boulder have the antennal lobes large and more flattened, much as in the typical *lobicornis*. The specimens from Helena are considerably paler and colored like the eastern *sabuleti*, but the greater length of the antennal scape in the male shows that they should be placed with *lobicornis*, although they may represent a distinct variety.

20. *Myrmica scabrinodis* subsp. *schencki* Emery var. *tahoënsis* var. nov.

*Worker.* Length 3.3–4 mm.

Small; antennal scapes geniculately bent at the base and at the flexure with a small rounded lobe, appearing as an acute tooth when the scape is seen from the side. Frontal area very distinct, triangular. Frontal carinae large, lobular. Epinotal spines slightly shorter than the base of the epinotum, as long as their distance apart at the base, rather slender, distinctly curved downwards at their tips. Petiole in profile blunt and rounded above.

Head, thorax and pedicel very coarsely and in the main longitudinally rugose, the surface subopaque; frontal area opaque, finely and densely longitudinally rugulose; concavity of epinotum smooth and shining like the gaster.

Hairs rather long, abundant and suberect on the body and legs as in the typical *schencki* var. *emeryana* Forel.

Head and gaster black; mandibles, antennae, thorax, petiole and post-petiole deep brownish red; legs slightly more yellowish red.

*Female* (deälated). Length 4.5–5 mm.

Very similar to the worker; pronotum transversely, mesonotum and scutellum strongly, pleurae more feebly longitudinally rugose; petiole and postpetiole longitudinally rugose above, densely and finely punctate on the sides and below as in the worker. Color like that of the worker, except that the thoracic dorsum and some spots on the pleurae are black.

*Male.* Length 3.5–4 mm.

Antennae very short, the scapes especially, which are feebly bent at the base and not more than three times as long as broad and shorter than the three basal funicular joints together; club 4-jointed. Frontal area large, distinct, triangular. Sculpture and pilosity much as in the var. *emeryana.* Color dark piceous brown; head black; mandibles, tarsi and articulations of legs brownish yellow; palpi whitish. Wings pale hyaline throughout, not infuscated at the base as in *emeryana.*

Described from numerous workers, several males and two females from several localities about Lake Tahoe (Tallac, Angora Lake, Glen Alpine Springs, Fallen Leaf Lake). The colonies are small and nest under stones in shady places.

21. *Myrmica scabrinodis* subsp. *schencki* var. *monticola* var. nov.

*Worker.* Length 3–3.5 mm.

Differing from the vars. *emeryana* and *tahoënsis* in its smaller size, in color and in the shape of the antennal scapes, which are rectangularly bent at the base and furnished at the flexure with a large, rounded, shovel-shaped, transverse lobe, which is prolonged as a low membranous ridge for a short distance along the posterior edge of the joint. Frontal carinae suberect; frontal area small but distinct, triangular. Epinotal spines shorter than the base of the epinotum and than their distance apart at the base, straight, slender and acute. Sculpture of the mandibles, head, thorax and pedicel sharp but rather loose and the punctuation of the interrugal spaces very shallow, so that the surface is much more shining than in the other varieties of *schencki.* Rugae of the pro- and mesonotum very coarse, vermiculate, indistinctly longitudinal. Middorsal portion of postpetiole rather smooth and shining, somewhat as in the var. *detritinodis* Emery.

Pilosity rather long, abundant, coarse, erect and blunt on the body, appressed on the legs.

Color brownish yellow; scapes and legs of a clearer, paler yellow; head more brownish above; first gastric segment dark brown.

*Male.* Length 3.8–4.9 mm.

Antennae much as in the var. *tahoënsis,* the scape being distinctly shorter than the three basal funicular joints together. Surface of body shining, feebly sculptured; head finely and densely punctate, with indistinct rugae; the rugae on the thoracic dorsum also very faint. Protuberances of epinotum very blunt.

Color brown; head darker; mandibles, thoracic sutures, margins of gastric segments, antennal clubs, legs and tarsi, except the middle portions of the femora and tibiae, brownish yellow.

Described from a dozen workers and nine males taken by myself at Buena Vista, Colorado.

This is a very distinct form, which in the larger lobe of the antennal scapes approaches the typical European *schencki* more closely than do either of the varieties *tahoënsis* or *emeryana.* There are, however, in the Eastern States one or more large, dark, undescribed varieties which have a similar extensive antennal lobe.

22. *Myrmica* (*Neomyrma*) *bradleyi* Wheeler.

California: Glacier Point, Yosemite, 8000 ft. and Tallac, Lake Tahoe 6000 ft. (Wheeler).

This form was recently redescribed by Forel as *Aphaenogaster* (*Neomyrma*) *calderoni* from specimens taken by Calderon in the Lake Tahoe region. The types were taken by Prof. J. C. Bradley in Alta Meadow, Tulare Co., Cala., at an altitude of 9500 ft. It nests under stones in rather dry, sunny places and in habits closely resembles *M.* (*N.*) *mutica* Emery. The localities for this and two other species are here transcribed from my recent paper on the American species allied to *M. rubra* Latr. (Psyche, 21, 1914, pp. 118–122).

23. *Myrmica* (*Neomyrma*) *mutica* Emery.

Colorado: Denver; type locality (E. Bethel); Colorado Springs, Salida, Buena Vista and Wild Horse, 6000–7000 ft. (Wheeler); Canyon City (Rev. P. J. Schmitt).

New Mexico: (Ern. André).

Utah: Salt Lake County (R. V. Chamberlin), as the host of the peculiar xenobiotic ant, *Symmyrmica chamberlini* Wheeler.

Washington: Olympia (T. Kincaid); Ellensburg and Pullman (W. M. Mann).

Alberta: McLeod (C. G. Hewitt).

British Columbia: Dog Lake, Penticton (C. G. Hewitt).

24. *Myrmica* (*Neomyrma*) *aldrichi* Wheeler.

Idaho: Moscow (J. M. Aldrich).

25. *Myrmica* (*Neomyrma*) *hunteri* Wheeler.

Montana: Madison R. near Beaver Creek, 7500 ft. (S. J. Hunter).

26. *Leptothorax andrei* Emery.

Originally described from California. I possess several workers taken by Dr. W. M. Mann at Palo Alto in that state.

27. *Leptothorax eldoradensis* Wheeler.

Taken on Mt. Wilson near Pasadena by Prof. J. C. Bradley.

28. *Leptothorax schmitti* Wheeler.

Known only from Canyon City, Colo.

29. *Leptothorax neomexicanus* Wheeler.

Described from Manzanares, New Mexico.

30. *Leptothorax nitens* Emery.

Known from Utah, California and Colorado.

31. *Leptothorax nitens* var. *heathi* Wheeler.

Known only from California.

32. *Leptothorax nitens* var. *mariposa* var. nov.

*Worker*. Resembling the var. *heathi* in being brown, with yellow legs and antennae, but the thorax is opaque and densely punctate throughout, like the petiole and postpetiole. The punctures are decidedly coarser than in this variety and the typical form. The tips of the antennal scapes nearly reach the posterior corners of the head, being separated from them only by a distance equal to the greatest transverse diameter of the scape.

Several workers found nesting under the edges of stones in dry places in Tenaya Canyon, Yosemite Valley, Cala.

33. *Leptothorax nitens* subsp. *occidentalis* Wheeler.

Described from Friday Harbor, Washington.

34. *Leptothorax melanderi* Wheeler.

Taken on Moscow Mt., Idaho by Prof. A. L. Melander.

35. *Leptothorax furunculus* Wheeler.

Taken in Williams Canyon, near Manitou, Colo. at an altitude of 7500 ft.

36. *Leptothorax tricarinatus* Emery.

Described from a single worker specimen taken by Pergande at Hill City, South Dakota. I have not been able to recognize it among my specimens of *Leptothorax*.

37. *Leptothorax nevadensis* Wheeler.

Described from King's Canyon, Ormsby County, Nevada, where it was taken by Prof. C. F. Baker. This locality is on the eastern shore of Lake Tahoe.

38. *Leptothorax nevadensis* subsp. *rudis* subsp. nov.

*Worker*. Length 2.6–3.3 mm.

Distinctly larger and more robust than the typical *nevadensis* and much more coarsely sculptured. Funicular joints 2–8 distinctly broader in proportion to their length. Head subopaque, finely and densely longitudinally rugose, with punctate interrugal spaces and sometimes with an interrupted shining median line. Frontal area shining, very finely striated. Mandibles coarsely punctate, striated at their bases. Thorax and petiole coarsely punctate-rugose, the rugae on the pleurae and often also on the pro- and epinotum longitudinal, on the mesonotum often vermiculate. Declivity of epinotum densely punctate and as opaque as the remainder of the thorax (more shining in the typical form). Postpetiole densely punctate and opaque. The epinotal spines are much stouter and blunter, and the petiolar node is much less compressed anteroposteriorly, its posterior surface being much more convex than in typical *nevadensis*. The color is considerably darker, the body being castaneous, with the head and gaster, except its incisures, blackish, the mandibles, clypeus, antennae and legs yellowish brown, the femora infuscated in the middle. Pilosity as in the typical form.

*Female* (deälated). Length: 3.5 mm.

Smaller than the female of typical *nevadensis*, with longer and more slender epinotal spines and the funicular joints 2–8 shorter. Sculpture of the head, thorax and petiole a little coarser. Petiolar node like that of the worker. In the typical form it is much compressed anteroposteriorly and has a sharp, transverse superior border. There is very little difference in color between the two forms.

Described from numerous workers and a single female taken from small colonies nesting under the edges of stones in Tenaya Canyon, Yosemite Valley, Cala. Seven workers from Angora Peak, 8600 ft., near Lake Tahoe, Cala., though differing in certain details of sculpture are nevertheless referable to this subspecies.

39. *Leptothorax rugatulus* Emery.

South Dakota: (Rergande).

Colorado: (Pergande); Cheyenne Canyon, near Colorado Springs (Wheeler).

Washington: Seattle (T. Kincaid).

Montana: Helena (W. M. Mann).

Study of much more material of this form and its varieties than I possessed when I wrote my "Revision of the North American Ants of the Genus Leptothorax" (Proc. Acad. Nat. Sci. Phila., 1903, pp. 215–260) convinces me that *rugatulus* is really a distinct species, as Emery maintained, and not a subspecies of *curvispinosus* Mayr. The latter is the most generally distributed and abundant *Leptothorax* in the Eastern and Central States as far west as Missouri, but *rugatulus* and its varieties are confined to the Western States. The two forms also differ in habits, *rugatulus* and its varieties nesting under stones and *curvispinosus* in hollow twigs and old galls.

40. *Leptothorax rugatulus* var. *cockerelli* Wheeler.

New Mexico: Las Vegas Hot Springs, type locality (T. D. A. Cockerell).

Arizona: Miller, Carr and Ramsay Canyons, Huachuca Mts. (Biedermann, Mann and Wheeler).

41. *Leptothorax rugatulus* var. *mediorufus* var. nov.

*Worker.* Length 2.5–3 mm.

A little larger, much more coarsely sculptured and of a much deeper color than the var. *cockerelli* and the subsp. *annectens*. Upper surface of head and gaster black; mandibles, clypeus, antennæ, thorax, petiole, postpetiole and a spot at the base of the first gastric segment ferruginous red; margins of gastric segments yellowish; legs yellowish brown, the femora infuscated in the middle. Rugosity of head, thorax and petiole coarse; head but slightly shining; thorax, petiole and postpetiole opaque, densely punctate. Declivity of epinotum transversely rugulose. Epinotal spines a little stouter but not more curved than in the typical *rugatulus*.

*Female.* Length 3.5 mm.

Decidedly larger than the female of *cockerelli*, with the body very dark brown, the antennæ, mandibles, legs and incisures of the gaster light brown. Surface of head, thorax and pedicel quite as coarsely sculptured as in the worker and much less shining than in *cockerelli* and the typical *rugatulus*. Wings whitish hyaline, with nearly colorless veins and pale brown stigma.

Described from many workers and three females from several colonies found near Lake Tahoe, Cala. (Tallac, Glen Alpine) and about Camp Curry in the Yosemite Valley. A series of workers and two winged females taken by Prof. J. C. Bradley at Volcano Creek in Southern California also belong to this form.

42. *Leptothorax rugatulus* Emery subsp. *annectens* Wheeler.

The four cotype workers taken at Boulder by Rev. P. J. Schmitt remain the only specimens I have seen of this form.

43. *Leptothorax rugatulus* subsp. *brunnescens* subsp. nov.

*Worker.* Length 1.6–2 mm.

Decidedly smaller than the preceding forms of the species and much more feebly sculptured, so that the surface of the head, thorax and pedicel is distinctly shining. The epinotal spines are shorter than their distance apart at the base, very feebly curved and slightly deflected at their tips. The postpetiole is nearly twice as broad as long, with prominent, but rounded anterior corners. The petiolar node seen from above is as broad as long, and as in the other forms broader behind than in front. In the other forms the segment is considerably longer. The color is dull yellowish brown, with the upper surface of the head and gaster, the summits of the petiolar and postpetiolar nodes and the middle portions of the femora darker brown.

Described from twenty workers taken by Dr. S. J. Hunter in Creede County, Colorado, at an altitude of 8844 ft.

44. *Leptothorax (Mychothorax) muscorum* Nyl. var. *sordidus* Wheeler.

Colorado: Boulder (P. J. Schmitt).

45. *Leptothorax (Mychothorax) muscorum* var. *septentrionalis* var. nov.

*Worker*. Length 2.5 mm.

Resembling the var. *sordidus*, but with the head and gaster dark brown or black above, the paler portions of the body of a deeper and more ferruginous red and the rugosity and punctuation decidedly coarser, so that the head, thorax, petiole and postpetiole are nearly opaque. The petiolar node is blunter and more rounded above in profile than in *sordidus* and the typical *muscorum* of Europe.

*Female*. Length 2.9 mm.

Dark brown; head black; venter yellowish, much of the pleurae lower surfaces of petiole and postpetiole, mandibles, legs and antennae, except the clubs, paler brown. Thorax subopaque, densely punctate, pronotum transversely, mesonotum longitudinally rugulose. Wings white, with colorless veins and brown stigma. Pilosity much as in the worker.

*Male*. Length 2.6–3 mm.

Black, legs and incisures of gaster dark brown. Wings as in the female. Head, thorax, petiole and postpetiole subopaque, densely rugulose-punctate. Pilosity abundant, short and white.

Described from numerous specimens of all three phases, which I took from several colonies nesting under stones on the southern slope of Tunnel Mt. at Banff, Alberta, a series of workers collected by Dr. C. Gordon Hewitt at Sulphur Springs, near Banff and a few workers which I found in the Yoho Pass, near Emerald Lake, British Columbia. At first sight this ant closely resembles *L. rugatulus* var. *mediorufus* but it can be readily distinguished by the feeble transverse mesoëpinotal impression and faintly indicated promesonotal suture.

46. *Leptothorax (Mychothorax) acervorum* Mayr subsp. *canadensis* Provancher.

Washington: Olympia (T. Kincaid).

Colorado: Florissant (Wheeler); Ward and Pikes Peak, 10,000 ft. (T. D. A. Cockerell).

Utah: Little Willow Creek, Salt Lake Co. (R. V. Chamberlin).

Maine: Orono (H. H. Severin).

New Hampshire: Franconia and summit of Mt. Washington (Mrs. A. T. Slosson).

This species was originally described from Canada. The following smaller and darker variety is also widely distributed through the Canadian Zone but seems to be rare and local:

47. *Leptothorax* (*Mychothorax*) *acervorum* subsp. *canadensis* var. *convivialis* Wheeler.

Wisconsin: Milwaukee, type locality (Wheeler).

New Mexico: Beulah, 8000 ft. (F. W. P. Cockerell); Top of Las Vegas Range, 11,000 ft. (T. D. A. Cockerell).

Connecticut: Colebrook (Wheeler).

Nova Scotia: Digby (J. Russell).

Newfoundland: Spruce Brook.

This form was described as *L. canadensis* subsp. *obscurus* by Viereck in a paper on the Hymenoptera of Beulah New Mexico (Trans. Amer. Ent. Soc. 29, 1903) which appeared a month later than my revision of the species of *Leptothorax*.

48. *Leptothorax* (*Mychothorax*) *acervorum* subsp. *canadensis* var. *kincaidi* Pergande.

Four workers and a deälated female taken by Mr. F. H. Whitney on the Upper Kugarok River, north of Nome, Alaska (65°!) are clearly referable to this variety, originally described from Metlakahtla. Both phases are larger (worker 3 mm.; female 4 mm.) and more coarsely sculptured than our other North American forms. In my workers the reddish brown thorax has a black crescent on the pronotum and the upper surface of the epinotum and the petiole and postpetiole are of the same color. The epinotal spines are long, thick and blunt, the antennal scapes reach only a little more than half the distance between the eyes and posterior corners of the head. The hairs on the legs are short, coarse and suberect.

49. *Leptothorax* (*Mychothorax*) *acervorum* subsp. *canadensis* var. *yankee* Emery.

British Columbia: Glacier (Wheeler); Rogers Pass and Prairie Hills, Selkirk Mts. and Carbonate (J. C. Bradley).

Alberta: Lake Louise and Moraine Lake, Valley of the Ten Peaks (Wheeler).

Colorado: Boulder (P. J. Schmitt and W. W. Robbins).
South Dakota and Utah: (Emery).
Michigan: Washington Harbor, Isle Royale (O. McCreary).

At Glacier and Lake Louise I found several colonies of this ant nesting under stones in rather damp places. The worker form, originally described from South Dakota, Utah and Colorado, differs from the typical *canadensis* and the preceding varieties in its somewhat finer sculpture and paler color, the mandibles, antennae, except their clubs, the thorax, pedicel and legs being reddish brown or red, the femora infuscated in the middle. The epinotal spines are rather long and pointed. The male measures 4 mm. and is distinctly larger than the worker (2.5–3 mm.) and only slightly smaller than the typical *acervorum,* from which it is almost indistinguishable. Comparison with Swiss specimens shows that the American variety has a darker pterostigma and smaller epinotal protuberances. The female is somewhat smaller and darker than the female of the typical *acervorum*.

50. *Leptothorax* (*Mychothorax*) *acervorum* subsp. *canadensis* var. *calderoni* Forel.

I have taken numerous workers and females of this form in the type locality (about Lake Tahoe, Cala.), where it is common in little nests in the bark of large pine logs and stumps, often in plesiobiosis with *Camponotus herculeanus* var. *modoc*. The worker has the color of the var. *yankee*. According to Forel the antennal scapes are longer than in *canadensis,* reaching nearly to the posterior corners of the head, but my specimens show considerable variation in this particular. Nor do I find that the worker *calderoni* is larger than the European *acervorum,* though it is larger than the var. *yankee*. My workers vary considerably in size, from 2.5–3.5 mm. The main difference which I detect between *yankee* and *calderoni* is in the proportions of the promesonotum, the length of this region in the latter form between the cervical ridge of the pronotum and the mesoëpinotal suture being more than 1½ times the breadth of the pronotum through the humeri, whereas in *yankee* it is distinctly less. Sculpture and pilosity are very similar in the two forms.

51. *Leptothorax* (*Mychothorax*) *acervorum* subsp. *crassipilis* subsp. nov.

*Worker*. Length: 2.5–3 mm.

Differing conspicuously from the preceding forms of *acervorum* in sculpture and pilosity. The surface of the head, thorax, petiole and postpetiole is distinctly though feebly shining owing to the more superficial punctuation. The rugae on the head and thorax are much more distinct, coarser and further apart. The blunt, erect hairs on the upper surface of the body, especially on the head, thorax and pedicel are much longer, coarser and glistening white. The hairs on the legs seem also to be somewhat coarser than in the various forms of *canadensis.* The spines of the epinotum are scarcely more than half as long as their distance apart at the base, acute and rather slender. The antennal scapes reach about half way between the eyes and the posterior corners of the head and the basal funicular joints are distinctly shorter and more transverse than in *canadensis* and its varieties. Color dark brown or castaneous, head and sometimes the gaster darker and more blackish; mandibles, antennae and legs pale brown; middle portions of femora, but not the antennal clubs, infuscated.

*Female.* Length: 3.5 mm.

Very similar to the worker, the rugosity of the head and thoracic dorsum and the pilosity of the head and thorax even a little coarser and more conspicuous. Body uniformly castaneous, except the pale incisures of the gaster. Wings grayish hyaline, veins pale brown, pterostigma dark brown.

*Male.* Length 3–3.5 mm.

Much smaller than the male of *canadensis* and *acervorum.* Dark brown; head black, mandibles and legs pale brown, tarsi paler. Wings white, with pale brown veins and stigma. Sculpture of head and thoracic dorsum distinctly more superficial than in *canadensis* var. *yankee* and *calderoni,* the head, especially, more shining. Pilosity not more abundant, but paler.

Described from numerous specimens of all three phases taken from small colonies under stones in several localities (Manitou, Cheyenne Creek, Red Rock Canyon, Williams Canyon) near Colorado Springs during July and August, 1903. This form is so distinct that it might be regarded as an independent species, but as it has the shape of *acervorum* and the median impression of the clypeus I prefer for the present to regard it as a subspecies.

52. *Leptothorax* (*Mychothorax*) *emersoni* Wheeler subsp. *glacialis* Wheeler.

Colorado: Florissant 8500 ft. (Wheeler).

As I have shown (Bull. Wis. Nat. Hist. Soc. 5, 1907, p. 78 *et seq.*), this subspecies lives in the colonies of *Myrmica brevinodis* var. *subalpina* in much the same manner as the typical *emersoni* of New England and Canada lives with *M. brevinodis* var. *canadensis.*

53. *Leptothorax* (*Mychothorax*) *emersoni* subsp. *hirtipilis* subsp. nov.
*Worker*. Length 2.5 mm.

Differing from the typical *emersoni* and the preceding subspecies in the following characters: the mesoëpinotal constriction is more pronounced, the pro- and mesonotum being somewhat more convex and at a higher level than the base of the epinotum. The sculpture is much coarser, the rugae on the head being very sharply defined even on the occiput and posterior corners; on the thoracic dorsum the rugae are vaguely longitudinal. The head, thorax and petiole are decidedly opaque. The pilosity is much coarser and more abundant, especially on the thorax and legs. The color is a little darker than that of *emersoni* and *glacialis*, with only the anterior border of the gaster yellowish.

A single specimen taken from a nest of *Myrmica brevinodis* var. *subalpina* on the southern slope of Tunnel Mt., at Banff, Alberta. This shows that the habits are symbiotic as in the other forms of the species.

54. *Leptothorax* (*Mychothorax*) *hirticornis* Emery subsp. *formidolosus* Wheeler.

Colorado: Flagstaff Mt., Boulder Co. (T. D. A. Cockerell).

South Dakota: Hill City (Pergande Coll. Nat. Mus.).

55. *Aphaenogaster subterranea* Latr. subsp. *occidentalis* Emery.

Washington: Pullman, type locality (Pergande); Pullman and Wawawai (W. M. Mann); Almota (A. L. Melander); Olympia (T. Kincaid).

Oregon: Ashland (W. Taverner).

California: Pacific Grove, Mt. Tamalpais, Yosemite and Lake Tahoe (Wheeler); Palo Alto and King's River Canyon (H. Heath); Corte Madera Creek, Santa Cruz Mts. (W. M. Mann); Mountain View.

Idaho: Moscow (J. M. Aldrich).

Utah: East Mill Creek, Salt Lake Co. (R. V. Chamberlin).

Colorado: Cheyenne Canyon near Colorado Springs and Boulder (Wheeler).

Montana: Helena (W. M. Mann).

British Columbia: Dog Lake, Penticton (C. G. Hewitt).

This subspecies is extremely common in both the Coast Range and the Sierras of California from sea-level to an elevation of 6000 ft. It appears to be equally common in Oregon and Washington but is much more sporadic in the other localities cited.

56. *Aphaenogaster subterranea* subsp. *valida* Wheeler.

Recently described from Cheyenne Canyon, near Colorado Springs.

57. *Aphaenogaster subterranea* subsp. *valida* var. *manni* var. nov.

*Worker.* Length 4–5 mm.

Differing from the typical *valida* in color, the body and antennae being yellowish brown throughout, the mandibles, clypeus, and legs paler and clearer yellow. The sculpture of the epinotum and mesopleuræ and sides of the pronotum much feebler and the upper surface of the pronotum more shining

Numerous workers from Pullman, Washington, (W. M. Mann). Other workers from the same locality but from different nests have the head and gaster darker than the thorax and thus represent transitions to the typical *valida*.

58. *Aphaenogaster subterranea* subsp. *borealis* Wheeler.

Recently described from worker specimens taken by Prof. J. C. Bradley at Lardo, Kootenay Lake, British Columbia. A number of workers taken by Dr. C. G. Hewitt at Arrowhead Lake, British Columbia, though slightly darker, are also referable to this subspecies.

59. *Aphaenogaster patruelis* Forel.

Known only from the Island of Guadeloupe off the coast of Lower California.

60. *Aphaenogaster patruelis* var. *bakeri* Wheeler.

This variety was taken several years ago on Catalina Island off the coast of Southern California by Prof. C. F. Baker.

61. *Aphaenogaster patruelis* var. *carbonaria* Pergande.

According to Forel, this form, originally described as a species, is merely a somewhat darker and more coarsely sculptured variety of *patruelis*. The types were taken by Eisen at Sierra Laguna and El Chinche in Lower California.

62. *Aphaenogaster mutica* Pergande.

Known from Lower California, Northern Mexico and Western Texas.

63. *Aphaenogaster texana* Emery.

Recorded from Texas, Arizona and Kansas.

64. *Aphaenogaster texana* var. *furvescens* Wheeler.
Described from the Huachuca Mts., Arizona.

65. *Aphaenogaster fulva* Roger subsp. *aquia* Buckley var. *rudis* Emery.

Several workers, a deälated female and a male of this form were taken by me at Boulder, Colo. July 29, 1906. Like other forms of the species it is common in the Central and Eastern States and reaches the western limit of its range on the eastern slopes of the Rocky Mts.

66. *Aphaenogaster fulva* subsp. *aquia* var. *azteca* Emery.

A form closely related to the preceding but more coarsely sculptured. It was described from Mexico without more precise locality.

67. *Aphaenogaster uinta* sp. nov.

*Worker*. Length 4.5–5 mm.

Head subrectangular, a little longer than broad, as broad in front as behind, with straight sides and rounded posterior corners and with a distinct pit-like impression in the median line on the vertex between the eyes. These are rather large, convex and situated near the median transverse diameter of the head. Mandibles with straight external borders and convex tips, with three larger apical and several more indistinct basal teeth. Clypeus moderately convex, its anterior border rather deeply notched in the middle. Frontal area distinct; frontal carinae small, erect in front, continued behind into slightly converging ridges. Antennae slender; scapes surpassing the posterior border of the head by somewhat less than $\frac{1}{6}$ their length, curved at the base and slightly thickened at the apex; funiculi with a distinct 4-jointed club; first funicular joint longer than second; joints 2–7 subequal, nearly twice as long as broad, joints 8–10 subequal, less than twice as long as broad, terminal joint distinctly longer. Thorax slender; pro- and mesonotum together forming a convex, hemispherical mass, the anterior border of the mesonotum not projecting above the pronotum, sloping and concave behind. Seen from above the mesonotum is narrow, more than twice as long as broad. Mesoëpinotal constriction rather deep. Epinotum long, somewhat less than twice as long as high, in profile with the base perfectly straight and horizontal, and on each side passing into the declivity, with a rectangular projection, representing the spine of other species. Seen from above the dorsal surface of the base is longitudinally impressed in the middle. Petiole short, its peduncle shorter than the node, which is nearly as high as the length of the segment, conical in profile, with strongly concave anterior and abrupt and feebly convex posterior slope. Seen from above the petiole through the node is about one half as broad as its length. Postpetiole shaped like the petiolar node in profile but somewhat lower, from above a little longer than broad and a little broader than the petiole. Gaster rather large, broadly elliptical. Legs moderately slender; spurs distinct on the median and hind tibiae.

Shining; mandibles densely striated, clypeus and antennal scapes finely longitudinally rugulose. Anterior portion of head to a short distance behind the eyes finely punctate and feebly longitudinally rugose. Posterior portion of head more shining, more sparsely and more superficially punctate. Pro- and mesonotum above smooth and shining, feebly shagreened; pleurae and epinotum punctate-rugulose and subopaque, the rugules on the base of the epinotum very fine, transverse. Petiole and postpetiole finely shagreened, feebly shining. Gaster very smooth and shining, superficially but distinctly shagreened, with sparse piligerous punctures. Legs moderately shining, finely shagreened.

Hairs yellowish; coarse, sparse and erect on the body, very short and subappressed on the antennal scapes and legs.

Yellowish red; legs more yellowish; gaster black or very dark brown, posterior borders of segments and anal region testaceous.

*Female.* Length about 7 mm.

Head scarcely longer than broad, distinctly broader behind than in front, with more rounded posterior corners than in the worker. Thorax through the wing-insertions as broad as the head through the eyes. Epinotum with two strong, acute spines, which are as long as broad at their bases. In profile the declivity is concave and distinctly shorter than the nearly straight, sloping base. Both the petiole and postpetiole, with their nodes, more compressed anteroposteriorly than in the worker. Wings rather long (8 mm.).

In sculpture, pilosity and color resembling the worker, but the head more opaque and more rugose behind. Pronotum transversely rugulose, mesonotum and scutellum and portions of mesopleurae smooth and shining; epinotum subopaque, its base transversely, its sides longitudinally rugulose. Mesonotum with three elongate brown blotches. Wings grayish hyaline, with pale brown veins and conspicuous dark brown stigma.

*Male.* Length nearly 4 mm.

Head a little longer than broad, flattened, rounded behind, with straight, marginate occipital border, large eyes and ocelli and very short cheeks. Mandibles small, with 5 or 6 teeth. Clypeus convex, its border straight and entire in the middle. Antennae slender, scapes as long as the first and second funicular joints together; joints of club strongly constricted at their proximal ends so that this portion of the antennae is moniliform. Mesonotum convex, distinctly longer than broad; scutellum as long as broad; base of epinotum straight in profile, gradually sloping to the posterior swellings which are feebly developed, rounded above and angulate behind, but not pointed. Nodes of petiole and postpetiole low and rounded. Legs slender.

Mandibles subopaque, very finely and indistinctly striate; clypeus smooth and shining; head subopaque, obscurely punctate-rugulose. Remainder of body smooth and shining, except the posterior swellings of the epinotum, which are subopaque and irregularly rugose.

Pilosity much as in the worker, but the long, erect hairs on the gaster finer.

Piceous; clypeus and legs pale brown; head black; mandibles and antennae sordid yellow. Wings as in the female, but a little more whitish.

Described from seventeen workers, one female and one male taken by Dr. R. V. Chamberlin at East Mill Creek, Salt Lake County, Utah.

This form is evidently quite distinct from any of our other North American species of *Aphaenogaster* though most closely related to *subterranea*. The worker and female of *uinta* can be distinguished from the various forms of this species by their color, the greater length of the scapes and funicular joints and the much larger eyes, the worker by its more rectangular head, peculiar epinotum, more conical postpetiole and larger gaster, the female by its smaller head, shorter epinotal spines and much darker pterostigma, the male by the very different epinotum, longer mesonotum, more shining and much less densely sculptured head and the paler body and appendages.

68. *Stenamma nearcticum* Mayr.

This species is known only from male and female specimens. What Mayr took to be the worker belongs to *brevicorne*. The types were from California. My collection contains a male and female from Corvallis, Oregon.

69. *Stenamma brevicorne* Mayr subsp. *diecki* Emery.

British Columbia: Yale, type locality (G. Dieck).
Illinois: Rockford (Wheeler).

Pennsylvania: Beatty (P. J. Schmitt).
Connecticut: Colebrook (Wheeler).

70. *Stenamma brevicorne* subsp. *heathi* Wheeler.
California: King's River Canyon (H. Heath).
Easily recognized by its uniform light red color and coarse sculpture.

71. *Stenamma brevicorne* subsp. *sequoiarum* subsp. nov.

*Worker.* Length 3–3.3 mm.

Resembling the subsp. *heathi* but larger and of the same color as *diecki*, with even coarser sculpture than the former, the rugae on the head being stronger and those on the pronotum very coarse and sparse, more longitudinal and less reticulate. The postpetiole is evenly and sharply longitudinally rugose and the rugae at the extreme base of the gaster are very distinct. The base of the epinotum is coarsely and vermiculately rugose. Head broader and the postpetiole distinctly longer than broad, its node lower and less convex than in *diecki* and *heathi*. The basal funicular joints are broader and slightly longer than in the other subspecies of *brevicorne*. Hairs on the body less abundant and more appressed, especially on the gaster and tibiae. Surface of the head, thorax and pedicel distinctly shining as in *heathi* and somewhat more opaque than in *diecki*.

*Female* (deälated). Length 3.6 mm.

Resembling the worker; larger than the female *diecki*, with more robust thorax and the whole body paler and more reddish. The sculpture is coarser, the upper surface of the mesonotum and scutellum more sharply longitudinally rugose. Funicular joints longer, hairs on the legs more appressed.

Described from a single female and numerous workers taken from several colonies nesting under stones among the large red-wood trees in Muir Woods on Mt. Tamalpais, Cala. A series of workers taken by Prof. H. Heath several years ago at Pacific Grove, Cala. appear to connect this subspecies with *diecki*. They are somewhat smaller than the specimens of *sequoiarum* and have a more convex postpetiole, which is longitudinally striate only on the sides and smooth and shining above. The head is somewhat more elongate and the basal funicular joints narrower and shorter. Both these specimens and those of *sequoiarum* may represent the unknown workers of *S. nearcticum*.

72. *Stenamma manni* Wheeler.

Known only from worker and female specimens taken by Dr. W. M. Mann at Chico in Hidalgo, Mexico.

## DOLICHODERINAE.

73. *Liometopum apiculatum* Mayr.

Mexico: Volcan de Colima, 7500 ft. (C. H. T. Townsend); Pinos Altos, Chihuahua and Ciudad de Durango, 8100 ft. (cited in Biol. Centr. Amer.); Guerrero Mill, Hidalgo, 9000 ft. (W. M. Mann).

Arizona: Huachuca Mts. 5000 ft. (Biedermann).

New Mexico: Las Vegas (Wheeler); Las Vegas Hot Springs, 6226 ft. and Romeroville (T. D. A. Cockerell); High Rolls, 6550 ft., Alamogordo, 4320 ft. (G. V. Krockow) and Beulah, 8000 ft. (H. Viereck).

Texas: Paisano Pass, 5079 ft. and Fort Davis, 5400 ft. (Wheeler).

Colorado: Canyon City, 5329 ft. and Cotopaxi 6371 ft. (P. J. Schmitt); Manitou and Cheyenne Canyon, 7000 ft. (Wheeler).

This ant seems always to be associated with live-oaks. Its habits, so far as I have been able to observe them, have been described in my paper "The North American Ants of the Genus Liometopum" (Bull. Amer. Mus. Nat. Hist. 21, 1905, pp. 321–333).

74. *Liometopum apiculatum* subsp. *luctuosum* Wheeler.

Colorado: Cheyenne Canyon, 7000 ft. near Colorado Springs, type locality (Wheeler).

Arizona: Grand Canyon 4000–7050 ft. and Prescott, 5320 ft. (Wheeler).

California: Baldy Peak, San Gabriel Mts., 6500 ft. (Brewster, Joos, and Crawford); Tenaya Canyon, Yosemite, 5000 ft. (Wheeler).

Though rarer and more sporadic than the typical form of the species and *occidentale*, this subspecies seems to have a wide range. The few colonies seen in the Yosemite were running on pine trees. This seems to confirm the opinion I advanced in 1905 that *luctuosum* is definitely associated with conifers.

75. *Liometopum occidentale* Emery.

California: San Jacinto, 1533 ft. (type locality); Mariposa 1962 ft.; Pasadena and Yosemite 4000 ft. and Wawona (Wheeler); Baldy Peak, San Gabriel Mts. (Brewster, Joos and Crawford); Claremont (C. F. Baker and Wheeler); Coalinga, below 500 ft., Fresno County, Ontario and Alpine (J. C. Bradley).

Oregon: Corvallis.

I have recently found a few specimens of the hitherto unknown female and male of this ant from San Jacinto, Cala., in the Pergande

Collection (U. S. Nat. Mus.). They differ so much from the corresponding phases of *apiculatum* that *occidentale* can no longer be regarded as a mere variety of the former, but must be elevated to specific rank. Comparison of the female and male of *occidentale,* however, with those of the European *L. microcephalum* may show closer affinities with this species, as a variety of which Emery originally described *occidentale.* The following descriptions are drawn from two females and a male:

*Female.* Length 10–10.5 mm.; wings 11.5 mm.

Much smaller than *apiculatum* and with much shorter wings (length of *apiculatum* 12–14 mm.; wings 17–18 mm.) and differing also in the following characters: The head, excluding the mandibles, as long as broad; with the scapes not reaching to the posterior corners, the frontal groove very sharp and distinct and extending from the frontal area to the anterior ocellus. Thorax through the wing insertions not broader than the head, the flattened mesonotum distinctly longer than broad. Surface of the body shining, though coarsely shagreened and sparsely punctate. Hairs short and rather numerous, but much shorter and less abundant than in *apiculatum.* Color ferruginous brown, gaster darker, lower surface of head, thoracic sutures and legs paler and more yellowish. Wings whitish hyaline, not infuscated as in *apiculatum,* with paler veins and brown stigma and subcostal vein.

*Male.* Length 9 mm.; wings 10 mm.

Differing from the male *apiculatum* in its smaller size, shorter wings and antennae (length of *apiculatum* 9–11 mm.; wings 14 mm.), with the wings pale like those of the female, the gaster, legs, genitalia and antennae reddish brown and the hairs, especially on the head, thorax and legs conspicuously shorter and less abundant. The volsellae of the genitalia are shorter and their tips slightly crenate along the dorsal border, whereas this border is smooth in *apiculatum.*

*L. occidentale* is very abundant among live oaks at low altitudes in the Coast Range of California but less common in the Sierras. It seems not to occur on their eastern slopes, judging from my inability to find it in the Lake Tahoe Region. Only a few colonies were seen in the Yosemite; at Wawona it was more abundant.

76. *Tapinoma sessile* Say.

Washington: Almota (A. L. Melander); Orcus Island, San Juan Island, Pullman and Ellensburg (W. M. Mann); Rock Lake.

California: San Jose and Palo Alto (H. Heath); Lompoc, Mt. San Jacinto, Harris, Humboldt County and summit of Mt. Wilson (J. C.

Bradley); Whittier and Azusa (W. Quayle); Yosemite and Lake Tahoe (Wheeler).

Washington: "Throughout the state" (W. M. Mann).

Idaho: Market Lake (J. M. Aldrich).

Nevada: King's Canyon, Ormsby Co. (C. F. Baker).

Colorado: Ward 9000 ft. (T. D. A. Cockerell); Buena Vista, Salida, Colorado Springs, Florissant and Boulder (Wheeler); Eldora, 8600 ft. and Swift Creek, (W. P. Cockerell); Creede Co. 8844 ft. (S. J. Hunter); Tolland and Boulder, 5000–10500 ft. (W. W. Robbins).

New Mexico: Las Vegas (Wheeler); Harvey's Ranch, Las Vegas Range, 10,000 ft. (Ruth Raynolds); Manzanares (Mary Cooper); Pecos and Albuquerque (T. D. A. Cockerell).

Arizona: Grand Canyon (Wheeler); Huachuca Mts., 5000 ft. (Biedermann and Wheeler).

Texas: Jefferson (E. S. Tucker).

Montana: Flathead Lake (C. C. Adams).

British Columbia: Golden (W. Wenman); Emerald Lake (Wheeler).

Alberta: Banff (Wheeler).

This ant is equally abundant and widely distributed in the whole region between the area covered by this list of localities and the Atlantic sea-board. It is an essentially eurythermal species, always nesting under stones, logs or bits of wood in open places. The large number of specimens which I have accumulated show considerable variations in size and coloration and some minor structural differences, so that one or more subspecies or varieties may have to be recognized when the material is more closely studied.

## CAMPONOTINAE.

77. *Prenolepis imparis* Say.

California: Piedmont, Alameda County and Berkeley (J. C. Bradley); Point Loma, near San Diego (P. Leonard); Palo Alto and San Jose (H. Heath); Santa Inez Mts. near Santa Barbara, Pasadena, San Diego, Claremont, San Gabriel Mts. and Yosemite Village (Wheeler).

Nevada: Ormsby County (C. F. Baker).

Oregon: Ashland (W. Taverner).

Arizona: Grand Canyon, 3670 ft. (Wheeler); Ramsay Canyon, Huachuca Mts. (W. M. Mann).

Colorado: Cheyenne Mt. near Colorado Springs (Wheeler).

Mexico: Colima, 7500 ft. (C. H. T. Townsend).

This species is also very common throughout North America east of the Mississippi River from Southern Ontario to St. Augustine, Florida, where it was taken by Prof. C. T. Brues. It belongs properly to the transition zone and is, according to my observations, always associated with oak trees. In the Eastern States the var. *testacea* Emery descends into the Upper and Lower Austral. It is one of the most abundant ants in the sandy pine-barrens of New Jersey and at low altitudes in the mountains of North Carolina. As Emery has shown, the European *nitens* Mayr is merely a subspecies of *imparis* with darker wings in the male and female (Deutsch. Ent. Zeitschr. 1910). This author has called attention to the remarkable distribution of the species, the subsp. *nitens*, the only known Old World form, being confined to Carinthia, Styria, the Balkan Peninsula, Asia Minor and the eastern shores of the Black Sea, whereas the typical form of the species has a very wide range in North America.

78. *Lasius niger* L. var. *sitkaënsis* Pergande.

Alaska: Sitka, type locality (T. Kincaid).

British Columbia: Glacier (Wheeler); Dowie Creek and Rogers Pass, Selkirk Mts. (J. C. Bradley).

Manitoba: Aweme (N. Criddle); Treesbank (C. G. Hewitt).

Ontario: Kenora (J. C. Bradley).

Washington: Olympia (T. Kincaid); Seattle (Wheeler); Pullman (W. M. Mann).

Oregon: Corvallis.

Idaho: Troy (W. M. Mann); Moscow (J. M. Aldrich).

Montana: Yellow Bay, Flathead Lake (C. C. Adams).

South Dakota: Elk Point (E. N. Ainslie).

California: Giant Forest, 6500 ft. (J. C. Bradley); Lake Tahoe, 6000–7000 ft. and Camp Curry and Glacier Point, 4000–8000 ft. Yosemite (Wheeler); King's River Canyon (H. Heath).

Colorado: Florissant, 8200 ft., Cheyenne Canyon and Williams Canyon, 8000 ft. (Wheeler); Denver (E. S. Tucker); Platte Canyon, 10,000 ft. and Rico, 10,000 ft. (E. J. Oslar).

Nova Scotia: Port Maitland (W. Reiff).

Maine: South Harpswell (Wheeler); Reed's Island, Penobscot Bay (A. C. Burrill).

As shown by the list of localities, this form is very widely distributed through the Canadian zone. The worker and female are larger than any of our other North American forms of *L. niger* (3.5–4 mm. and

8–9 mm. respectively) and quite as large as those of the typical Eurasian form of the species. The worker has the same abundant pubescence and erect hairs on the legs and scapes, but in the female the hairs are less abundant. The ocelli of the worker are very distinct. The body is yellowish brown, with the upper surface of the head, thorax and gaster darker and the appendages a little paler. The wings of the female measure 10–10.5 mm. and are uniformly pale yellowish brown, whereas those of the typical *niger* are colorless. The male *sitkaënsis* is scarcely smaller than the male *niger* dark brown and with the wings faintly tinged with the same color. This form passes by gradations into the smaller and darker variety, *neoniger* Emery and also approaches the true *niger* and the subspecies *alienus*. Thus the workers of some of the colonies found at Lake Tahoe and in the Yosemite are much like the European *niger*, whereas others are smaller and, except for their pilosity, might be confounded with large forms of *alienus* or its variety *americanus* Emery.

79. *Lasius niger* var. *neoniger* Emery.

Alberta: Banff (Wheeler).

California: Lake Tahoe (Wheeler).

South Dakota: Elk Point (C. N. Ainslie).

Colorado: Broadmoor, near Colorado Springs, Florissant and Salida (Wheeler); Ward, 9000 ft. and Steamboat Springs (T. D. A. Cockerell).

New Mexico: Viveash Ranch, 9000 ft. (Cockerell).

Washington: Pullman (W. M. Mann and R. W. Doane); Union City (J. C. Bradley).

This variety is also common in cool woods or at higher altitudes throughout the maritime provinces of Canada and the Eastern and Central States as far south as the Black Mts. of North Carolina. The worker and male measures only 2–2.5 mm., the female 6–7 mm. The wings of the female measure 8–9 mm. and both in this sex and the male are clear and hyaline. The body of the worker and female is dark brown or black and the erect hairs on the dorsal surface and on the legs and scapes are abundant and conspicuous.

80. *Lasius niger* subsp. *alienus* Forster var. *americanus* Emery.

Alberta: Banff (J. C. Bradley).

British Columbia: Glacier, Field and Emerald Lake (Wheeler); Carbonate (J. C. Bradley).

Colorado: Denver (E. Bethel); Silverton, 10,000 ft. (E. J. Oslar);

Manitou and Salida (Wheeler); Boulder (T. D. A. Cockerell); Canyon City (P. J. Schmitt).

Idaho: Julietta (J. M. Aldrich); Troy (W. M. Mann).

Utah: East Mill Creek, Salt Lake Co. (R. V. Chamberlin).

New Mexico: Gallinas Canyon (T. D. A. Cockerell); Las Valles (Mary Cooper).

Arizona: Grand Canyon, 7000 ft. (Wheeler).

California: Glacier Point, 8000 ft., Yosemite (Wheeler).

Very common throughout the Middle and Eastern States as far south as Georgia. That this variety should be attached to the European subsp. *alienus* and not to the typical *niger* is evident from the absence of erect hairs on the legs and antennal scapes and the sparse pilosity of the body, and also from the fact that the female, especially in the mountains of the western states, is indistinguishable in stature from the female of the true *alienus*, measuring nearly 8 mm., with wings 9–10 mm. long, although the females of the typical eastern *americanus* are often not more than 5–5.5 mm., with wings not exceeding 8 mm. Both forms may occur in some of the middle-western states, e. g. in Illinois and Wisconsin. The western form might be distinguished as a variety, for which the name *alieno-americanus* would be appropriate. The males, too, vary greatly in size and the differences of color among the workers of different colonies are considerable. In the Eastern States the workers of a form always found in dry sandy fields are very pale, almost drab-colored, whereas in adjacent woodlands the workers are always darker, varying from dark brown to black. Future study on the basis of a large amount of material will probably lead to the distinction of a number of forms of *L. niger* in North America, where the species is more variable than it is in Europe.

81. *Lasius* (*Formicina*) *brevicornis* Emery.

Montana: Elkhorn (W. M. Mann).

Colorado: Cheyenne, Canyon, near Colorado Springs (Wheeler).

New Mexico: San Geronimo (Mary Cooper).

This species, which is very common under stones on dry sunny hill slopes in New England and New York, is rare in the Western States.

The worker specimens from Colorado and New Mexico approach those of the following subspecies in the shape of the petiolar node and in having slightly smaller eyes than the typical form, but the differences are not sufficient to justify a new varietal name.

82. *Lasius* (*Formicina*) *brevicornis* subsp. *microps* subsp. nov.

*Worker.* Differing from the typical *brevicornis* in the much smaller and more nearly circular eyes which have only 11–13 ommatidia. The funicular joints of the antennae are slightly longer. The petiole is narrower, with straight sides and broadly and feebly emarginate superior border, whereas the typical *brevicornis* has the node entire with more rounded sides and border. The pubescence on the head and thorax is distinctly shorter so that the surface is more shining and the color of the body is not a pure but a more brownish yellow.

Described from numerous specimens taken from a large colony under a stone at Yosemite Village, 4000 ft., Cala.

83. *Lasius* (*Formicina*) *flavus* Fabr. subsp. *nearcticus* Wheeler.

Colorado: Topaz Butte, 9000 ft., near Florissant, and Salida (Wheeler).

This form is common throughout the Eastern and Middle States but evidently rare in the arid west, probably because of its preference for damp, shady situations. In the eastern states and Canada I have found it only in moist woods.

84. *Lasius* (*Formicina*) *flavus* subsp. *claripennis* subsp. nov.

*Worker*. Length 2.6–3 mm.

Similar to the typical *flavus* of Europe and the subsp. *nearcticus* but averaging smaller and with the antennae shorter, the scapes scarcely surpassing the posterior corners of the head. The color of the body is brownish yellow as in the true *flavus* and not pale yellow with whitish gaster as in *nearcticus*. The eyes are distinctly smaller and much as in the European subsp. *myops* Forel.

*Female*. Length 7 mm.

Differing from the female *flavus* and *nearcticus* in the shorter antennae, the scapes of which reach only to the posterior corners of the head. The wings are not infuscated at the base as in these forms, but clear and hyaline throughout and the posterior portion of the head and thoracic dorsum is dark brown, much darker than in *nearcticus* and perceptibly darker than in the typical *flavus*.

*Male*. Length 3 mm.

Differing from the male *flavus* and *nearcticus* in having slightly shorter antennae, in its smaller size and the darker, nearly black color of the body.

Described from numerous workers, four females and six males taken Aug. 20th from several colonies nesting under stones on the southern slope of Tunnel Mt. at Banff, Alberta. Several workers received from Farewell Creek, Southern Saskatchewan (E. G. Titus), three workers from Pullman, Washington, and a series of workers, males and females from Creede Co., Colo., 8844 ft. (S. J. Hunter) also belong to this subspecies. It is evidently quite distinct from the other forms of *flavus*. It resembles the subspecies *myops* in preferring hot, stone-covered slopes to moist, shady places as a habitat.

85. *Lasius* (*Formicina*) *umbratus* Nyl. subsp. *subumbratus* Viereck.

New Mexico: Beulah, 8000 ft., type-locality (T. D. A. Cockerell).

Colorado: Cheyenne Canyon near Colorado Springs, Williams Canyon near Manitou, and Boulder (Wheeler); Canyon City (P. J. Schmitt).

Utah: Little Willow Canyon, Salt Lake County (R. V. Chamberlin).

Arizona: Williams, 7000 ft. (Wheeler).

California: Angora Peak, near Lake Tahoe, 8000 ft. (Wheeler).

Ontario: Ottawa (Wheeler).

Quebec: Hull (Wheeler).

Maine: Reed's Island, Penobscot Bay (A. C. Burrill).

Nova Scotia: Digby (J. Russell); Bedford (W. Reiff).

The list of localities shows that this form has a very wide range. It is the most boreal of our forms of *umbratus* and is confined to the Canadian Zone.

86. *Lasius* (*Formicina*) *umbratus* subsp. *mixtus* Nyl. var. *aphidicola* Walsh.

This form, so abundant in many localities east of the Rocky Mts., is very rare further west. During the summers of 1903 and 1906 I found a few colonies near Florissant and Colorado Springs, Colorado. They nest by preference in moderately moist, shady places. This probably accounts for their almost complete absence from the arid portions of the country.

87. *Lasius* (*Formicina*) *umbratus* subsp. *vestitus* Wheeler.

Known only from a female specimen taken by Prof. J. M. Aldrich at Moscow, Idaho.

88. *Lasius* (*Formicina*) *humilis* sp. nov.

*Worker*. Length 1.5–1.7 mm.

Head as broad as long, a little narrower in front than behind, with broadly and feebly excavated posterior border and feebly and regularly convex sides. Eyes very small, somewhat larger than in the typical *brevicornis*, flat, with only about six ommatidia in their greatest diameter. Antennae slender; scapes extending beyond the posterior corners of the head, funiculi scarcely enlarged at their tips; joints 2–3 as broad as long, all the other joints distinctly longer than broad, the ninth and tenth being nearly 1½ times as long as broad. Clypeus very bluntly subcarinate. Frontal area large, triangular. Palpi rather long, the six joints of the maxillary pair gradually decreasing in length towards the tip as in other members of the subgenus. Thorax rather short, the pro- and mesonotum together as long as the epinotum; mesonotum as long as broad, the promesonotal suture not deeply impressed; mesoëpinotal constriction short but moderately deep; epinotum in profile with the convex base about ¼ as long as the flat, sloping declivity. Petiole narrow and rather high, much compressed anteroposteriorly, with flat anterior and posterior surfaces, straight, nearly subparallel sides and rather sharp, entire and evenly rounded superior border. Gaster broad, flattened dorsoventrally.

Surface shining; mandibles finely striated; remainder of body very finely and superficially shagreened.

Pubescence and hairs pale yellow, the former appressed, abundant and moderately long on the body and appendages, the latter blunt and erect, very sparse on the head, more numerous on the thoracic dorsum and still more abundant on the gaster.

Pale yellow; head and antennae a little darker; mandibles with reddish borders and teeth.

*Female.* Length 3.5 mm.

Head subrectangular, slightly broader than long, with rather deeply and broadly excised posterior border and straight cheeks. Eyes large, convex, more than half as long as the cheeks. Antennal scapes surpassing the posterior corners of the head by about ⅙ their length; all the funicular joints distinctly longer than broad. Thorax not broader through the wing-insertions than the head through the eyes, flattened above; mesonotum nearly as long as broad. Petiole with more convex sides and blunter superior border than in the worker, this border feebly emarginate in the middle in some specimens. Wings rather long.

Sculpture and pubescence much as in the worker, erect hairs on the thorax and gaster apparently less numerous and conspicuous.

Color like that of the worker, but the occipital portion of the head, the pro- and mesonotum, scutellum and dorsal surface of the gaster pale brown. Wings grayish, not infuscated at their bases, with pale brown veins and stigma.

Described from nine workers and three females taken by Dr. W. M. Mann at Pyramid Lake, Nevada.

This ant might be regarded as an extreme subspecies of *umbratus*, but the worker and female are decidedly smaller even than the corresponding phases of the eastern subsp. *minutus* Emery, the female, indeed, being smaller than that of any other North American *Lasius*. The different proportions of the funicular joints seem to justify a specific name, as the joints 9 and 10 are very distinctly longer. The eyes of the worker are smaller, the promesonotal suture is much less deeply impressed and the mesonotum much less convex and projecting, the mesoëpinotal impression shallower than in *umbratus* and the petiolar border not so sharp.

89. *Lasius* (*Acanthomyops*) *occidentalis* Wheeler.

Colorado: Colorado Springs and Ute Pass (Wheeler).

New Mexico: Pecos and Trout Spring, Gallinas Canyon (T. D. A. Cockerell); Manzanares (Mary Cooper); Albuquerque (W. H. Long).

This small species is not known to occur east of the Rocky Mts. and appears to have the most limited range of any species of the subgenus.

90. *Lasius* (*Acanthomyops*) *murphyi* Forel.

North Carolina: Morganton, type locality (Forel).

New York: Cold Spring Harbor, L. I. and Bronxville (Wheeler).

Ontario: Toronto.

Colorado: Boulder (P. J. Schmitt and T. D. A. Cockerell).

Montana: Helena (W. M. Mann).

This ant appears to belong to the warmer and dryer portions of the Transition Zone and to be rare in all parts of its range. It forms large colonies under stones in open woods.

91. *Lasius* (*Acanthomyops*) *latipes* Walsh.

California: Mt. Tamalpais (C. G. Hewitt); Mountain View.

Washington: Pullman and Wawawai (W. M. Mann); Almota (A. L. Melander); Rock Lake.

Idaho: Julietta (J. M. Aldrich).

Utah: Salt Lake Co. (R. V. Chamberlin).

Colorado: Manitou and Florissant (Wheeler); Boulder (P. J. Schmitt and T. D. A. Cockerell).

New Mexico: Las Vegas (T. D. A. Cockerell); Albuquerque (W. H. Long).

Illinois: Rockford (Wheeler).

Pennsylvania: Enola.

New Jersey: Weasel Mt. and Lakehurst (Wheeler).

New York: Bronxville and White Plains (Wheeler).

Connecticut: Colebrook (Wheeler).

Massachusetts: Franklin and Boston (Wheeler); Needham (A. P. Morse).

As shown by the list of localities, this species has a very wide range, from Boston to San Francisco. It is rather sporadic and nests under large stones in dry fields and pastures.

92. *Lasius* (*Acanthomyops*) *interjectus* Mayr.

Washington: Pullman (W. M. Mann).

Colorado: Manitou, Cheyenne Canyon and Colorado Springs (Wheeler); Longmont (P. J. Schmitt).

New Mexico: Las Valles (Mary Cooper).

Montana: Flathead Lake (C. C. Adams).

Specimens of all three phases from these localities are indistinguishable from those of the Central and Eastern States where the species is much more common. It nests in rather dry, sunny places. The basal border of the mandibles in the worker and female, is distinctly denticulate, a peculiarity which I have not observed in our other species of the subgenus *Acanthomyops*.

93. *Lasius* (*Acanthomyops*) *interjectus* subsp. *californicus* subsp. nov.

*Worker*. Length 2.6–3 mm.

Much smaller than the worker of the typical form, which measures 4–5 mm., with the basal border of the mandibles very indistinctly denticulate and the funicular joints of the antennae a little shorter and the scapes much shorter, extending only slightly beyond the posterior corners of the head. The sides of the head are distinctly less convex. The mesoëpinotal constriction is much feebler and the base of the epinotum much less convex, narrower and rounded. Color, sculpture and pilosity much as in the typical form.

*Female*. Length 7.5 mm.

Somewhat smaller than the typical form and of a different color, the thorax, petiole and gaster being rich reddish castaneous, the head, base of first gastric segment and the appendages red. The infuscation of the bases of the wings is scarcely perceptible. The petiole is much broader and more deeply excised than in the female of the typical *interjectus* and the funicular joints of the antennae are distinctly shorter.

Described from eleven workers taken by myself from a single colony in Palmer's Canyon, San Gabriel Mts., near Claremont, Cala., at an altitude of about 2000 ft., and a single female from the same mountains (F. Grinnell).

94. *Lasius* (*Acanthomyops*) *interjectus* subsp. *coloradensis* subsp. nov.

*Worker.* Very similar to the preceding subspecies and of the same size but with distinctly larger eyes and finer and conspicuously more abundant erect hairs on the head, thorax and especially on the gaster. The proportions of the scape and funicular joints and the shape of the thorax and petiole are the same as in *californicus.*

*Female.* Length 4.5–5 mm.

Decidedly smaller than the female of *californicus,* with the head as well as the thorax deep castaneous. Mandibles, antennae and legs brownish yellow, the femora somewhat infuscated in the middle. Wings grayish hyaline, not darker at the base, the veins and stigma pale. Petiole less deeply emarginate above and much narrower than in *californicus.* Erect hairs on body more abundant.

*Male.* Length 3 mm.

Much smaller and more pilose than the male of the typical *interjectus,* with uniformly grayish, hyaline wings. The superior petiolar border is noticeably blunter and the funicular joints are distinctly shorter.

Described from a dozen workers and as many females taken by myself at Manitou, Colo. (type locality), Aug. 9, 1903, six workers, seven females and two males taken by Mr. E. Bethel at Denver, a single deälated female taken by Prof. Cockerell at Las Vegas, New Mexico and three workers from Manzanares in the same state (Mary Cooper).

95. *Lasius* (*Acanthomyops*) *interjectus* subsp. *arizonicus* subsp. nov.

*Worker.* Length 3.5–4.5 mm.

Larger than *californicus* and *coloradensis* but somewhat smaller than the typical *interjectus* and of a slightly paler yellow color. The proportions of the antennal scape and funicular joints and the shape of the thorax are much the same as in *interjectus,* but the petiole is much smaller and narrower. The eyes are considerably larger and more convex. The erect hairs on the head, thorax and gaster are much fewer, there are usually no hairs on the gula, and the pubescence on the body and especially on the gaster is much shorter than in typical *interjectus* so that the surface appears very glabrous and shining.

I took many workers of this species in Miller Canyon, Huachuca Mts., 5000 ft., during November 1910 and received additional material from Mr. Biedermann, who took it in Carr Canyon, and from Dr. W. M. Mann who took it in Ramsay Canyon in the same mountain range. The subspecies is very easily recognized by its larger eyes, peculiar pilosity and very smooth surface.

96. *Lasius* (*Acanthomyops*) *interjectus* subsp. *mexicanus* Wheeler. I have recently described this subspecies from specimens of all three phases taken by Dr. Mann at Guerrero Mill in the State of Hidalgo, Mexico, at an altitude of 8500–9000 ft.

97. *Lasius* (*Acanthomyops*) *claviger* Roger.

I have seen only a few specimens of this very common eastern species from the west, a worker taken at Old Pecos Pueblo, New Mexico, by Prof. Cockerell and several workers and females taken by Dr. Mann at Helena, Montana. In Massachusetts and Connecticut *claviger* is the most abundant *Acanthomyops*. The small subspecies of *interjectus* described above, show that the species is by no means so distinct from *claviger* as we had supposed. The small form described by Emery from the Eastern States as *claviger* var. *subglaber* should be regarded as a subspecies. I have taken all three phases of it at Rockford, Ill., and on Great Blue Hill, near Boston, Mass., and have seen specimens from the District of Columbia (cotypes from Pergande), Delaware Co., Pa. (Cresson) and Algonquin, Ill. (W. A. Nason). At first sight this form closely resembles *interjectus* subsp. *coloradensis* in size, but the workers and females of this form are much more pilose, the antennal funiculi are distinctly less clavate and the female is much darker.

98. *Formica sanguinea* Latr. subsp. *subnuda* Emery.

British Columbia: Field and Emerald Lake (Wheeler).

Alberta: Lake Louise and Banff (Wheeler).

California: Fallen Leaf Lake and Glen Alpine Springs, near Lake Tahoe (Wheeler); Sugar Pine, Madera County (J. C. Bradley).

Idaho: Troy (W. M. Mann).

Washington: Seattle (Wheeler).

Colorado: San Juan Mts., 12,000 ft., Bullion Peak, Park Co., 13,000 ft., and Gibson's Gulch, Hayden Peak, 10,000 ft. (E. J. Oslar); Tolland (W. W. Robbins).

In my "Revision of the Ants of the Genus Formica" I have cited this ant from many other localities from British Columbia to Connecticut. It is very widely distributed in the Transition and Canadian Zones. Of the numerous colonies seen in California and British America during the summer of 1915 few contained slaves (*F. fusca*).

99. *Formica sanguinea* subsp. *puberula* Emery.

Arizona: Graham Mts. (E. G. Holt).

Recorded from various localities in South Dakota, Colorado, Utah, Washington, Montana, New Mexico, Texas, Missouri and Illinois.

100. *Formica sanguinea* subsp. *obtusopilosa* Emery.

An imperfectly known subspecies described from a single worker taken in New Mexico.

101. *Formica munda* Wheeler.

Known from several localities in Colorado, New Mexico, South Dakota, Montana and Alberta where it occurs below elevations of 7000 ft.

102. *Formica munda* var. *alticola* var. nov.

*Worker*. Length 4.5–5 mm.

Differing from the typical *munda* in having the red portions of the body of a much deeper shade, and the petiole and dorsal portion of the head infuscated. The erect hairs on the head and thorax are distinctly more abundant than in *munda*.

Described from seventeen specimens taken by Mr. E. J. Oslar in Jefferson County, Colorado at an altitude of 9500 ft. This is clearly an alpine variety. One of the specimens is a pseudogyne, with very convex pro- and mesonotum and well-developed scutellum and metanotum, but without traces of wings.

103. *Formica emeryi* Wheeler.

Known only from Broadmoor, near Colorado Springs, Colo.

104. *Formica manni* Wheeler.

Idaho: Boise (A. K. Fisher).

Originally described from several localities in Washington and Owen's Lake, California.

105. *Formica perpilosa* Wheeler.

Occurring at rather low elevations in Colorado, New Mexico, Arizona, Western Texas and Northern Mexico.

106. *Formica bradleyi* Wheeler.

Colorado: San Miguel, 12,000 ft. and Bullion Peak, Park Co. 12,000 ft. (E. J. Oslar).

Montana: Missoula.

Previously known only from workers taken at Georgetown, Colo. and Medicine Hat, Alberta. It is evidently an exclusively alpine species. The two specimens from Missoula are females. They measure nearly 7 mm. and are colored like the worker, red throughout, except the posterior borders of the gastric segments which are fuscous. The surface of the body is subopaque. The petiole is cuneate in profile, broad below, with flat anterior and posterior and rather sharp, emarginate superior border. The notch in the middle of the anterior clypeal border is very distinct. The wings are grayish hyaline, with brown stigma and yellowish brown veins.

107. *Formica rufa* L. subsp. *obscuripes* Forel.

British Columbia: Glacier (Wheeler).

Alberta: Banff (Wheeler).

Manitoba: Treesbank (C. G. Hewitt).

Montana: Beaver Creek (S. J. Hunter).

Wyoming: Cheyenne (Fanny T. Hartman): Rock River (S. J. Hunter).

Colorado: Creede, 8844 ft. (S. J. Hunter); Tolland (W. W. Robbins); Jefferson (A. K. Fisher).

California: Tallac, Lake Tahoe (Wheeler).

Washington: The eastern part of the state (W. M. Mann).

The attempts in my "Revision" to dissipate the confusion in regard to our American forms of *rufa*, prove to have been unsuccessful and I must here make another attempt. I believed that I could recognize four forms of this species, the subsp. *obscuripes* Forel, the subsp. *aggerans* (a new name for Emery's *rubiginosa* (*nom. praeocc.*)), the var. *melanotica* Emery and a var. *whymperi* described by Forel as belonging to *obscuripes* but at the time unknown to me. I have since found this variety in British Columbia and am able to state that it does not belong to *rufa* or *obscuripes* but is a distinct species of the *microgyna* group and is very close to the form I described as *F. adamsi* (*vide infra*). The var. *melanotica* is a very definite color variety of *obscuripes* and need not be discussed. Doubt remains then only in regard to the typical *obscuripes* and *aggerans*. Although I studied much material from a number of localities I was unable to distinguish these forms satisfactorily for the reason that both were inadequately described by Emery and Forel and because the latter published conflicting descriptions of *obscuripes*. His original description (C. R. Soc. Ent. Belg. 1886, p. 2) runs as follows: "Ouvrière. Long. 3.8–8 mm. Très semblable a la *F. rufa i. spec.* d'Europe. Mais elle est plus petite; les grandes ouvrières sont d'un rouge plus clair et presque ou entièrement sans tache sur la tête et le thorax, tandis que les pattes et l'ecaille sont d'un brun noirâtre. Les petites ouvrières sont beaucoup plus foncées et tachées de brun sur la tête et le thorax. L'abdomen est mat, noir, et a une pubescence grise un peu plus forte que chez la *F. rufa i. spec.*, tandis que la pilosité est plutôt un peu plus faible.—Green River, Wyoming (Scudder)." Recurring to this form in connection with his description of *whymperi* (Ann. Soc. Ent. Belg. 48, 1904, p. 152) Forel says: "Emery rattache *l'obscuripes* comme variété a *l'obscuriventris*. Mais *l'obscuriventris* est beaucoup plus poilue et a les tibias garnis de poils dressés, ce qui n'est pas le cas de *l'obscuripes*, dont les tibias n'ont qu' une pubescence adjacente et dont le corps n'a que très peu de poils dressés, beaucoup moins que chez la *pratensis* d'Europe et même que chez la *rufa* typique. Je maintiens donc *l'obscuripes* comme race ou sous-espèce distincte."

Finding that there was considerable variation in the pilosity of my series of *rufa* from different colonies and localities and relying on these descriptions, I naturally referred the more pilose forms to Emery's *rubiginosa* and the less pilose forms to Forel's *obscuripes*. Since the publication of my "Revision" Forel returns to the subject with the following statement (Deutsch. Ent. Zeitschr. 1914, p. 619):

"Ich habe leider bei Gelegenheit der Beschreibung dieser Rasse (*obscuripes*) und bei derjenigen der var. *whymperi* For. die Tatsache übersehen, dass die Originaltypen der *obscuripes*, die ich noch in Anzahl besitze, nicht, wie ich angegeben hatte, ohne abstehende Haare, sondern sehr deutlich, obwohl meistens spärlich abstehend behaart sind. Die Behaarung wechselt, wie Wheeler von *aggerans* angibt, ziemlich stark bei den verschiedenen Individuen und ich hatte bei der Beschreibung ein solches angesehen, wo sie fast oder ganz fehlten, ebenfals ist der Hinterleib matt mit feiner rauher Pubescens, so dass ich mit dem besten Willen keinen Unterschied zwischen *obscuripes* und *aggerans* finden kann und diese beiden Formennamen als synonym betrachten muss; auch die Haare des Hinterleibs sind gleich. Dagegen bleiben die v. *melanotica* Em. und *whymperi* For. (letztere ohne abstehende Haare) bestehen.—Lake Tahoe, Kalifornien."

As I collected many specimens of this ant at Lake Tahoe I am undoubtedly in possession of specimens which Forel now pronounces to be his *obscuripes*. These also undoubtedly belong to the form I called *aggerans* in my "Revision." They are much more hairy than the European *rufa* and if the Wyoming types of *obscuripes* had been like these it is difficult to see how Forel could have overlooked their striking pilosity and have penned the two descriptions above quoted. Pending a more exhaustive study I am willing, however, to attribute both forms to *obscuripes* and to regard it as a subspecies in which there is considerable variability in pilosity (as in the typical *rufa* of Europe). But as the term *aggerans* was suggested to replace Emery's *rubiginosa* (a preoccupied name), we have still to determine what Emery meant by this form. As he possessed cotypes of *obscuripes* when he wrote the description of *rubiginosa*, the latter was evidently something different. After carefully rereading Emery's description I conclude that he must have had a distinct variety of *obscuriventris* which I believe I am now able to recognize (*vide infra* under the forms of *truncicola*).[3]

108. *Formica rufa* subsp. *obscuripes* var. *melanotica* Emery.
Colorado: Boulder (W. W. Robbins).
Washington: Tacoma (C. C. Adams).
Alberta: Pincher and Lethbridge (C. G. Hewitt).
Manitoba: Treesbank (C. G. Hewitt).

This form, which I have recorded also from Wisconsin, Illinois, South Dakota, Nebraska, Wyoming, Washington and British Columbia, is merely a very dark form of *obscuripes* with only the head red in the largest workers. The true *obscuripes* does not range eastward of the Rocky Mountains.

109. *Formica truncicola* Nyl. subsp. *integroides* Emery.
California: Lake Tahoe (Wheeler).

Recorded previously from several localities both in the Coast Range and in the Sierras of California. The colonies which I saw in large pine logs and stumps near Tallac and Fallen Leaf Lake were very populous like those of the vars. *haemorrhoidalis* and *coloradensis* in the Rocky Mts. The nests were banked with considerable quantities of vegetable detritus.

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[3] Since this paragraph was written I have found two worker cotypes of *F. obscuripes* from Green River, Wyo. in the Pergande Collection (U. S. Nat. Mus.). They have very distinct suberect hairs on the legs, but the pubescence on the gaster seems to be shorter and finer than in workers from Lake Tahoe.

110. *Formica truncicola* subsp. *integroides* var. *tahoënsis* var. nov.

*Worker.* Length 4–6 mm.

Resembling the subsp. *integroides* but differing in its decidedly smaller average size, in the shape of the head, larger eyes, pilosity and coloration. The head is narrower in front with much less convex, anteriorly converging cheeks and straight posterior border, so that it is distinctly trapezoidal with less rounded posterior corners. Surface of body opaque, mandibles lustrous, frontal area shining. The erect hairs are extremely sparse on the head and thorax, usually restricted to a few scattered hairs on the clypeus and upper surface of the head. Pubescence grayish, abundant, especially on the gaster. Gaster black, with red anal region. In large workers the head, thorax, petiole and appendages are red, with the apical half of the antennal funiculus infuscated, but often even the largest workers have the ocellar triangle, the upper portion of the petiole and a spot on the pro- and mesonotum blackish and the coxae, femora and tibiae dark red or fuscous. The infuscation of the red portions of the body may be even more extensive in small individuals.

*Female.* Length 8.5–9.5 mm.

Differing from the female *integroides* in lacking erect hairs on the upper surface of the thorax and pedicel, the gula and posterior portions of the head. The head, thorax, petiole and appendages are uniformly red in some specimens, in others the metanotum and posterior portion of the scutellum are black and there may be three elongate black blotches on the mesonotum or only a single anteromedian blotch. Wings grayish hyaline, with their basal halves distinctly infuscated; veins and stigma brown.

Described from numerous workers and four deälated females taken from several colonies near Lake Tahoe, Cala. (Tallac, Glen Alpine Springs, Fallen Leaf Lake, Angora Peak), and a single female taken by Prof. C. F. Baker in Ormsby County, Nevada. In the almost complete absence of erect hairs on the head, thorax and petiole, this variety resembles the var. *haemorrhoidalis*, but the worker averages distinctly smaller, has a differently shaped head and the smaller and even some of the larger workers are more or less spotted with black. It is less hairy than the typical *integroides* and the gaster is darker. Like the other forms of the subspecies it forms large colonies in stumps and logs which it banks with much vegetable detritus.

111. *Formica truncicola* subsp. *integroides* var. *propinqua* var. nov.

*Worker.* Very similar to the preceding both in size and coloration, but with more numerous erect hairs on the head, gula, thoracic dorsum and petiolar border than even in the typical *integroides*, and these hairs are coarse and blunt on the thorax. The head is a little less narrowed in front than in the var. *tahoënsis*, the cheeks feebly convex, the eyes of the same size.

Described from numerous specimens taken from several colonies in the same localities as the var. *tahoënsis*. As the differences of pilosity of these two varieties appear to be constant throughout the colonies, it seems necessary to regard them as distinct. It is not improbable that they really belong to different altitudes, *propinqua* preferring the hot moraine region (6000 ft.) between Fallen Leaf Lake and Lake Tahoe and *tahoënsis* the greater elevations (7000–7500 ft.), but as I did not distinguish the two forms in the field, owing to their great similarity in size and color, I am unable to make a positive statement on this matter.

112. *Formica truncicola* subsp. *integroides* var. *coloradensis* Wheeler.

Colorado: Bullion Peak, Park County 12,000 ft., Gibson's Peak, 10,000 ft. and Wilson Peak, 13,000 ft. (E. J. Oslar); Creede 8844 ft. (S. J. Hunter); Tolland (W. W. Robbins).

Known also from other localities in Colorado, from New Mexico and Idaho.

113. *Formica truncicola* subsp. *integroides* var. *haemorrhoidalis* Emery.

Colorado: Creede, 8844 ft. (S. J. Hunter).

This form seems to be more widely distributed than *coloradensis* as it is known to occur in Colorado, South Dakota, Idaho, Nevada and Washington.

114. *Formica truncicola* subsp. *integroides* var. *ravida* Wheeler.

Known only from Elkhorn and Helena, Montana (W. M. Mann).

115. *Formica truncicola* subsp. *integroides* var. *subfasciata* var. nov.

*Worker*. Length 6–8.5 mm.

Averaging considerably larger and apparently much more feebly polymorphic than the typical *integroides*. Erect hairs on the upper surface of the head, thorax and petiole much less numerous. The red color of the body is paler and clearer and without traces of infuscation even in the smaller workers. Gaster with the base of each of the segments dull red. Tips of antennal funiculi scarcely infuscated. Anus red as in the other forms of the subspecies.

Described from numerous workers taken by Mr. Fordyce Grinnell in Mill Creek Canyon, Wilson Peak, 7500 ft., San Bernardino Mts., Southern California.

116. *Formica truncicola* subsp. *integra* Nyl. var. *subcaviceps* var. nov.

*Worker.* Length: 6–7.5 mm.

Differing from the typical *integra* in the following characters: Posterior border of head in largest workers more deeply excavated, almost as deeply as in *F. exsectoides* Forel. Whole body and especially the gaster more opaque. Gula and posterior corners of head with numerous, delicate, short, erect hairs. Smallest workers distinctly infuscated, entirely dark brown. Median workers with darker legs and petiolar border.

*Male.* Length 7 mm.

Differing from the male *integra* in having the petiole more compressed, with sharp, broadly excavated superior border, the head, thorax and petiole covered with abundant, short, delicate, black hairs and in the coloration of the legs and wings. The femora are black, the tibiae and metatarsi yellow, with indications of infuscation in the middle of the fore and middle tibiae. Wings distinctly paler than in the typical *integra.*

Described from a single male and three workers from Medford, Oregon (C. M. Keyes) and two dozen workers taken by Dr. C. G. Hewitt at Dog Lake, Penticton, British Columbia. Six workers taken by Dr. W. M. Mann on San Juan Island, Washington, belong to the same variety but have the upper border of the petiole sharper and more compressed anteroposteriorly.

117. *Formica truncicola* subsp. *mucescens* Wheeler.

Colorado: various localities between 7000 and 8000 ft.

118. *Formica truncicola* subsp. *obscuriventris* Mayr.

Montana: Flathead Lake (C. C. Adams).

Also known from the Eastern and Central States, Ontario, Colorado and British Columbia.

119. *Formica truncicola* subsp. *obscuriventris* var. *aggerans* Wheeler.

British Columbia: Emerald Lake (Wheeler); Carbonate, Columbia R. 2600 ft. (J. C. Bradley).

Utah: Promontory Point (A. Wetmore).

Series of workers from these localities agree perfectly with Emery's very brief description of the var. *rubiginosa.* The only difference I can detect between this form and the typical *obscuriventris* is the greater infuscation of the thorax and petiole of even the largest workers. Emery records the form from Nebraska, Colorado and Dakota.

120. *Formica foreliana* Wheeler.

Known only from the Huachuca Mts., Arizona, where it was taken by Mr. C. R. Beidermann at altitudes between 4500 and 5600 feet.

121. *Formica ciliata* Mayr.

Colorado: West Cliff, 7864 ft. (T. D. A. Cockerell).

Recorded from various localities in Colorado and Montana.

Several workers taken by Mr. E. J. Oslar in the San Miguel Mts. of Colorado at an elevation of 11,000 ft. seem to belong to this species, but the head is covered with short erect hairs and not naked as in typical ciliata. As there is a possibility that these specimens may represent a new species with aberrant female form like *ciliata, comata, criniventris*, etc., I await further material before introducing another name.

122. *Formica comata* Wheeler.

Known from Colorado and South Dakota.

123. *Formica criniventris* Wheeler.

Recorded from Boulder, Colo. and Helena, Montana.

124. *Formica oreas* Wheeler.

Taken in various localities in Colorado and New Mexico.

125. *Formica oreas* Wheeler var. *comptula* Wheeler.

Known from Pullman, Washington and Elkhorn, Montana.

126. *Formica dakotensis* Emery.

Alberta: Banff (Wheeler).

Colorado: Creede, 8844 ft. (S. J. Hunter).

Previously known from South Dakota, British Columbia and Nova Scotia.

127. *Formica dakotensis* var. *montigena* Wheeler.

Montana: Nigger Hill, Powell Co. (W. M. Mann).

Recorded from Colorado, New Mexico, Montana and Idaho.

128. *Formica dakotensis* var. *saturata* var. nov.

*Worker*. Length 4.5–5 mm.

Averaging a little smaller than the other forms of the species and of a much deeper color, the head, thorax, petiole and appendages being rich blackish red, nearly as dark as the gaster, the cheeks and anterior portion of the head sometimes a little paler. The pilosity is like that of the typical *dakotensis*, the erect hairs being exceeding scarce on the head and thorax and lacking on the gula.

Described from a dozen workers taken by Dr. W. M. Mann at Helena, Montana.

129. *Formica microgyna* Wheeler.

Known only from Manitou and Florissant, Colo. (7000–8100 ft.).

130. *Formica microgyna* var. *recidiva* Wheeler.

Colorado and New Mexico.

131. *Formica microgyna* subsp. *rasilis* Wheeler.

Colorado: Buena Vista (Wheeler).

Recorded previously from several localities in Colorado, New Mexico, Utah and Washington.

132. *Formica microgyna* subsp. *rasilis* var. *spicata* Wheeler.

Known only from Florissant, Colorado, 8100 ft.

133. *Formica microgyna* subsp. *rasilis* var. *pullula* Wheeler.

Taken at Flathead Lake, Montana, by Prof. C. C. Adams.

134. *Formica microgyna* subsp. *rasilis* var. *nahua* Wheeler.

Taken by Dr. W. M. Mann at Guerrero Mill (9000 ft.) and Velasco in Hidalgo, Mexico.

135. *Formica microgyna* subsp. *rasilis* var. *pinetorum* var. nov.

*Worker*. Length 3.5–6 mm.

Very similar to the var. *spicata* but differing in the darker, more blackish gaster, its much more abundant, obtuse hairs and the greater tendency to infuscation of the red regions of the body in the large and median workers. In the latter the ocellar region is black and there is a very distinct, elongate triangular black spot on the mesonotum, with dark clouds on the pronotum and occiput. In small workers the infuscation is more extensive on the head and pronotum and the legs and antennae are darker. The red of the body in larger workers is rather pale and yellowish. The erect hairs are coarse and obtuse and are present on the gula, where none appears in *spicata.*

Described from numerous specimens taken from several colonies on Angora Peak, near Lake Tahoe, California, between 7500 and 8600 ft. These colonies were rather populous and were living under stones and logs banked with vegetable detritus.

136. *Formica microgyna* subsp. *californica* subsp. nov.

*Worker.* Length 3.5–6.5 mm.

Differing from the other forms of *microgyna* in the sculpture of the integument and in pilosity. The surface of the head, thorax and petiole is so finely and superficially shagreened as to be distinctly shining, and the gaster resembles that of *dakotensis* and *obscuriventris* though more opaque. The pubescence is very short and indistinct and there are no erect hairs on the head, thorax and petiole and only a few on the clypeus. The erect hairs on the gaster are blunt, yellow and sparse. Large workers have the head, thorax, petiole and appendages uniformly red, the gaster black; the median workers have traces of infuscation on the ocellar region, and mesonotum. Small workers have the head and petiole above extensively blackened, the thorax clouded with black even on the sides and the coxae and legs, except the knees and tarsi, fuscous. The petiolar node has a sharp border and in many of the small workers is produced upward as a blunt point in the middle.

Described from numerous workers taken at Glen A'pine Springs, near Lake Tahoe, Cala. (6500 ft.). From one colony of this subspecies on July 26 I took a number of diminutive females resembling those of *microgyna* subsp. *rasilis,* but unfortunately the vial containing them was lost.

137. *Formica microgyna* subsp. *californica* var. *hybrida* var. nov.

*Worker.* Length 3.5–6.5 mm.

Intermediate between the typical *californica* and the var. *pinetorum,* the color and sculpture being that of the former, the pilosity that of the latter, the pubescence, especially on the gaster, being intermediate.

Numerous workers from several colonies found in the same localities as the var. *pinetorum* on Angora Peak near Lake Tahoe. These may represent a true hybrid form but as I have no proof of their genetic origin, I have preferred to give them a varietal name.

138. *Formica whymperi* Forel.

This form, as above stated, was described by Forel as a mere variety of *F. rufa obscuripes*. During August 1915 I found several colonies of it on the shores of Emerald Lake in British Columbia. It evidently belongs to the *microgyna* group and is specifically the same as my *F. adamsi* described from Isle Royale, Mich. The colonies are rather small and nest under stones and logs which they bank with accumulations of vegetable detritus. The worker of the form which I take to be the same as the type is larger than *adamsi*, measuring 3.5–6 mm. The petiole is blunter and thicker and is produced upward in a blunt point, the hairs on the head and thorax are somewhat less numerous, the dark portion of the gaster is black and not dark brown as in *adamsi* and the black markings of the head and thorax are more pronounced and more sharply outlined in the large workers. *F. adamsi* is, therefore, to be retained as a variety of *whymperi*.

139. *Formica whymperi* var. *alpina* Wheeler.

This variety must also be referred to *whymperi*. I have recorded it from Pikes Peak, 10500–11000 ft., (type locality), Troy, Idaho and Cape Breton Island, but further examination leads me to doubt whether the specimens from the two latter localities really belong to this form. I am not even certain that *whymperi* and *microgyna* are specifically distinct. Both of these forms, with their subspecies and varieties constitute a very difficult complex which can be satisfactorily analyzed only with the aid of more material and with a better knowledge of the males and females than we possess at present.

140. *Formica nevadensis* Wheeler.

Known only from a single female taken in Ormsby County, Nevada by Prof. C. F. Baker. As this county is on the eastern shore of Lake Tahoe we might expect the specimen to be the female of one of the three Californian forms of *microgyna* described above, but this cannot be the case owing to the peculiar abundant pilosity on the body and antennal scapes of the Nevada specimen and the very smooth and shining gaster.

141. *Formica exsectoides* Forel var. *hesperia* Wheeler.

Known only from the vicinity of Colorado Springs.

142. *Formica exsectoides* subsp. *opaciventris* Emery.

Colorado: Creede, 8844 ft. (S. J. Hunter).

Wyoming: Yellowstone National Park (J. C. Bradley).

Montana: Beaver Creek, 6300 ft. (S. J. Hunter).
Previously recorded only from Colorado.

143. *Formica ulkei* Emery.
Manitoba: Treesbank (C. G. Hewitt).
Recorded from South Dakota, Illinois, Nova Scotia and New Brunswick.

144. *Formica fusca* L.
British Columbia: Field and Emerald Lake (Wheeler).
Alberta: McLeod and Jasper (C. G. Hewitt); Lake Louise, Moraine Lake in the Valley of the Ten Peaks, and Banff (Wheeler).
Manitoba: Aweme (N. Criddle).
Washington: Mt. Renier (J. C. Bradley).
California: Kern Lake (J. C. Bradley); Lake Tahoe, 6200–9000 ft. (Wheeler); Camp Curry and Glacier Point, Yosemite, 4000–8000 ft. and Muir Woods, Mt. Tamaplais (Wheeler).
Arizona: San Francisco Mts., 12,000 ft. (W. M. Mann).
Colorado: Creede, 8844 ft. (S. J. Hunter); Chimney Gulch, Golden 14,000 ft.; Bullion Peak, Park County, 14,000 ft.; Clear Creek, Jefferson County, 9500 ft. and Wilson's Peak, 12,000–14,000 ft. (E. J. Oslar).
Montana: Flathead Lake (C. C. Adams); Beaver Creek, 6300 ft. (S. J. Hunter).
Wyoming: Yellowstone National Park (J. C. Bradley).

In my "Revision" I have given a long list of localities of this species, which is the most eurythermal and therefore the most widely distributed of all the species of *Formica* in North America as well as in Eurasia. In the Western States it varies considerably in size and pubescence and in the coloration of the legs and antennae, but I deem it inexpedient to give these varieties names at the present time. Many of them seem to represent transitions between the typical form and the following four varieties:

145. *Formica fusca* var. *subsericea* Say.
I am not sure that this form occurs in the Western States. Specimens referred to this variety on account of their more abundant pubescence are cited from Arizona and Colorado in my "Revision."

146. *Formica fusca* var. *argentea* Wheeler.
Oregon: Ashland (W. Taverner).
California: Angora Peak near Lake Tahoe, 7500–8600 ft. (Wheeler).

Washington: Seattle (Wheeler); Pullman (W. M. Mann); Mt. Renier (J. C. Bradley).

Idaho: Boisé (A. K. Fisher).

Arizona: Graham Mts. (E. G. Holt); San Francisco Mts., (A. K. Fisher).

Colorado: Salida (Wheeler).

Widely distributed through the Transition Zone from the Pacific to the Atlantic Coast.

147. *Formica fusca* var. *marcida* Wheeler.

California: Summit of Angora Peak, near Lake Tahoe, 8650 ft. (Wheeler).

I have recorded this variety from British Columbia, Alberta, Manitoba and Washington. It is a small, depauperate, alpine form. On the bare summit of Angora Peak I found it nesting in little craters in spots from which the snow had recently receded (July 26th). The colonies were small and the ants very active. Prof. Bradley took this variety under very similar conditions at Moraine Lake in the Valley of the Ten Peaks, Alberta. The female measures only 6.5 mm. and has the gaster much more pubescent and much less shining than in the typical *fusca*.

148. *Formica fusca* var. *subaenescens* Emery.

California: Angora Peak, Lake Tahoe, 8600 ft. (Wheeler).

Previously recorded from California, Washington, Idaho, Utah, Colorado, New Mexico, Montana, Alberta, British Columbia and portions of the Middle and Atlantic States.

149. *Formica fusca* var. *gelida* Wheeler.

Recorded from Colorado, New Mexico, Arizona, California, Oregon, Washington, Alaska, British Columbia, Alberta and Saskatchewan.

This is an alpine variety, probably the most stenothermal of all the varieties of *fusca*. In my "Revision" I cited it also from Ontario, Quebec, Labrador, Newfoundland, Nova Scotia, Michigan and New Hampshire, but I have recently shown (Psyche, Dec. 1915, p. 205) that the specimens with this more eastern distribution really constitute a distinct variety, which I have described as var. *algida*.

150. *Formica fusca* var. *neorufibarbis* Emery.

British Columbia: Glacier, Field and Emerald Lake (Wheeler).

Alberta: Lake Louise, Moraine Lake in the Valley of the Ten Peaks (Wheeler); Jasper (C. G. Hewitt).

Washington: Mt. Renier (J. C. Bradley).

California: Lake Tahoe, 6000–7000 ft. and Glacier Point, Yosemite (Wheeler).

Previously cited from various localities in South Dakota, Utah, Montana, Idaho, Oregon, Washington, British Columbia and Alberta.

I found many colonies in the localities above recorded and secured all three phases so that I am able to improve on the description given in my "Revision."

The workers vary greatly in size in each colony, from 3.5–5.5 mm. The largest have the head, including the clypeus, palpi and antennae black, with the scapes, cheeks and mandibles deep red or castaneous, the gaster black and shining, with very short grayish pubescence, the thorax, petiole and legs opaque and immaculate red. The medium-sized workers have a black spot on the pro- and one on the mesonotum; the smallest workers have the whole thorax, petiole and legs dark brown. The petiole is broad, much compressed anteroposteriorly, with broadly rounded, rather sharp border.

The female measures 7–8 mm. and is colored like the worker, except that the sides of the pronotum and the pleurae are clouded with fuscous and the metanotum and posterior border of the scutellum and three large, elongate blotches on the mesonotum are black. The thorax is as shining as the head. The petiole is very broad, very much compressed anteroposteriorly, with flat anterior and posterior surfaces and sharp, broadly rounded superior border. Wings clear grayish hyaline with pale brown veins and dark brown stigma.

The male measures 7–7.5 mm. and is larger than the male of *gelida*. It differs also in having the legs rich yellow, with the base of the femora slightly infuscated and the gaster more shining, with much shorter pubescence and much fewer erect hairs on the head and thorax.

This ant nests by preference in old logs in hot sunny places, but both at Lake Tahoe and in British Columbia I often found it nesting under stones. Several pseudogynes were taken in both localities. They are small (3–4 mm.) and have the dorsal surface of the pro- and mesonotum and the scutellum black.

The specimens cited in my "Revision" as belonging to *gelida* from Blue Lake, Humboldt Co., and Alta Peak, Cala. and from Kassiloff Lake, Kenai Peninsula, Alaska, have the color of the var. *neorufibarbis* but the pubescence of *gelida*. They may be regarded as representing a form intermediate between the two varieties. Owing to its constancy and the pronounced variation in size of the workers of the same colony, *neorufibarbis* should, perhaps, rank as a subspecies.

151. *Formica fusca* var. *neoclara* Emery.

Colorado: Creede, 8844 ft. (S. J. Hunter); Hall's Valley, Park Co., 10,500 ft. and Gibson's Gulch, Hayden Peak, 12,000 ft. (E. J. Oslar).

Recorded only from localities in Colorado. As I have never seen this ant above 7000–8000 ft. the elevations given on Oslar's labels seem excessive. The variety really belongs to higher levels in what Cockerell calls the "sub-alpine zone."

152. *Formica fusca* var. *blanda* Wheeler.

The types of this variety are from Olympia, Washington. Further study shows that the specimens cited in my "Revision" from Seattle, Wash. and Lemon Cove, Tulare Co., Cala. do not belong to it but are pale forms of *cinerea* (*vide infra*). The two workers from the Yosemite are also doubtful as they may be immature specimens of *fusca* var. *marcida*.

153. *Formica fusca* subsp. *pruinosa* subsp. nov.

*Worker*. Length 3.5–4 mm.

Differing but little in size in the same colony and allied to var. *neoclara*, but with narrower, less flattened gaster. The petiole is similar, with broad, blunt superior border, nearly always distinctly emarginate in the middle. Head scarcely longer than broad, narrowed in front, with straight sides and posterior border. Eyes rather large. Epinotum obtusely angular, with subequal base and declivity.

Surface of body finely shagreened, uniformly shining, except the clypeus and anterior portion of the head, which are coarsely shagreened and opaque.

Whole body uniformly covered with very short, dense, silvery pubescence. Head with only a few pairs of erect hairs on its dorsal surface, thorax and petiole without any; gaster with short, sparse, obtuse hairs.

Gaster dark brown, head black; clypeus and mandibles dark brown, cheeks yellowish brown; thorax, coxae and legs yellowish or reddish brown, the thorax and coxae spotted with dark brown, the spots sometimes fusing so that only the sutures are yellowish or reddish. Petiole often infuscated above. Antennal scapes and base of funiculi red, the tip darker.

*Female*. Length 6.5–7 mm.

Color, pilosity and sculpture much as in the worker; frontal area yellowish red; pronotum of the same color, with its posterior border and a few spots on the sides fuscous; remainder of thorax fuscous or blackish, with a few small reddish spots at the anterior end of the mesonotum and on the pleurae. Coxae yellowish red, like the legs. Thorax with erect hairs on the dorsal surface; those on the gaster longer than in the worker and pointed. Petiole broad, much compressed anteroposteriorly, its anterior and posterior surfaces flat, its superior border transverse and emarginate in the middle. Gaster long and narrow, more than twice as long as broad. Wings grayish hyaline, with pale brown veins and dark brown stigma.

*Male.* Length 7–7.5 mm.

Very similar to the male of the var. *neoclara* but with the coxae and bases of the femora black, the external genitalia more infuscated and the thorax more robust and broader through the mesonotum. Wings like those of the female but with darker veins.

Described from many workers, two males and three females taken from several nests at Emerald Lake, British Columbia, Aug. 12–15 (type locality), a deälated female and several workers from Field, B. C., numerous workers from a single colony which I found at Banff, Alberta and three workers taken at Beaver Creek, Montana (6300 ft.) by Dr. S. J. Hunter.

This form may represent a distinct species, but as the following variety and the var. *blanda* seem to connect it with the var. *neoclara* I have preferred to regard it, at least provisionally, as a subspecies of *fusca*. At Emerald Lake the colonies of *pruinosa* were found only on the open flat delta at the north end of the lake in a rather moist spot traversed by the icy streams from the Emerald Glacier. The nests were peculiar, being small, loose mounds of spruce needles 8 to 12 inches in diameter and of about the same height, built about the trunks of the scattered and stunted bushes. The colonies were very populous. The two seen at Field and Banff were nesting under stones, in the latter locality at the base of Tunnel Mt.

154. *Formica fusca* subsp. *pruinosa* var. *lutescens* var. nov.

*Worker.* Differing from the typical *pruinosa* in color and in averaging a little smaller. Body and appendages pale brownish yellow, the gaster pale brown, the head behind the frontal area dark brown, the thorax and coxae spotted with pale brown. Antennae scarcely infuscated at the tip. The petiole is narrower and its superior border distinctly blunter than in the typical *pruinosa*, though usually emarginate in the middle. There are no differences in sculpture and pilosity.

Described from numerous workers taken by Dr. W. M. Mann at

Wawawai (type locality), Kiona and Ellensburg, Washington. The specimens from Ellensburg are somewhat darker, with the thorax and petiole uniformly pale brown like the gaster, thus representing a transition to the var. *blanda*.

155. *Formica rufibarbis* Fabr. var. *occidua* Wheeler.

California: Berkeley (Wheeler).

Recorded from many localities in the Coast Range of California and from Wawawai, Washington. As I did not find this ant in the Yosemite or about Lake Tahoe, I infer that it does not occur in the Sierras, at least at elevations above 4000 ft. or east of California.

156. *Formica rufibarbis* var. *gnava* Buckley.

Recorded from various localities in Texas, New Mexico, Arizona, Southern California, Colorado, Utah and Mexico, as far south as the State of Hidalgo. In the northern portion of its range this variety occurs only at low altitudes in warm, shady canyons.

157. *Formica cinerea* Mayr var. *altipetens* Wheeler.

Colorado: Chimney Gulch, Golden, 9500 ft. (E. J. Oslar).

Montana: Beaver Creek, 6300 ft. (S. J. Hunter).

Previously known from Florissant and Cheyenne Mt., Colo., where I found it at elevations between 7000 and 8200 ft., and Pachuca in Hidalgo, Mexico, 9000 ft.

158. *Formica cinerea* var. *neocinerea* Wheeler.

Illinois: Hyde Park, Chicago (Wheeler).

Recorded from Illinois, Indiana, South Dakota, Colorado and California.

159. *Formica cinerea* var. *canadensis* Santschi.

"*Worker*. Length 4.5–6 mm.

Black. Anterior portion of head, antennae, excepting the terminal funicular joints, legs, excepting the coxae and often the middle of the femora, base of the petiole brownish red. Pubescence a little less abundant than in the type. Epinotum a little more angular. Petiole as in the var. *neocinerea* Wheeler, from which it differs, as also from the var. *altipetens* Wheeler, in the entirely black color of the thorax, which makes it resemble *F. fusca* L. var. *subaenescens* Emery.

*Female*. Length 9–9.5 mm.

The front of the head is nearly black; all the remainder of the body black; antennae and legs as in the worker.

Canada: Saskatchewan (Frey 1909), five workers and six females."

As I have not seen this variety, I quote Santschi's description (Ann. Soc. Ent. Belg. 57, 1913, p. 435). In certain particulars it seems to resemble the form described below as *F. hewitti*, but I cannot suppose that Santschi would have described this form as a variety of *cinerea*.

160. *Formica cinerea* var. *lepida* Wheeler.

Lower California: La Ensenada (F. X. Williams).

California: Lemon Cove, Tulare Co. and Blue Lake, Humboldt Co. (J. C. Bradley).

Washington: Seattle (T. Kincaid and Wheeler).

I found workers of this variety running on the sidewalks in Seattle. Two deälated females taken in the same city by Kincaid measure nearly 9 mm. They are colored like the workers, with the sutures and parapsidal furrows of the thorax blackish.

161. *Formica cinerea* subsp. *pilicornis* Emery.

California: Jacumba (J. C. Bradley).

Recorded only from numerous localities in the Coast Range of California, from San Francisco to San Diego.

162. *Formica sibylla* Wheeler.

California: Yosemite Valley, from Yosemite Village, 4000 ft., to Glacier Point, 8000 ft., and Tallac, Fallen Leaf Lake and the Moraine east of Angora Peak, near Lake Tahoe (Wheeler).

The types of this interesting species were taken by Prof. C. F. Baker in King's Canyon, Ormsby County, Nevada, on the eastern shore of Lake Tahoe. In the Californian localities above cited I saw numerous colonies, each comprising a rather small number of workers and nesting in craters 6 to 8 inches in diameter in sandy soil fully exposed to the sun. The workers, which run very rapidly, were seen outside the nests only during the early morning and late afternoon hours of the hot days of July and August. I failed to secure the hitherto unknown female. In life the worker has a peculiar bronzy appearance owing to the dense and rather long grayish yellow pubescence covering the whole body. It is readily distinguished from the forms of *fusca* by the numerous long, erect hairs on the gula, and from the forms of *cinerea* by the absence of erect hairs on the thorax and petiole, their sparse development on the head and gaster, the less angular epinotum, more slender antennae and less curved scapes.

163. *Formica hewitti* sp. nov.

*Worker.* Length 5–6 mm.

Resembling a large *F. fusca.* Head a little longer than broad, narrower in front than behind, with straight sides and posterior border. Clypeus sharply carinate, its anterior border produced, rounded, entire. Frontal carinae straight, diverging behind. Antennae as in *F. fusca.* Epinotum in profile with subequal, straight base and declivity, meeting at a pronounced obtuse angle. Petiole convex in front, flat behind, very broad, its superior margin straight and truncated, rather sharp, its sides straight, converging below. Gaster large, elongate.

Head in front and thorax coarsely shagreened. Cheeks with elongate punctures. Mandibles lustrous, densely striate-punctate. Frontal area smooth and shining. Thorax opaque; head somewhat shining, especially behind; gaster more shining, very finely shagreened.

Hairs yellowish, erect, sparse on the upper surface of the head, one or more pairs also on the gula. Upper surface of pro- and mesonotum with numerous short, obtuse hairs. Those on the gaster short, obtuse. Legs naked, except for several erect hairs on the flexor surfaces of the femora. Pubescence grayish, short, uniform over the whole body, conspicuously long, but sparse on the sides of the gula.

Black; thorax dark brown or piceous, mandibles, scapes, three basal joints of funiculus, petiole, coxae and legs deep red.

*Female.* Length 7 mm.

Very similar to the worker in sculpture, color and pilosity, except that the thorax and petiole are black and the mesonotum and scutellum are shining. The petiole is very broad and compressed anteroposteriorly, its superior border rather sharp, straight and entire. The erect hairs on the mesonotum and scutellum are longer than in the worker and pointed. Wings colorless, with pale yellowish veins and pale brown stigma.

*Male.* Length 6.5 mm.

Closely resembling the male of the typical *fusca,* but the head, thorax and petiole are much more pilose and with a few erect hairs on the gula, the antennae and mandibles are entirely black, as are also the coxae and basal halves of the femora. The wings are clearer, with paler veins and stigma. The petiole is somewhat broader, with a much more compressed and more deeply emarginate superior border.

This species, dedicated to my friend Dr. C. Gordon Hewitt, is described from numerous workers, three females and a single male which I took from several nests under large stones at Emerald Lake (type locality), at Field, British Columbia, and at Laggan, Alberta. Three workers taken by Prof. C. C. Adams at Flathead Lake, Montana, and three workers taken by Dr. S. J. Hunter at Beaver Creek, 6300 ft. in the same state also belong to this species, though the thorax in the former is more reddish and the cheeks and clypeus are deep red. Some workers from a single colony at Field seem to represent a hybrid form between *hewitti* and the typical *fusca*. Only an occasional worker in this series has one or two erect hairs on the gula and the color, pubescence and sculpture is more like certain forms of *fusca* that are intermediate between the type and the var. *subsericea*. Like the following species and *F. sibylla, hewitti* is a puzzling form, since, owing to its peculiar pilosity, it cannot be assigned either to *fusca* or to *cinerea*.

164. *Formica subcyanea* Wheeler.

This species is known only from the state of Hidalgo, Mexico, where it was taken in several localities at elevations of about 9000 ft. by Dr. W. M. Mann. The worker and female are readily distinguished by the very opaque, blue-black surface of the body, entirely black legs, antennae and mandibles and the sparse, erect hairs on the gula.

165. *Formica subpolita* Mayr.

California: Mt. Tamalpais (Wheeler).

Oregon: Ashland (W. Taverner).

The typical form of this species is known only from the coast of California, Oregon, Washington and British Columbia. I failed to find it in the Yosemite or about Lake Tahoe though it ascends to an elevation of at least 6400 ft. in the Coast Range in Southern California.

166. *Formica subpolita* var. *camponoticeps* Wheeler.

California: Yosemite Village, 4000 ft. (Wheeler).

This variety has been recorded from several localities in Washington. The Californian specimens agree in all particulars with the types. Each of the numerous colonies which I found nesting under stones on the dry slope of the canyon wall near Yosemite Village contained workers of very different sizes. These colonies were all much less populous than those of the typical *subpolita*, which prefers a moister environment. In several of the nests I took mature larvae of a Coccinellid (*Brachyacantha* sp.) resembling those which occur in the nests of *Acanthomyops*.

167. *Formica subpolita* var. *ficticia* Wheeler.

Montana: Helena and Elkhorn Mts. (W. M. Mann); Flathead Lake (C. C. Adams); Missoula; Gallatin Co., 6500 ft.

Colorado: Boulder and Buena Vista (Wheeler).

The worker of this variety is much less pilose than the typical *subpolita* and the var. *camponoticeps*. In color it is like the latter, with the upper surface of the head darker, but the head of the largest workers is not so large and rectangular and more like that of the typical form of the species.

168. *Formica* (*Proformica*) *neogagates* Emery.

Utah: Promontory Point (A. Wetmore); Salt Lake (T. H. Parks).

The typical form of this species is widely distributed through the Transition Zone from the New England States to Washington and as far north as Quebec, British Columbia and Alberta, but is not known from California.

169. *Formica* (*Proformica*) *neogagates* subsp. *lasioides* Emery.

Recorded from South Dakota, Colorado and Massachusetts.

170. *Formica* (*Proformica*) *neogagates* subsp. *lasioides* var. *vetula* Wheeler.

California: Lake Tahoe, 6000–8000 ft. and Glacier Point, Yosemite, 8000 ft. (Wheeler).

Alberta: Banff (C. G. Hewitt).

Colorado: Chimney Gulch, Golden 9500 ft., and San Juan Mts. 12,000 ft. (E. J. Oslar).

Montana: Beaver Creek, 6300 ft. (S. J. Hunter).

More widely distributed than the typical *neogagates*. All the workers and females from California have the scapes even more distinctly hirsute than many specimens from the Eastern and Central States. This ant is very common at Lake Tahoe and is the summer host of *Xenodusa montana* Casey. I found the larvae of this beetle in the nests both at Tahoe and Glacier Point, but failed to find any pseudogynes in these localities. The winter hosts of the *Xenodusa* are *Camponotus herculeanus* var. *modoc* and *C. laevigatus*.

171. *Formica* (*Proformica*) *neogagates* subsp. *lasioides* var. *limata* Wheeler.

Recorded from Colorado and New Mexico.

172. *Formica* (*Neoformica*) *pallidefulva* Latr. subsp. *schaufussi* Mayr var. *incerta* Emery.

This variety, which is very common throughout the Central and Atlantic States, does not extend westward beyond the Eastern slopes of the Rocky Mts. It has been taken in Colorado and New Mexico.

173. *Formica* (*Neoformica*) *pallidefulva* subsp. *nitidiventris* Emery. With the same distribution and western limits as the preceding.

174. *Formica* (*Neoformica*) *pallidefulva* subsp. *nitidiventris* var. *fuscata* Emery.

Also known to occur as far west as New Mexico, but more abundant in the Eastern States.

175. *Formica* (*Neoformica*) *moki* Wheeler.

Recorded from Arizona and Utah.

176. *Polygerus lucidus* Mayr subsp. *montivagus* Wheeler.

Colorado: Colorado Springs (Wheeler).

The typical *lucidus* is known only from the Eastern and Central States as far west as South Dakota.

177. *Polygerus rufescens* Latr. subsp. *breviceps* Emery.

Colorado; Breckenridge (P. J. Schmitt); Florissant, Ute Pass and Colorado Springs (Wheeler).

New Mexico: Old Pecos Pueblo (T. D. A. Cockerell).

Kansas: Osage City (A. C. Burrill).

Illinois: Algonquin (W. A. Nason); Galesburg (M. Tanquary).

Montana: Elkhorn Mts. (W. M. Mann).

California: Santa Cruz (H. Heath); Kern Lake (J. C. Bradley); Fallen Leaf Lake and Glen Alpine, near Lake Tahoe (Wheeler).

Washington: Pullman (W. M. Mann).

In extending to Illinois this subspecies overlaps the distribution of *lucidus* in the Mississippi Valley. My observations on the slave-raids of *breviceps* at Lake Tahoe are published in the Proc. N. Y. Ent. Soc. 24, 1916, pp. 107–118.

178. *Polyergus rufescens* subsp. *breviceps* var. *montezuma* Wheeler.

Mexico: Pachuca in Hidalgo (W. M. Mann).

179. *Polyergus rufescens* subsp. *breviceps* var. *umbratus* Wheeler.

California: Brookdale (H. Heath).

180. *Polyergus rufescens* subsp. *breviceps* var. *fusciventris* var. nov.

*Worker*. Length 4 mm.

Differing from the typical *breviceps* in its smaller size, more opaque and more coarsely shagreened surface, in having the petiolar node distinctly shorter and more compressed anteroposteriorly and the posterior $\frac{2}{5}$ of the first gastric and the whole of the succeeding segments except the anal region, fuscous.

Described from a single worker taken by Prof. T. D. A. Cockerell at the Half Way House on Pike's Peak, Colo. Several workers taken by Dr. C. G. Hewitt at Treesbank, in Southern Manitoba, though slightly larger, also belong to this variety, which is clearly transitional to the subsp. *bicolor* Wasm. The slaves accompanying these specimens belong to the typical *Formica fusca*.

181. *Polyergus rufescens* subsp. *mexicanus* Forel.

This form, described from Mexico, without precise locality, is hardly distinct from the subsp. *breviceps*, to judge from a couple of cotypes received from Prof. Forel.

182. *Polyergus rufescens* subsp. *bicolor* Wasmann.

Wisconsin: Prairie du Chien, type locality (H. Muckermann).
Illinois: Rockford (Wheeler).
Montana: Yellow Bay, Flathead Lake (C. C. Adams).

183. *Polyergus rufescens* subsp. *laeviceps* Wheeler.

California: Mt. Tamalpais, 1000 ft. (Wheeler); Laws (A. Wetmore).

184. *Camponotus laevigatus* F. Smith.

California: Yosemite Village, 4000 ft. and Tallac, Lake Tahoe (Wheeler).
Washington: Seattle (Wheeler).
Montana: Flathead Lake (C. C. Adams).
Colorado: Meeker (W. W. Robbins).

Previously recorded from numerous localities in California, Oregon, Washington, Idaho, Montana, Colorado, Utah, New Mexico, Arizona and Northern Mexico. At Seattle I found its large colonies in huge pine stumps less than 100 feet above sea-level. Further south it is distinctly boreal, rarely, if ever, descending below 4000–5000 ft. and occurring as high as 11,000 ft. on Alta Peak, Cala.

185. *Camponotus herculeanus* L. var. *whymperi* Forel.

Alaska: Seward (F. H. Whitney).
British Columbia: Emerald Lake and Glacier (Wheeler); Arrowhead (C. G. Hewitt).
Alberta: Laggan, Lake Louise and Moraine Lake (Wheeler).
Washington: Mt. Renier (J. C. Bradley).
Colorado: Bullion Peak, Park Co. 12,000 ft.; Chimney Gulch,

Golden, 9500 ft.; San Miguel Mts. 12,000 ft.; Halls Valley, Park Co., 10,500 ft.; Wilson Peak, 13,000 ft. and Bear Creek, Morrison (E. J. Oslar).

Also recorded from many other localities in the Canadian Zone from Alaska and British Columbia to Labrador and Maine.

186. *Camponotus herculeanus* var. *modoc* Wheeler.

California: Nevada Falls, Yosemite (Talcott Williams); Yosemite Village 4000 ft. to Glacier Point 8000 ft. and Lake Tahoe, 6000–9000 ft. (Wheeler).

Washington: Seattle (Wheeler).

Colorado: Creede, 8844 ft. (S. J. Hunter).

British Columbia: Vancouver (Wheeler).

This ant like *C. laevigatus*, descends to sea-level in Washington and western British Columbia, but further south it is subalpine. Huge colonies of it were found at Lake Tahoe nesting in old pine logs and stumps.

187. *Camponotus herculeanus* subsp. *ligniperda* Latr. var. *noveboracensis* Fitch.

British Columbia: Agassiz (C. G. Hewitt).

Common east of the Rocky Mts. It is recorded also from Colorado, Washington, and Oregon.

188. *Camponotus schaefferi* Wheeler.

Known only from the Huachuca Mts., Arizona, where it was taken at altitudes of about 5000 ft. by C. R. Biedermann.

189. *Camponotus texanus* Wheeler.

From low elevations in Central Texas (Travis Co.).

190. *Camponotus sayi* Emery.

Arizona: Graham Mts. (E. G. Holt).

Known also from Phoenix and Prescott in the same state.

191. *Camponotus hyatti* Emery.

California: Palo Alto (W. M. Mann).

Originally taken at San Jacinto, Cala.

192. *Camponotus hyatti* var. *bakeri* Wheeler.

Catalina Island, Cala. (C. F. Baker).

193. *Camponotus fallax* Nyl. var. *nearcticus* Emery.

California: Angora Peak, Lake Tahoe, 7000 ft. (Wheeler).

Recorded also from Washington, Oregon, Idaho, California, and common throughout the Central and Eastern States.

194. *Camponotus fallax* var. *minutus* Emery.

I have referred some specimens from Vancouver to this variety, which, like the preceding is common in the Central and Eastern States.

195. *Camponotus fallax* var. *decipiens* Emery.

Known only from Indiana, Kansas, Colorado and Utah.

196. *Camponotus fallax* subsp. *rasilis* Wheeler.

Ranges through Arizona, Texas and Louisiana to Florida.

197. *Camponotus fallax* subsp. *rasilis* var. *pavidus* Wheeler.

Having much the same distribution as the preceding.

198. *Camponotus fallax* subsp. *subbarbatus* Emery.

According to Emery this form has been taken at Los Angeles, Cala. It is recorded also from Virginia, New Jersey and Illinois.

199. *Camponotus fallax* subsp. *discolor* Buckley.

Very common in Texas and known to extend up the Mississippi Valley to Oklahoma, Missouri and Illinois.

200. *Camponotus fallax* subsp. *discolor* var. *clarithorax* Emery.

This variety was originally described from San Jacinto, and Los Angeles, Cala. I have seen specimens from San Diego, Whittier, Felton, Santa Cruz Mts. and Three Rivers and have taken it in the Santa Inez Mts., near Santa Barbara. It ranges eastward as far as Illinois and Pennsylvania.

201. *Camponotus anthrax* Wheeler.

Known only from the Santa Inez Mts., near Santa Barbara, Cala., where I found it nesting under stones at an altitude of about 1000 ft.

202. *Camponotus* (*Myrmoturba*) *maculatus* Fabr. subsp. *vicinus* Mayr.

Arizona: Grand View, Grand Canyon (Wheeler).

California: Tenaya Canyon, Yosemite, 5000 ft. and Lake Tahoe (Tallac, Glen Alpine Springs and moraine east of Angora Peak), 6000–7000 ft. (Wheeler); Alpine (J. C. Bradley); San Gabriel Mts. near Claremont and Point Loma, near San Diego (Wheeler).

Previously known from several localities in California, Nevada, Oregon, Washington, Idaho, New Mexico and British Columbia.

203. *Camponotus* (*Myrmoturba*) *maculatus* subsp. *vicinus* var. *plorabilis* Wheeler.

Recorded from California, Nevada and Idaho.

204. *Camponotus* (*Myrmoturba*) *maculatus* subsp. *vicinus* var. *luteangulus* Wheeler.

British Columbia: Vancouver Island (Pergande Coll.), Dog Lake, Penticton (C. G. Hewitt).

Montana: Flathead Lake (C. C. Adams).

Arizona: Palmer's Canyon, Huachuca Mts. (Wheeler).

This variety is widely distributed as it is known also from other localities in Arizona, from Washington and Idaho. It seems, however, to be rare and sporadic.

205. *Camponotus* (*Myrmoturba*) *maculatus* subsp. *vicinus* var. *semitestaceus* Emery.

California: San Jacinto Mts. (Fordyce Grinnell); Claremont (Metz); Friant (R. V. Chamberlin); Ramona (J. C. Bradley).

In Emery's "Beiträge," p. 672 this ant was originally described from "Plummer County, Cala., 5000 ft." but no such county exists. Examination of cotypes in the Pergande Collection recently acquired by the National Museum, shows that the specimens came from Plumas County.

206. *Camponotus* (*Myrmoturba*) *maculatus* subsp. *vicinus* var. *nitidiventris* Emery.

Colorado: Chimney Gulch, Golden and Bear Creek, Morrison (E. J. Oslar); Golden (E. Bethel).

Arizona: Grand Canyon (Wheeler).

Recorded also from many other localities in Colorado, Wyoming, New Mexico and Northern California.

207. *Camponotus* (*Myrmoturba*) *maculatus* subsp. *vicinus* var. *maritimus* Wheeler.

California: Santa Cruz Island (R. V. Chamberlin); Santa Cruz Mts. and Santa Inez Mts. near Santa Barbara (Wheeler).

This form occurs also on Catalina Island and along the coast at Pacific Grove, Cala.

208. *Camponotus* (*Myrmoturba*) *maculatus* subsp. *vicinus* var. *infernalis* Wheeler.

California: Tenaya Canyon and Camp Curry, Yosemite 4000–5000 ft. and Lake Tahoe, 6000–7000 ft. (Wheeler); Wilson Peak, San Bernardino Mts., 7500 ft. (Fordyce Grinnell).

Arizona: Williams (Wheeler).

This variety occurs also in New Mexico where I have taken it at Las Vegas. Forel has recently redescribed it (from a single worker minor!) from Lake Tahoe as var. *subrostrata* (Deutsch. Ent. Zeitschr. 1914, p. 620).

209. *Camponotus* (*Myrmoturba*) *maculatus* subsp. *dumetorum* Wheeler.

California: Berkeley (Wheeler).

This is a common form in the chaparral of the San Gabriel and Santa Inez Mts. of Southern California up to an altitude of 2000 ft. Forel has recently redescribed it (from a single worker media!) under the name of *C. maculatus maccooki* var. *berkeleyensis* from Berkeley, Cala. (Deutsch. Ent. Zeitschr. 1914, p. 619).

210. *Camponotus* (*Myrmoturba*) *maculatus* subsp. *maccooki* Forel.

California: Palo Alto (W. M. Mann); San Ysidro and Carpinteria, near Santa Barbara and Tenaya Canyon, Yosemite 5000 ft. (Wheeler).

Previously recorded from other localities in California, Washington and Oregon. It is confined to the Pacific Coast and western slopes of the Sierra-Cascade Range. I failed to find it about Lake Tahoe.

211. *Camponotus* (*Myrmoturba*) *maculatus* subsp. *sansabeanus* Buckley.

Arizona: Miller Canyon, Huachuca Mts. (Wheeler).

Previously recorded from Texas (as far east as Austin), New Mexico, Arizona and Colorado.

212 *Camponotus* (*Myrmoturba*) *maculatus* subsp. *sansabeanus* var. *torrefactus* Wheeler.

Originally described from the Grand Canyon, Arizona, 7000 ft. and East Mill Creek, Salt Lake Co., Utah.

213. *Camponotus* (*Myrmoturba*) *maculatus* subsp. *bulimosus* Wheeler.

Known only from the canyons of the Huachuca Mts., Ariz. 5000–

6000 ft., where it was repeatedly taken by Dr. W. M. Mann, Mr. C. R. Biedermann and myself.

214. *Camponotus* (*Myrmoturba*) *fumidus* Roger var. *festinatus* Buckley.

Recorded from several localities in Texas, west of Austin and San Antonio, and from Arizona and Mexico as far south as Cuernavaca.

215. *Camponotus* (*Myrmoturba*) *fumidus* var. *fragilis* Pergande.

Taken by Dr. G. Eisen at San Jose del Cabo and San Fernando, Lower California.

216. *Camponotus* (*Myrmoturba*) *fumidus* var. *spurcus* Wheeler.

Described from Western Texas and the Huachuca Mts. Ariz.

217. *Camponotus* (*Myrmoturba*) *vafer* Wheeler.

Known only from the Huachuca Mts., Ariz.

218. *Camponotus* (*Myrmoturba*) *acutirostris* Wheeler.

Originally described from Alamogordo in the foot-hills of the Sacramento Mts., New Mexico (G. v. Krockow) and Box Canyon in the same region (A. G. Ruthven).

219. *Camponotus* (*Myrmoturba*) *acutirostris* var. *clarigaster* Wheeler.

Known only from the Grand Canyon, Ariz., 3000 ft.

220. *Camponotus* (*Myrmoturba*) *ocreatus* Emery.

Emery described this ant as a subspecies of *maculatus* from the Panamint Mts., Cala. Not having seen worker specimens I long suspected that it might prove to be identical with my *acutirostris* from Arizona. Recently I found a fine series of cotypes of *ocreatus* in the Pergande Collection (U. S. Nat. Mus.) and these show that the form is not only distinct from the typical *acutirostris* but that it is an independent species. The worker major of *ocreatus* has a much broader head, the clypeus is broader than long and its median lobe, though somewhat acute, is much shorter, less projecting and less angular than in *acutirostris*. The frontal carinae of the latter are much more lyriform and the color is different. In *ocreatus* the whole head, including the mandibles, clypeus, scapes and first funicular joint, is black, as are also the tibiae, the tips of the femora and the dorsal surface of the pronotum and anterior portion of the mesonotum. The cheeks lack erect hairs and foveolae. My subsp. *primipilaris*, however, evidently belongs to *ocreatus* and not to *acutirostris*. It is larger than the typical *ocreatus*, has much the same coloring of the legs and thorax and a similar clypeus, but the gaster is darker, the mandibles and clypeus are largely red and only the base of the first funicular joint is black. The Pergande Collection also contains specimens of the typical *ocreatus* labeled "St. Francis Mts., Mexico."

221. *Camponotus* (*Myrmoturba*) *ocreatus* subsp. *primipilaris* Wheeler.

Known only from Nogales and the Huachuca Mts. of Arizona, 5000–6000 ft. It has been repeatedly taken in the latter locality by Dr. W. M. Mann, Mr. C. R. Biedermann and myself.

222. *Camponotus* (*Myrmamblys*) *bruesi* Wheeler.

From Fort Davis, Texas, Chihuahua and Guadalajara, Mexico.

223. *Camponotus* (*Myrmobrachys*) *mina* Forel.

Known only from Cape St. Lucas at the tip of Lower California.

224. *Camponotus* (*Myrmobrachys*) *mina* subsp. *zuni* Wheeler.

Taken at Tucson, Arizona.

225. *Camponotus* (*Colobopsis*) *ulcerosus* Wheeler.

Known only from the canyons of the Huachuca Mts., Arizona, (5500–6000 ft.).

226. *Camponotus* (*Colobopsis*) *yogi* Wheeler.

Taken by Mr. Percy Leonard on Point Loma, near San Diego, California, nesting in twigs of manzanita.

227. *Camponotus* (*Colobopsis*) *abditus* For. var. *etiolatus* Wheeler.

Known from various localities in Western and Central Texas where it nests in pecan twigs and live oak galls.

228. *Camponotus* (*Colobopsis*) *pylartes* Wheeler.

Texas and Louisiana, nesting in twigs.

229. *Camponotus* (*Colobopsis*) *pylartes* var. *hunteri* Wheeler.

A pretty color variety of the preceding taken at Victoria, Texas, in twigs of pecan trees. This and the preceding form belong more properly to the Louisianian fauna.

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# *THE ELECTRICAL RESISTANCE OF METALS UNDER PRESSURE.*

By P. W. Bridgman.

Investigations on Light and Heat made and published with aid from the Rumford Fund.

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## Introduction.

In this paper the effect of pressure combined with temperature on the electrical resistance of 22 metals is investigated. The list of metals includes nearly all that are sufficiently permanent in the atmosphere and can be made into wire. The pressure range is from atmospheric pressure to 12000 kg. per cm.$^2$, and the temperature range from 0° to 100°.

The subject is one of considerable importance for the electron theory of metals. Previous discussion has been concerned mostly with explaining the effect of temperature on resistance, and very little with the effect of pressure. There seems to be no reason for this except the lack of experimental material; the pressure effect is certainly as significant as the temperature effect, and a study of it should throw much additional light on the mechanism. The old form of electron theory due to Drude was not competent to explain the pressure effect, although it dealt fairly well with the temperature effect. If the experimental facts with regard to pressure had been sufficiently known it would not have required considerations of specific heat to show that Drude's theory cannot be correct. Only quite recently has the theoretical bearing of the effect of pressure begun to be discussed[1]. I hope in this paper to present enough material for a more extended discussion, and in the latter part of the paper I shall try to indicate some of the significance of the results. Even in its present incomplete form, however, the electron theory will evidently need other data than the effect of pressure and temperature on resistance. One of the most important of such other effects is that of pressure on thermo-electromotive force; I hope to give such data in a succeeding paper for some of the metals investigated here.

The effect of pressure on electrical resistance has of course been measured a number of times before. The data here cover a considerably wider range, both of material and pressure. None of the previous work covers a pressure range of more than 3000 kg., and except for a single isolated instance with 100% error, I know of no measurements of the variation of pressure coefficient with temperature. The data here bring out the to me unexpected fact that the variation of pressure coefficient with temperature is very much less than the variation of resistance itself. I have also tried to improve on previous work in respect to the purity of the materials. Where possible I have given chemical analyses, and in all cases the temperature coefficient. The temperature coefficient is a very good indication of the purity of the metal, being almost always higher for the purer material. The pressure coefficient shows no such consistent variation, but may sometimes increase and sometimes decrease in the presence of impurity.

The most important previous work on the effect of pressure has been done by Lisell[2] up to 3000 kg., Lafay[3] to 2000 kg., Barus[4] and Palmer[5] to 2000 kg., and Beckman[6] with Lisell's apparatus. For a full bibliography the papers of Lisell and Beckmann may be consulted.

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[1] E. Grüneisen, Verh. D. Phys. Ges. **15**, 186–200 (1913).

[2] E. Lisell, Om Tryckets Inflytande på det Elektriska Ledningsmotståndet hos Metaller samt en Ny Metod att Mäta Höga Tryck, Upsala, (1902).

[3] Lafay, Ann. de Chim. et Phys. **19**, 289–296 (1910). C. R. **149**, 566–569, (1909).

[4] C. Barus, Bull. U. S. Geol. Sur. No. 92 (1892). Amer. Jour. Sci. **40**, 219 (1890).

The plan of this paper is to first give the data for individual substances with a description of the details of preparation and the characteristic features of each substance, and then a discussion of the significance of the results.

## Experimental Methods.

The apparatus is in all essentials the same as that previously used and described in a number of papers[7]. It consists of two parts, an upper and a lower cylinder connected by a stout tube. In the upper cylinder pressure is produced by the descent of a piston driven by a ram. The upper cylinder also contains the coil of manganin wire which gives the pressure by its change of resistance. The calibration of the manganin coil and the details of the upper cylinder have already been fully described. During the experiment, the upper cylinder was kept thermostatically at a constant temperature of 40°. The lower cylinder contains the metal whose resistance is to be measured, and is placed in a second thermostat independent of that controlling the upper cylinder. The resistance of the lower coil was measured on the same Carey Foster bridge as was the manganin wire; connection to the bridge was by mercury switches in paraffine blocks. The lower cylinder is shown in Figure 1. It consists of a cylindrical piece of Chrome Vanadium steel pierced axially with a $\frac{9}{16}$ inch hole, enlarged and threaded to receive the connecting pipe at one end and at the other the plug through which the insulating leads connect with the wire under measurement. Lower cylinders of two different lengths were used according as the wire to be measured was insulated, and so could be coiled into a narrow space, or was bare and had to be wound in the spiral grooves of a core about 9 cm. long. The usual accidents were encountered in making these cylinders; several broke during use because of defective steel.

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[5] A. deF. Palmer, Amer. Jour. Sci. **4**, 1, (1897) and **6**, 451 (1898).

[6] B. Beckman, (a) Inaug. Dis. Upsala, (1911).
(b) Ark, f. Mat., Astr., och Fys. **7**, No. 42, (1912).
(c) Ann. Phys. **46**, 481–502 (1915).
(d) Ann. Phys. **46**, 931–941 (1915).
(e) Phys. Zs. **16**, 59–62 (1915).

[7] P. W. Bridgman, Proc. Amer. Acad. **47**, 321–343 (1911), and **49**, 627–643 (1914).

It was my original intention to make several apertures in the lower cylinder, so that measurements could be made on several coils at once, but this scheme was given up because two such cylinders, after much work had been put on them, proved defective. It seems to be the part of experimental economy when much effort in making the apparatus may go for nothing because of defective material, to make the apparatus of as simple design as possible, even at the expense of extra labor in obtaining the readings. Positive results were obtained with one of these preliminary cylinders, however, in which two coils could be placed simultaneously. Measurements were made on two coils of iron, and at another time on two coils of copper. The agreement between the members of these two pairs of coils was perfect within the sensitiveness of the measurements, which for these coils was about 1/3000 of the change of resistance. This is gratifying because it shows that the effect of pressure on resistance may be reproduced with not more error than that in the absolute measurement of pressure.

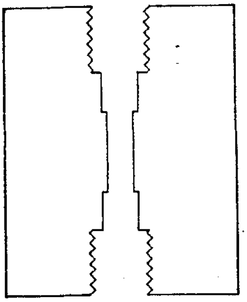

FIGURE 1. The cylinder to contain the wire under measurement. At the top, connection is made to the pressure pump; at the bottom, the insulating plug with the wire is inserted.

The resistance was measured by a null method on a Carey Foster bridge. The galvanometer circuit was permanently closed to avoid thermo-electric effects, and the battery circuit was closed only when making measurements. The current was, however, kept so low that the battery circuit could be kept permanently closed with no change of resistance due to heating. It should be emphasized that this avoids a source of error present in the work of Lisell[2] and Beckman.[6] They allowed the current to flow continuously through the wire. Any change in the thermal conductivity of the oil under pressure would change the heating effect and so partly mask the change of resistance due to pressure. If thermal conductivity of a liquid decreases under pressure, as seems plausible, this source of error would make their pressure coefficient too low; which as a matter of fact is the direction of discrepancy between most of their results and mine. The magnitude of the error from this source varies with the material and the size of the wire. Lisell states that for lead his error from this source is probably not as much as 1%, but for zinc he is willing to admit the probability of an error as large as 10%. Beckman, working with Lisell's apparatus, makes no mention of this source of error, and would seem not to have sufficiently guarded against it.

Temperature control was one of the most troublesome difficulties of the preliminary work. The effect of temperature on resistance is very large compared with that of pressure; for some substances 1° C may make a change in resistance 3000 or 4000 times as large as 1 kg. pressure. The upper thermostat in which was the manganin measuring coil gave no trouble; it was sufficient to keep the temperature of this within 1°. For the lower thermostat a bath of water violently stirred was used, with a very sensitive regulator consisting of a spiral of thin walled copper tubing filled with ethyl benzoate and connected to a regulating mercury column in a glass capillary by means of a steel intermediary part to which the copper was soldered and in which the glass was directly sealed without cement. By means of a Beckmann thermometer and adjustment by hand of the mercury contact if necessary, temperature could be kept constant within two or three thousandths of a degree during a run. Error from slow drift was avoided in those cases where the temperature coefficient was high compared with the pressure coefficient by using as the comparison coil another coil of the same substance and resistance as the coil subjected to pressure, placed in a glass tube in the bath in close proximity to the cylinder. For those substances with larger pressure coefficient this precaution was not necessary, but a comparison coil of manganin at room temperature was sufficiently good. Of course the temperature of 0.0° was obtained in an ice bath, and no temperature trouble was ever found here. Difficulty of temperature control probably accounts for the fact that most previous measurements have been made only at 0.0°. It was not possible to run the thermostat with water at 100°. Instead, the maximum temperature was usually set at about 99° and troublesome evaporation avoided by covering the surface with a thin layer of oil.

The manganin resistance gauge was calibrated from time to time during the measurements, which extended over six months after the preliminary work. There was a slow and uniform secular change in the constant of the coil amounting in all to about $\frac{2}{3}\%$. Of course correction was made for this change. A new detail in the method of calibration gave somewhat sharper results than possible by the method described in the previous paper. By using petroleum ether to transmit pressure to the freezing mercury instead of kerosene, it is possible to avoid any effects due to viscosity of the transmitting medium, and obtain results more quickly and sharply. The freezing pressure of mercury at 0.0° may be reached from either above or below within the limits of sensitiveness, about 1/3500, in fifteen minutes. The manganin slide wire of the Carey Foster bridge was also repeatedly calibrated. This showed a secular change, due to wear, of about $\frac{1}{2}\%$ during the six months of the runs.

The wires to be experimented on were usually wound either non-inductively on themselves in the form of anchor rings of approximately 1.5 cm. external diameter, or were wound non-inductively in a double thread cut on a bone core. It is essential that the method of winding be such that the pressure is transmitted freely to all parts of the coil without any mechanical hindrance from the frame on which it is wound. This object is obviously at once attained when the wire is wound on itself without a core, constrained only by a wrapping of silk thread to keep it in shape, but this method is feasible only when the wire can be covered with silk insulation without damage. If the wire is soft like lead, it cannot be covered without damage, and it must be wound bare on some sort of a core. Several attempts were made before a suitable material for a core was found. At first, hard rubber was used, but this is so compressible that at the highest pressures the wire drops out of the grooves, and is so expansible that at the highest temperatures the wire is stretched. A hard rubber shell on a steel core does not work because the unequal compressibility of the rubber and steel causes the rubber to crack. Bakelite was tried without success. I was afraid to use mica because of the sharp bends unavoidably introduced into the wire during handling. Finally bone was found to be satisfactory from the points of view of both sufficiently low compressibility and thermal expansion. A double 12 or 18 thread of square section was cut in the lathe on the surface of a cylinder

1.7 cm. diameter and 9 cm. long, and the wire was wound loosely in the groove.

Another mechanical effect apart from that offered by the constraints is due to the viscosity of the transmitting medium. This was of course particularly prominent at low temperatures. It may be almost entirely avoided by using petroleum ether to transmit pressure at 0° and 25°. At higher temperatures, pure kerosene may be used without sensible error.

When these various precautions have been taken to avoid extraneous mechanical and temperature effects, results may be obtained of a constancy and regularity much greater than I had anticipated. In the majority of cases pressure could be applied to 12000 kg. and removed with a change of zero of less than 0.3% of the total change, or a constancy of the total resistance of 0.01%, and in many cases the change was imperceptible. This change was not the effect of pressure alone, but was the sum of all effects, including temperature drift in the thermostat and changes in the bridge due to changes in room temperature. Nevertheless, 0.3% of the change might be 15 or 20 times the sensitiveness of the measurements, and in all such cases a correction for the zero drift was applied proportional to the time.

The wires were attached to the terminals of the insulating plug with silver solder in most cases where the melting point was high enough; the softer metals were attached usually with soft solder. In a few cases other methods of contacting were employed, which will be described in detail later. In all cases in which metallic connection can be made by fusing, there need be no trouble whatever at the contacts.

Other minor corrections are for the effect of pressure on the leads of the insulating plug (only 0.2 mm. of bridge wire at the maximum) and a temperature correction for the leads, which was determined experimentally, and the correction for lack of uniformity of the bridge wire.

The purity of the metals used is a matter of great importance. The harder metals were drawn from sizes below 0.04 inch through diamond dies. Before passing to the diamond dies the trace of iron that might have been rubbed in from the larger steel dies was removed by etching off with acid at least 8% of the diameter. Softer metals were extruded at one operation through a steel die. Special tests with the very delicate potassium thiocyanate method showed no perceptible iron introduced during the extrusion. Chemical analyses are given for all those metals for which I could obtain it. As giving the best indication of the purity, careful measurements were made of the temperature coefficient. In many cases the materials seems purer than any on which measurements have been previously published. Detailed data from which the purity may be judged are given under the individual substances. The resistance at 25° intervals in terms of that at 0°C as unity is tabulated in the following. The average temperature coefficient between 0° and 100° may be read directly from the resistance at 100°.

In comparing these results with those of others it is necessary to keep in mind that the relation between resistance and temperature is not linear, but for most substances the resistance increases more rapidly at higher temperatures. This introduces a slight correction. For instance, Jaeger and Diesselhorst[8] made measurements at 18° and 100°, extrapolated linearly to find the resistance at 0°, and tabulated the coefficient obtained in this way as the average coefficient between 0° and 100°. The value thus found is evidently too high, and so is not strictly comparable with the results found here. The correction, which is always slight, may be computed in any case from the tables given below.

After winding the coils they were in most cases seasoned by exposing to several changes of temperature between 0° and 140° or 150° in an air bath. In addition to this, the coil was frequently seasoned for pressure by several times exposing to 12000 kg. This seasoning for pressure turned out to my surprise, however, to be hardly necessary. A number of substances showed no perceptible change of resistance after their first exposure to pressure. This is an interesting point, as it indicates with a high degree of sensitiveness the perfect elasticity of volume under hydrostatic pressure. Further details of seasoning are given under the individual substances.

Measurements of the change of resistance were made at intervals of 1000 kg. at 0°, 25°, 50°, 75°, and 100°. Two readings were made at the maximum and two zero readings, one before and one after the run. At each temperature, therefore, 15 readings were made. The straight line connecting the mean of the two points at 12000 with the mean zero was computed, and at each pressure the difference between the observed and computed value was found. These differences were then smoothed graphically.

Readings of resistance were made with increasing and decreasing pressure in order to avoid any error due to the direction of change of pressure. These readings were made alternately; at 0, 1000, 2000, 4000, 6000, 8000, 10000, 12000, with increasing pressure, and at 12000, 11000, 9000, 7000, 5000, 3000, and 0 with decreasing pressure. There is no perceptible hysteresis. Figure 2 for lead shows this. This entire absence of hysteresis was very gratifying; I had not expected results so favorable. At the two lower temperatures of the earlier runs a small effect in a direction opposite to that of hystersis was sometimes found. This was traced to the viscosity of the kerosene transmitting pressure; it entirely disappeared on using the less viscous petroleum ether to transmit pressure at 0° and 25°.

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[8] W. Jaeger und H. Diesselhorst, Phys. Tech. Reichsanstalt, Wiss. Abh. **3**, 269–425 (1900).

After every change of pressure some time is necessary before the next reading can be made, because of temperature disturbance due to the heat of compression. This change of temperature is in many cases so great as to entirely mask the effect of change of pressure; immediately after changing pressure the resistance of most substances changes in a direction opposite to that of the final change. Some substances, like lead, in which the ratio of pressure coefficient to temperature coefficient is high, do not show the initial reversal, but in most cases the immediate change may be 5 or 10 times as great as the final change and in the opposite direction. This effect is very troublesome, as it may need as much as 30 or 45 minutes to reach temperature equilibrium after each change of pressure. Without some trick of procedure a run at a single temperature might occupy seven or eight hours and is excessively tedious. The time to reach equilibrium may be very much shortened by running the pressure beyond the desired final mark and then, after most of the heat of compression has been dissipated, bringing the pressure back to the desired mark. The heating effect during this second change of pressure is opposite in direction from the

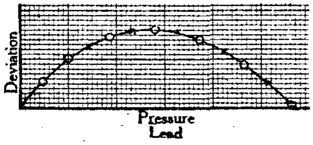

FIGURE 2. The deviation from linearity for lead against pressure, both on an arbitrary scale. The purpose of the diagram is to show the entire absence of hysteresis; the circles show the measurements with increasing pressure, and the crosses with decreasing pressure. One small division corresponds to $\frac{1}{3}$% of the total change of resistance produced by the maximum pressure.

initial change. With a little practice a secondary change of such magnitude may be found that temperature equilibrium is reached in 5 to 7 minutes.

When the apparatus was in good running order and the thermostat had reached equilibrium, a complete run on one substance at one temperature could usually be made in about two hours. Including all manipulations of adjusting temperature of the thermostat and waiting for zero equilibrium, runs at two different temperatures could be easily made on a single substance in a working day.

## GENERAL CHARACTER OF RESULTS.

The effect of pressure on all the metals tried, with the exception of antimony and bismuth, is to decrease the resistance. To a first approximation, the relation between pressure and resistance is linear. The average pressure coefficient is the datum which is of most immediate interest. The average coefficient up to 12000 kg. for any temperature was found by connecting the average zero with the average of the two maximum points (which in many cases were nearer 13000 than 12000 kg.) by a straight line, calculating and plotting the deviations of the observed points from this straight line, passing through these deviation points a smooth curve (one is shown for lead in Figure 2), and from this smooth deviation curve reading off the value of the deviation at 12000, which may then be combined with the slope of the straight line to give the average coefficient to 12000. The coefficient found in this way is called the "observed" average coefficient. This "observed" coefficient is a function of temperature. The "observed" coefficient at each of the five temperatures was then plotted against temperature, a smooth curve drawn through the five points, and the value given by the smooth curve at any temperature taken as the best value of the coefficient for that temperature. In the following the smoothed average coefficient is given in tables as a function of temperature, and the experimental points are given in diagrams from which an estimate may be formed of the experimental accuracy.

To a second approximation the relation between pressure and resistance is not linear, but the initial rate of decrease of resistance is in all cases greater than that at higher pressures. The departure from linearity may vary with the substance from 0.8% to 5% of the total change of resistance under 12000 kg. For some substances which depart little from linearity, the departure is symmetrical about the mean pressure, and may be represented within the limits of error by a second degree equation of the form Ap (12000 — p). In these cases the manner of departure from linearity is entirely specified by giving the maximum departure. The initial slope of the pressure-resistance curve is the average slope plus four times the maximum percentage deviation, and the final slope is the average slope minus four times the maximum deviation. The departure from linearity is of course a function of temperature. The experimental values of the departure were plotted against temperature and smooth curves drawn through the points. In the tables the smoothed values of maximum departure are given as functions of temperature, and in a diagram the experimental values are given, from which the accuracy of the departure from linearity may be estimated.

For most substances, however, the departure from linearity is not symmetrical, and the relation between pressure and change of resistance cannot be represented by a second degree equation. Furthermore, the manner of variation from linearity is different for different metals, so that it is not possible to represent the behavior of all metals by a formula containing only two constants. Lisell[2] and Beckman[6] found a two constant formula sufficient. Lisell's formula was a simple second degree expression, $R = R_0(1 + \gamma p + \delta p^2)$, and Beckman's was exponential, $R = R_0 e^{ap+bp^2}$. Any formula, however, must give correctly at least the average coefficient, the maximum deviation from linearity and the pressure of maximum deviation. At higher pressures all three of these data are unrelated, so that two constants will certainly not suffice. It might be possible to find a three constant formula which would work for all the metals within the limits of error, but I have preferred to exhibit graphically the deviations from linearity of each substance. The deviation curves are functions of temperature, so that to completely represent the data within my range a curve at each one of the five temperatures is necessary. The deviation curves have been smoothed as follows. First, smooth deviation curves were drawn through the experimental points at each temperature. The maximum deviation and the pressure of maximum deviation were next each plotted against temperature and smooth curves drawn through these points. These smoothed maximum deviations and pressures of maximum deviation are listed in the Tables, and the experimental values of maximum deviation are shown in the curves as functions of temperature. The smoothed deviation curves for each temperature were then further adjusted graphically so that the maximum falls at the values indicated by the smooth curves. This adjustment of the maximum was always slight and could be made with little uncertainty. Finally, each one of these smoothed adjusted deviation curves was uniformly changed in scale by the factor necessary to make its maximum coincide with the maximum deviation found from the smoothed curves of maximum deviation against temperature, and these curves are given in the following as the "deviation curves."

It will conduce to clearness to give an example or two showing the combined use of the tables and the deviation curves. Let us find, for example, the resistance of tin at 5000 kg. and 50° in terms of its resistance at 0° and 0 kg.* Turning to Table II, we find the average pressure coefficient at 50° is $0.0_5936$ and the resistance at 0 kg. at 50° is 1.2179. If the relation between pressure and resistance were linear, the resistance at 50° would be $1.2179\ (1 - 5000 \times 0.0_5936) = 1.1609$. But from the deviation curve, Figure 4, we find the deviation at 50°, and 5000 kg. to be 0.0046. The actual resistance at 50° and 5000 kg. is therefore $1.1609 - 0.0046 = 1.1563$. Or let us find the initial pressure coefficient of lead at 75°. From Table V, the average coefficient at 75° is $0.0_41243$. By drawing a tangent to the deviation curve, Figure 7, at 75° at the origin we find that the deviation for 1 kg. at 75° is $0.0_5302$. But the initial resistance at 75° is 1.3127, so that the initial deviation at 75° for 1 kg. in terms of unit resistance at 75° is $0.0_5302/1.3127 = 0.0_5230$. Adding this to the average coefficient gives $0.0_41243 + 0.0_5230 = 0.0_41473$ for the initial pressure coefficient of lead at 75°.

The column in the tables headed "coefficient at 12000 kg." requires a word of explanation, the meaning of "coefficient" not always being unambiguous. This means the instantaneous rate of change of resistance with pressure at the temperature in question divided by the resistance at 0 kg. at the temperature in question. In other words, it is the slope of the line plotting resistance against pressure, drawn to such a scale that the resistance at the temperature in question and 0 kg. is taken as unity. Later in this paper I shall discuss another "coefficient" as 12000 kg., this time "the instantaneous coefficient." By this will be meant the rate of change of resistance with pressure divided by the actual resistance at 12000 kg. and the temperature in question.

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* The pressures in the tables and diagrams are *gauge* pressures. To get absolute pressures, add approximately one kg. The difference between absolute and gauge pressure is in almost all cases far within the limits of error.

## DETAILED DATA.

The detailed data for individual substances follow. These are arranged in order of melting point, except for the anomalous substances bismuth, antimony, and tellurium.

*Indium.* A sample only one gram in amount was available from Merck, without analysis. This metal is as soft or softer than lead. It was extruded into a wire of 0.006 inch diameter in a die of special construction. Indium oxidizes much less rapidly than lead; after extrusion the surface of the wire is brightly polished and remains so for at least several weeks when exposed to the air. It was wound loosely on a bone core of the dimensions already given. Its actual resistance at 0° was 11.7 ohms. Connections were made by soft soldering with a miniature copper, using a fusible solder of melting point slightly above 100°. There is some difficulty in making a successful soldered connection because of the low melting point of the indium, which is about 155°. It alloys very rapidly with any ordinary solder, forming an alloy of much lower melting point than any of the constituents. It must be caught by the solder with a single well directed touch.

The melting point of the alloy at the soldered connections limited the range over which measurements could be made. Successful runs were made at 0°, 25°, and 50°, but at 75° the soldered connections dropped off. Difficulty because of alloying also made it necessary to omit the usual temperature seasoning; this in any event is not so necessary for a low melting metal as for a higher one. No irregularity to be ascribed to lack of seasoning was to be found. After the very first application of pressure to 12000 kg. at 0° there was a permanent change of zero of 1.3% of the total pressure effect. This may well have been a viscosity effect from the petroleum ether, since indium is the softest of all the substances tried and so is particularly susceptible. At 25° the permanent change of zero after 12000 kg. was only 0.3% of the pressure effect, and at 50° only 0.003%. The maximum deviation of any reading from the smooth curves is 0.3% of the total effect, and the average numerical deviation 0.13%.

The smoothed results are collected in Table I and the experimental points are shown in Figure 3. The average coefficient does not depart more than 0.2% from linearity with temperature. The deviation from linearity is not symmetrical about the mean pressure; it is so great that the initial slope is from 25 to 33% greater than the mean and the final slope from 12 to 16% less.

## TABLE I.

### INDIUM.

| Temp. °C. | Resistance | Pressure Coefficient At 0 kg. | Pressure Coefficient At 12000 kg. | Pressure Coefficient Average 0–12000 kg. | Maximum Deviation from Linearity | Pressure of Maximum Deviation |
|---|---|---|---|---|---|---|
| 0° | 1.0000 | $-0.0_41226$ | $-.0_5891$ | $-.0_41021$ | −.0058 | 5600 |
| 25 | 1.1002 | 1297 | 896 | 1051 | 70 | 5600 |
| 50 | 1.2015 | 1368 | 911 | 1081 | 83 | 5600 |

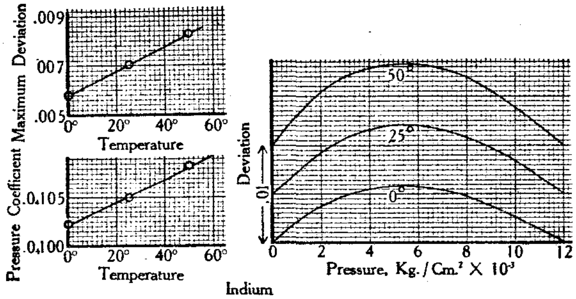

FIGURE 3. Indium, results for the measured resistance. The deviations from linearity are given as fractions of the resistance at 0 kg. and 0°C. The pressure coefficient is the average coefficient between 0 and 12000 kg.

The average temperature coefficient found above, extrapolated for the range 0° to 100°, is 0.00407. This is considerably lower than the only other value I have been able to find, 0.00474 by Erhard.[9]

The general character of the results for indium is as follows. The average pressure coefficient of resistance increases linearly with temperature, but the increase is less than one third as much as the increase of resistance. The departure from linearity, on the other hand, increases at higher temperatures more rapidly than the initial resistance. This means that if resistance is plotted as ordinates against pressure for each of several temperatures and then if the scale of the ordinates of the several curves is changed so as to make the resistance at 0 kg. independent of the temperature, the curves corresponding to the higher temperatures will drop off more sharply, and will also have greater curvature. This is entirely as one would expect, particularly if the curves are asymptotic to zero resistance at infinite pressure.

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[9] T. Erhard, Wied. Ann. **14**, 504 (1881).

*Tin.* This was Kahlbaum's best, grade "K." It was extruded to 0.008 inch diameter and wound bare on a bone core. Its resistance at 0° and 1 kg. was 12.3 ohms. Connections were made by soldering with a fusible alloy of tin and lead of melting point about 180°. It was seasoned by heating to 120°, and by a preliminary application of 12000 kg. at 25°. Extrusion instead of drawing proved necessary. It was possible to draw the wire down to 0.01 inch, but there were many irregularities and it was not possible to wind it with silk insulation. Several unsuccessful preliminary attempts were made with the bare extruded wire on a hard rubber core. There were large initial irregularities due to the large thermal expansion of the hard rubber which entirely disappeared on using bone. With the final set-up the permanent change of zero after a run to 12000 was never more than 0.1% of the pressure effect, except at 0°, where the change jumped to 1%, probably because of viscosity of the transmitting medium.

Runs successful in every way were made at all five temperatures. The smoothed results are collected in Table II, and the experimental points are shown in Figure 4. Except for a single discordant point,

TABLE II.

TIN.

| Temp. °C. | Resistance | Pressure Coefficient |  |  | Maximum Deviation from Linearity | Pressure of Maximum Deviation |
|---|---|---|---|---|---|---|
|  |  | At 0 kg. | At 12000 kg. | Average 0–12000 kg. |  |  |
| 0° | 1.0000 | $-.0_41044$ | $-.0_5833$ | $-.0_59204$ | −.0036 | 5770 |
| 25 | 1.1080 | 1055 | 836 | 9280 | 415 | 5800 |
| 50 | 1.2179 | 1062 | 839 | 9357 | 47 | 5820 |
| 75 | 1.3306 | 1064 | 841 | 9434 | 525 | 5850 |
| 100 | 1.4473 | 1062 | 844 | 9510 | 58 | 5870 |

the maximum deviation of any point from the smoothed curves is 0.4% of the pressure effect, and the average numerical deviation is 0.07%, or, excluding the run at 0°, where the effect of viscosity was apparent, 0.03%. The deviation curves are sensibly not symmetrical, although the pressure of maximum deviation is very nearly the mean pressure.

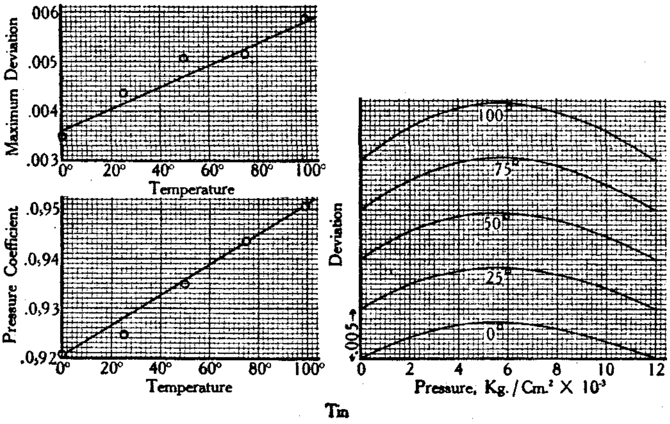

FIGURE 4. Tin, results for the measured resistance. The deviations from linearity are given as fractions of the resistance at 0 kg. and 0° C. The pressure coefficient is the average coefficient between 0 and 12000 kg.

The average temperature coefficient of tin between 0° and 100° was 0.00447. This is not quite as high as the value of Jaeger and Diesselhorst [8] for tin from the same source as this, which by their analysis had less than 0.03% lead. Their value for tin rod was 0.00459 (corrected for range as explained in the introduction).

The initial value of the pressure coefficient at 0° has been found by Beckman [6] to be $-0.0_592$. This is to be compared with $-0.0_4104$ given above. The average coefficient to 12000 kg. agrees very closely with Beckman's value.

The general character of the results is the same as for indium. If the curves of resistance against pressure are so changed in scale that the resistance at 0 kg. for each temperature is the same, then the curves for the higher temperatures are the steeper and have the greater curvature.

*Thallium.* This was electrolytically prepared from two samples of metal which originally came from Merck and Eimer and Amend, and was supposed to be chemically pure in its original state. The metal was first converted into the nitrate, and measurements made on the polymorphic transitions under pressure. From the nitrate it was converted to the iodide. These two conversions have been described in a previous paper.[10] The iodide was then converted to the sulfate by heating with c. p. sulfuric acid, and it was finally electrolyzed onto a platinum electrode from aqueous solution of the sulfate. After all these metamorphoses any impurity of the original metal should have been effectively removed. The purity of this electrolytic thallium was tested by comparing its temperature coefficient of resistance with untreated c. p. thallium from Merck. The mean coefficient at 0° from readings between 25° and 96° of electrolytic thallium was 0.005177 against 0.004898 of Merck's. The electrolytic is therefore appreciably purer.

Thallium was formed into wire about 0.013 inch diameter by cold extrusion. It was wound loosely on a bone core and connections made by soldering with "fine" solder (2 parts tin to 1 part lead). Its initial resistance was 9.47 ohms at 0°. Thallium becomes coated rather rapidly in the air with brownish oxide. The layer of oxide in time becomes so deep as to very appreciably increase the resistance of the wire. There is also a slow formation of oxide on standing in the kerosene transmitting the pressure. The rate of formation of oxide is not great enough to introduce appreciable error during a single run; at 100° and 75°, where the rate of formation is most rapid, there was no perceptible permanent change of zero after the runs. When the apparatus stands over night, however, between runs, there is sufficient formation to introduce appreciable error into the temperature coefficient determined from successive zeroes. The runs at 25° and 0° were made 9 days after those at 50°, 75°, and 100°, and there was a break in the resistance of 5% in this interval due to oxidation. For this reason the relation between temperature and resistance has been taken as linear in the table and the value chosen for the mean coefficient is that found from the comparison of electrolytic with Merck's thallium.

The wire was seasoned before measurements by subjecting to 9000 kg. at 95° and 12000 at 50°.

The smoothed results are shown in Table III and the experimental values in Figure 5. The maximum deviation of any point from the smooth curve is 0.7% of the total pressure effect, and the average numerical deviation is 0.13%. The curves of deviation from linearity are not symmetrical about the mean pressure; these curves also are given in Figure 5.

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[10] P. W. Bridgman, Proc. Amer. Acad. **51**, 593 (1916), and **52**, 151 (1916).

There are no previous measurements of the pressure coefficient.

## TABLE III.

### THALLIUM.

| Temp. °C. | Resistance | Pressure Coefficient At 0 kg. | Pressure Coefficient At. 12000 kg. | Pressure Coefficient Average 0–12000 kg. | Maximum Deviation from Linearity | Pressure of Maximum Deviation |
|---|---|---|---|---|---|---|
| 0° | 1.0000 | $-.0_41319$ | $-.0_41017$ | $-.0_41151$ | −.00580 | 5470 |
| 25 | 1.1292 | 1358 | 1023 | 1165 | 702 | 5530 |
| 50 | 1.2585 | 1393 | 1028 | 1183 | 823 | 5580 |
| 75 | 1.3877 | 1425 | 1028 | 1203 | 945 | 5640 |
| 100 | 1.5170 | 1456 | 1024 | 1226 | 1066 | 5700 |

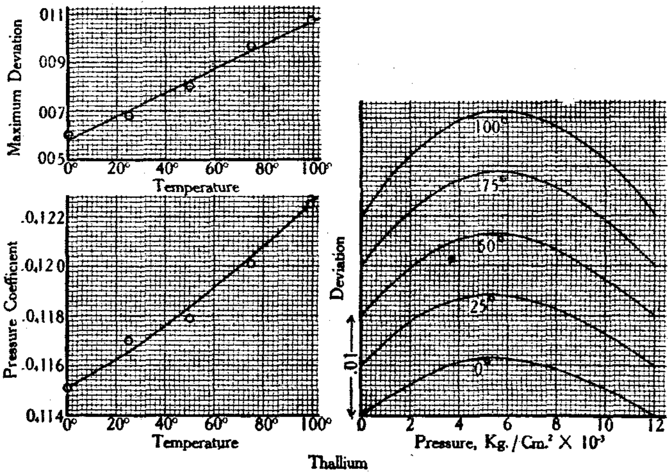

FIGURE 5. Thallium, results for the measured resistance. The deviations from linearity are given as fractions of the resistance at 0 kg. and 0°C. The pressure coefficient is the average coefficient between 0 and 12000 kg.

For the temperature coefficient between 0° and 100° there are the values .00398 by Dewar and Fleming,[11] and .00458 by Matthiesen and Vogt.[12] These are both considerably lower than .00517, the value found above.

The general character of the results is the same as for indium and tin. When the resistance-pressure curves are plotted to the same initial scale, the curves for higher temperatures are steeper and have the greater curvature.

*Cadmium.* This material was from Kahlbaum, grade "K." It was made into wire by extruding when hot into a wire 0.06 inch diameter. The surface was then deeply etched to remove any possible impurity of iron introduced by the extrusion. From 0.06 inch it was drawn to the final size, 0.005 inch, through diamond dies and wound with one layer of silk insulation. The drawing and winding were done by the New England Electrical Works, Lisbon, N. H. It was seasoned after winding into the coreless toroid for pressure measurements by keeping at 130° and 0° alternately for 30 minutes over a space of 8 hours and by 4 applications of 12000 kg. at 100°. Connection was made by soldering with "fine" solder. The resistance at 0° was 33.5 ohms.

TABLE IV.

CADMIUM.

| Temp. °C. | Resistance | Pressure Coefficient |  |  | Maximum Deviation from Linearity | Pressure of Maximum Deviation |
|---|---|---|---|---|---|---|
|  |  | At 0 kg. | At. 12000 kg. | Average 0–12000 kg. |  |  |
| 0° | 1.0000 | $-.0_4 1063$ | $-.0_5 746$ | $-.0_5 8940$ | −.00567 | 5880 |
| 25 | 1.1012 | 1082 | 765 | 9104 | 604 | 5880 |
| 50 | 1.2057 | 1095 | 778 | 9212 | 640 | 5880 |
| 75 | 1.3133 | 1102 | 786 | 9257 | 676 | 5880 |
| 100 | 1.4240 | 1106 | 790 | 9270 | 713 | 5880 |

The smoothed results are shown in Table IV and the experimental points in Figure 6. Except for one point due to viscosity at 0°, the maximum departure of any reading from the smooth curve was 0.6% of the total pressure effect, and the average numerical departure, including every point, was 0.05%. A preliminary sample gave a few readings at 75° agreeing within the limits of sensitiveness with those finally obtained. Measurements on this preliminary sample were terminated by leak due to a defective cylinder. The deviations from linearity are very nearly, but not quite, symmetrical and parabolic. The deviation curves are also reproduced in Figure 6.

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[11] J. Dewar and J. A. Fleming, Phil. Mag. **36**, 271–299 (1893).
[12] A. Matthiesen and C. Vogt, Phil. Mag. **26**, 242 (1863).

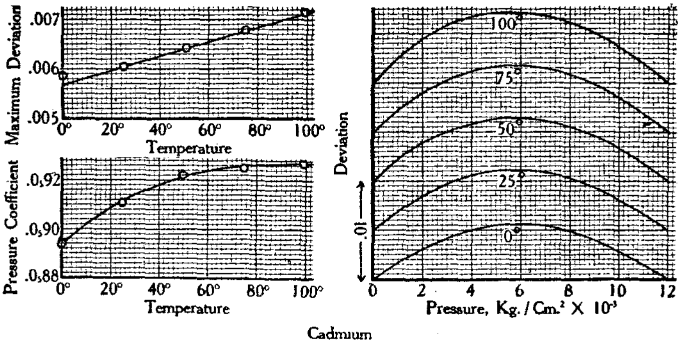

FIGURE 6. Cadmium, results for the measured resistance. The deviations from linearity are given as fractions of the resistance at 0 kg. and 0°C. The pressure coefficient is the average coefficient between 0 and 12000 kg.

The average temperature coefficient between 0° and 100° at 0 kg. found above is 0.00424. This may be compared with values of Jaeger and Diesselhorst[8] for cadmium from the same source, showing by analysis less than 0.05% of Pb, Zn, or Fe. Their value for cadmium rod is .00421, and for wire 0.00396. (I have corrected both of their data by a factor of 1% to compensate for the difference of range.) The cadmium used above is therefore probably a little purer than that of Jaeger and Diesselhorst. Beckman[6a] found 0.00425 for the average coefficient of the cadmium on which he made pressure measurements. The initial pressure coefficient at 0° has been found by Beckman to be $-0.0_592$, against $-0.0_41063$ above. My average coefficient between 1 and 12000 kg. is $-0.0_5894$ and is in much better agreement with Beckman's initial value. Beckman found a variation of the coefficient of 1.7% over 2700 kg.

The general character of the results is somewhat different from those of the three previous metals. When the pressure-resistance curves are scaled to the same initial resistance, the curves for the higher temperatures are steeper, but are *less* curved.

*Lead.* I owe this material to the kindness of Mr. C. Wadsworth, who prepared it under the direction of Professor T. W. Richards in connection with determinations of the atomic weight of lead of ordinary and radio-active origin. It showed by spectroscopic analysis not more than 1 part in 300,000 of Ag and Cu, and no trace of any other metal. It was made into wire 0.013 inch diameter by cold extrusion, wound loosely on a bone core as usual, and connected to the leads with "fine" soft solder. Its initial resistance at 0° was 11.0 ohms. It was seasoned by preliminary applications of 12000 kg. at 50° and 100°.

TABLE V.

LEAD.

| Temp. °C. | Resistance | Pressure Coefficient At 0 kg. | Pressure Coefficient At. 12000 kg. | Pressure Coefficient Average 0–12000 kg. | Maximum Deviation from Linearity | Pressure of Maximum Deviation |
|---|---|---|---|---|---|---|
| 0° | 1.0000 | $-.0_41442$ | $-.0_41044$ | $-.0_41212$ | $-.00660$ | 5560 |
| 25 | 1.1022 | 1452 | 1045 | 1222 | 749 | 5590 |
| 50 | 1.2065 | 1462 | 1047 | 1232 | 838 | 5620 |
| 75 | 1.3127 | 1473 | 1049 | 1243 | 927 | 5650 |
| 100 | 1.4207 | 1483 | 1051 | 1253 | 1016 | 5690 |

The temperature coefficient of this excessively pure lead was compared with that of Kahlbaum's "K" lead, formed into wire of the same dimensions by extrusion in the same way. The coefficient of Kahlbaum's lead was 0.2% lower than this, thus again confirming this test of purity.

An unsuccessful attempt, terminated by a flaw in the cylinder, was made about two months before the successful runs. This first run was on Kahlbaum's "K" lead, drawn from 0.06 to 0.01 inch and silk insulated by the New England Electrical Works. It was very fragile, with many places of incipient break. One break was made in winding, and it had to be soft soldered in the center. A successful run was made on this specimen at 100° before the cylinder broke; the average coefficient of this specimen at 100° was 0.5% lower than that of the pure specimen.

The smoothed results are collected in Table V and the experimental points are shown in Figure 7. The maximum departure of any single point from the smooth curve was 0.2% of the total pressure effect, and the average numerical departure was 0.026%. The deviation from linearity is very nearly symmetrical and parabolic, but there are distinct failures of symmetry in the usual direction. It is very noticeable at the higher temperatures that the curvature of the deviation curves is greatest near the maximum. This is unmistakably indicated by the data. The deviation curves are also given in Figure 7.

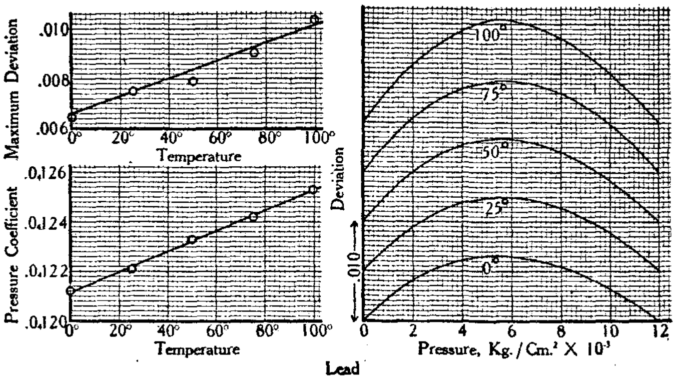

FIGURE 7. Lead, results for the measured resistance. The deviations from linearity are given as fractions of the resistance at 0 kg. and 0°C. The pressure coefficient is the average coefficient between 0 and 12000 kg.

The temperature coefficient of the lead given in the table is 0.004207. The coefficient of a piece of the same lead, which had never been subjected to pressure, was found to be 0.00441. This value was found by extrapolation of the readings between 25° and 96°. It is this which is strictly comparable with the value 0.00428 of Jaeger and Diesselhorst[8] for Kahlbaum's "K" lead.

The initial pressure coefficient at 0° is given by Lisell[2] as $-0.0_4140$. Williams[13] found the relation between resistance and pressure to be linear over a range of 700 kg. and the value of the coefficient to be $-0.0_4138$. The value found above by graphical extrapolation from the deviation curves is $-0.0_4144$. The agreement of these values is better than in the majority of cases.

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[13] W. E. Williams, Phil. Mag. **13**, 635–643 (1907).

The general character of the results is normal. When the resistance-pressure curves for the different temperatures are scaled to the same initial resistance, the curves for the higher temperatures are steeper and are slightly more curved.

*Zinc.* This was Kahlbaum's grade "K." Spectroscopic analysis by Mr. A. E. Becker showed traces of cadmium, iron and lead. It was extruded hot into wire of 0.006 inch diameter and wound loosely on a bone core. Connections were made with ordinary soft solder. The initial resistance at 0° was 10.3 ohms. It was seasoned by several exposures to 120° at atmospheric pressure, and by three preliminary applications of 12000 kg. at 100°. A second sample of the same wire was used for the points at 0° and 25°. This was seasoned by subjecting to 120° and to 12000 kg. at room temperature.

The smoothed values are shown in Table VI and the experimental values in Figure 8. The maximum departure of any point from a smooth curve was 0.9% of the total pressure effect, and the average numerical departure was 0.3%. These results are somewhat less regular than usual. Without making a bone core of new design it was not feasible to use a length great enough for the best results. Within the limits of error the deviations from linearity are symmetrical and parabolic.

TABLE VI.

ZINC.

| Temp. °C. | Resistance | Pressure Coefficient | | | Maximum Deviation from Linearity | Pressure of Maximum Deviation |
|---|---|---|---|---|---|---|
| | | At. 0 kg. | At 12000 kg. | Average 0–12000 kg. | | |
| 0° | 1.0000 | $-.0_5540$ | $-.0_5400$ | $-.0_54700$ | −.00210 | 6000 |
| 25 | 1.1017 | 533 | 394 | 4634 | 231 | 6000 |
| 50 | 1.2050 | 529 | 389 | 4590 | 252 | 6000 |
| 75 | 1.3098 | 526 | 387 | 4562 | 273 | 6000 |
| 100 | 1.4159 | 524 | 385 | 4544 | 294 | 6000 |

The mean temperature coefficient of this wire, 0.00416, is distinctly higher than the best wire of Jaeger and Diesselhorst,[8] for which they give 0.00402 (uncorrected for range). The zinc of Jaeger and Diesselhorst was from the same source as this and showed on analysis not more than 0.01% of Pb, Cd, or Fe. The initial pressure coefficient at 0° is given as $-0.0_559$ by Lisell,[2] against $-0.0_5540$ found graphically from the deviation curve above. Lisell does not mention the source of his zinc, nor does he give its analysis or temperature coefficient. He did not regard it as one of his purer materials, and says that he estimates the error in his value to be not over 10%.

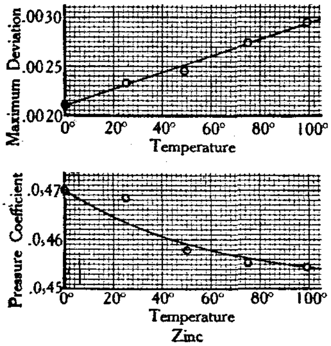


FIGURE 8. Zinc, results for the measured resistance. The deviations from linearity are given as fractions of the resistance at 0 kg. and 0°C. The deviations are symmetrical about the mean pressure, so that it is not necessary to give the detailed deviation curves as in the preceding figures. The pressure coefficient is the average coefficient between 0 and 12000 kg.

The general character of the results is as follows. When the resistance-pressure curves for the different temperatures are scaled to the same initial resistance the curves at higher temperatures are *less* steep, but show very little difference of curvature.

*Magnesium.* The material available was commercial magnesium from Eimer and Amend, for which I have no analysis. It was extruded hot into wire 0.007 inch diameter and wound loosely on a bone core. The initial resistance was 5.6 ohms. It is a matter of great difficulty to make any sort of electrical connections; no commercial method is known. I was able to make a connection with aluminum solder which appeared good to the eye and gave satisfactory results up to 8000 kg. But beyond this the solder broke loose, probably because of unequal compression, and no further results of any constancy were possible. I also made indifferent connections with a spring clip. The error in using the clip is introduced by the heavy coating of oxide, which forms rapidly on magnesium in the air, and has very high insulating qualities. With this clip a series of regular results were obtained between 12000 and 6000 kg., showing the same linear relation between pressure and resistance as the other run, but no other points of this run were at all good. It did not seem worth while to try for points at other temperatures than 0°. The pressure coefficient at 0° of this sample of magnesium may be taken to be $-0.0_555$ within perhaps 1 or 2%. No attempt was made to find the deviations from linearity.

There are no previous measurements of the pressure coefficient over any range. For the temperature coefficient I found .00390 between 0° and 20°. This may be compared with 0.00381 between 0° and 100° by Dewar and Fleming.[11] The material was not very impure, evidently.

*Aluminum.* This is one of most difficult of metals to get entirely pure. I obtained some considerably purer than that on which measurements are usually made through the courtesy of the Aluminum Company of America. Their analysis was Fe 0.23%, Si 0.24% Cu 0.06%, Al 99.47%. It was provided by them in the shape of ½ inch rod. I extruded it hot from this size to 0.06 inch diameter and etched the surface to remove iron. It was then drawn down to 0.005 inch through diamond dies and single covered with silk insulation by the New England Electrical Works. About 30 ft. of it was wound on itself into a coreless toroid and seasoned by carrying several times back and forth between 0° and 120° over an interval of 5 hours. Its initial resistance at 0° was 18.7 ohms.

Great trouble was found in making connections to this fine wire. Commercial aluminum solder was tried without success, the fine wire being completely alloyed through and eaten off at the temperature necessary to make a good connection. Finally connections were made with a spring clip but with results not entirely satisfactory. Consistent irregularities of as much as 4% of the total effect were found at the last three points with decreasing pressure at all temperatures, due probably to some slight slipping of the clip. The zero, however, was recovered satisfactorily; the permanent changes of zero were 1.4, 0.0, 0.6, 0.2, and 0.7% of the total effect. Except for the points at which consistent departures were found the greatest departure of any point from a smooth curve was 0.5% of the total effect. With a perfectly satisfactory method of making connections it might possibly pay to repeat these measurements searching for a polymorphic transition. The possibility of such a transition with a very slight discontinuity of resistance is not entirely ruled out by these measurements, although it is exceedingly improbable.

## TABLE VII.

### ALUMINUM.

| Temp. °C. | Resistance | Pressure Coefficient At 0 kg. | Pressure Coefficient At. 12000 kg. | Pressure Coefficient Average 0–12000 kg. | Maximum Deviation from Linearity | Pressure of Maximum Deviation |
|---|---|---|---|---|---|---|
| 0° | 1.0000 | $-.0_5416$ | $-.0_5347$ | $-.0_53815$ | $-.00104$ | 6000 |
| 25 | 1.1077 | 410 | 349 | 3794 | 101 | 6000 |
| 50 | 1.2159 | 405 | 351 | 3781 | 97 | 6000 |
| 75 | 1.3245 | 401 | 354 | 3772 | 93 | 6000 |
| 100 | 1.4337 | 397 | 356 | 3766 | 90 | 6000 |

The smoothed results are collected in Table VII, and the experimental values are shown in Figure 9. Within the limits of error the deviations from linearity are symmetrical and parabolic, and are sufficiently characterized by the values of the table.

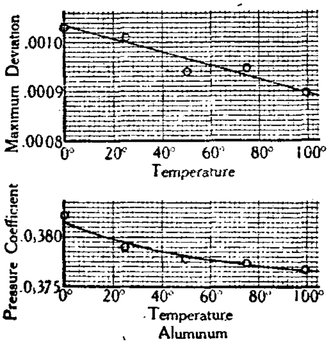


FIGURE 9. Aluminum, results for the measured resistance. The deviations from linearity are given as fractions of the resistance at 0 kg. and 0°C. Since the deviations are symmetrical, there is no need for giving the detailed deviation curves. The pressure coefficient is the average coefficient between 0° and 12000 kg.

The mean temperature coefficient of the sample above was 0.00434, which is considerably higher than the best value of Jaeger and Diesselhorst,[8] 0.0039 (uncorrected). Their aluminum showed by analysis 0.5% Fe, and 0.4% Cu, and was therefore much less pure than the sample above. The initial pressure coefficient at 0° is given as $-0.0_537$ by Williams[13] against $-0.0_5416$ from the deviation curves above. Williams does not give the analysis or temperature coefficient of his sample.

The results are somewhat unusual. When the resistance-pressure curves for different temperatures are scaled to the same initial resistance, the curves for the higher temperatures are very slightly less steep, and very much less curved.

*Silver.* This was obtained in the form of wire 0.06 inch in diameter from the United States Mint at Philadelphia. The original material from which the wire was drawn was "proof" silver, in which no impurity whatever could be detected by any chemical test. The drawing was done with steel dies. After drawing, it was found that 0.03% of impurity had been introduced, which was probably mostly steel from the dies. I owe this examination of the effect of drawing to the interest and kindness of the late Dr. D. K. Tuttle. It seems safe to assume that the impurity so introduced was in the outer layers. I therefore removed the outer layers by scraping with glass, reducing the diameter from 0.06 to 0.04 inch. It is very probable that nearly if not all the 0.03% of impurity was removed in this way, although I made no analysis of the wire after scraping. From the size 0.04 inch it was further drawn to 0.003 inch through diamond dies and wound with single silk insulation by the New England Electrical Works. It was annealed after the final drawing. For the pressure experiment it was wound on itself into a coreless toroid of 31.5 ohms resistance at 0°. It was seasoned in the same way and at the same time as the cadmium. Connections to the electrodes were made with silver solder.

The smoothed results are shown in Table VIII and the experimental points in Figure 10. The permanent change of zero after each run averaged 0.1% of the total effect; the maximum departure of any point from a smooth curve, except for one point at 0° where the viscosity was high, was 0.5%, and the average numerical departure was 0.1%. The departures from smoothness were almost entirely due to the viscosity of the kerosene used as a transmitting medium. This run was made before petroleum ether had been tried. The curves of deviation from linearity are not symmetrical or parabolic although nearly so. They also are shown in Figure 10.

## TABLE VIII.

### SILVER.

| Temp. °C. | Resistance | Pressure Coefficient | | | Maximum Deviation from Linearity | Pressure of Maximum Deviation |
|---|---|---|---|---|---|---|
| | | At 0 kg. | At 12000 kg. | Average 0–12000 kg. | | |
| 0° | 1.0000 | $-.0_5308$ | $-.0_5308$ | $-.0_53332$ | −.00088 | 5500 |
| 25 | 1.1011 | 363 | 311 | 3343 | 96 | 5660 |
| 50 | 1.2024 | 362 | 313 | 3352 | 995 | 5830 |
| 75 | 1.3044 | 359 | 315 | 3358 | 97 | 5990 |
| 100 | 1.4074 | 355 | 318 | 3362 | 905 | 6150 |

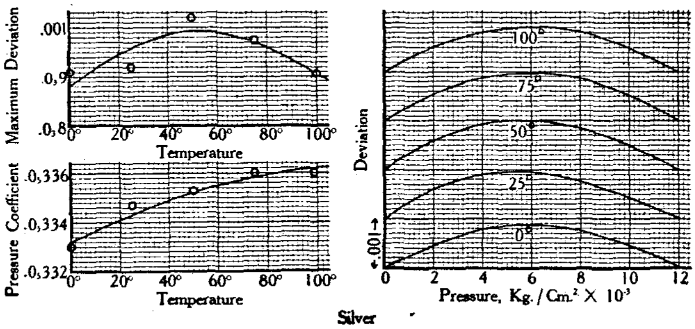


FIGURE 10. Silver, results for the measured resistance. The deviations from linearity are given as fractions of the resistance at 0 kg. and 0°C. The pressure coefficient is the average coefficient between 0 and 12000 kg.

The average temperature coefficient between 0° and 100° was 0.00407, against 0.00400 found by Jaeger and Diesselhorst[8] for silver 99.98% pure. The initial pressure coefficient at 0° found from the deviation curve above was $-0.0_5358$, against $-0.0_535$ by Lisell.[2] Lisell's silver was not excessively pure; it contained 0.08% of Cu and traces of carbon.

The results are somewhat unusual in general character. When the pressure-resistance curves of different temperatures are scaled to the same initial resistance, the steepness increases slightly at the higher temperatures as is normal, but the curvature becomes less at an accelerated rate as temperature increases. The shift of the pressure of maximum deviation from linearity to a value above the mean pressure at the higher temperatures is also unusual.

*Gold.* The gold, like the silver, was furnished by the United States Mint at Philadelphia in the form of wire 0.06 inches in diameter drawn down through steel dies from bars of "proof" metal in which no impurity whatever could be detected chemically. Dr. Tuttle found that the amount of impurity introduced in the drawing was practically the same in amount as for the silver. I scraped the surface of this wire with glass, just as the silver, reducing the diameter to 0.04 inch. From this size it was drawn to 0.004 inch through diamond dies and covered with single silk by the New England Electrical Works. It was wound into a coreless toroid of 23.5 ohms initial resistance and seasoned at the same time as the cadmium. Connections were made by silver soldering.

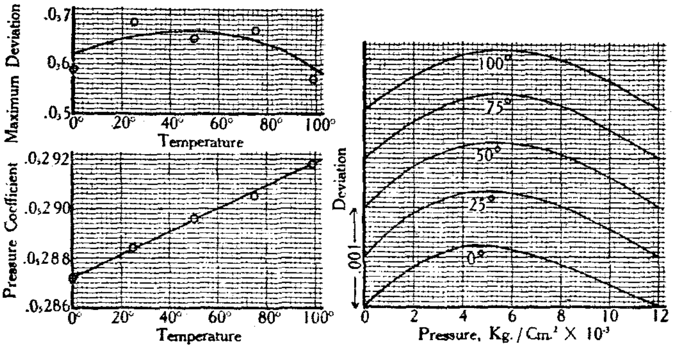
The smoothed results are collected in Table IX, and the experimental values of the average coefficient and maximum deviation from linearity in Figure 11. Kerosene was used as the transmitting medium throughout, and most of the departure from smooth curves is to be attributed to its viscosity. The average zero shift after a

TABLE IX.

GOLD.

| Temp. °C. | Resistance | Pressure Coefficient At 0 kg. | Pressure Coefficient At 12000 kg. | Pressure Coefficient Average 0–12000 kg. | Maximum Deviation from Linearity | Pressure of Maximum Deviation |
|---|---|---|---|---|---|---|
| 0° | 1.0000 | $-.0_5312$ | $-.0_5276$ | $-.0_52872$ | −.00062 | 4440 |
| 25 | 1.0977 | 310 | 275 | 2883 | 66 | 4910 |
| 50 | 1.1963 | 308 | 276 | 2895 | 665 | 5390 |
| 75 | 1.2960 | 306 | 279 | 2907 | 64 | 5860 |
| 100 | 1.3968 | 304 | 282 | 2918 | 59 | 6340 |

run was 0.16% of the total effect. Except for several points at 0°, where the viscosity is high, the maximum departure of any point from a smooth curve was 0.3% of the total effect, and the average numerical departure was 0.08%. The curves of deviation from linearity are distinctly not symmetrical, and show an unusually large progressive change in the location of the maximum with increasing temperature. The deviation curves are shown in Figure 11 also.



Gold

Figure 11. Gold, results for the measured resistance. The deviations from linearity are given as fractions of the resistance at 0 kg. and 0°C. The pressure coefficient is the average coefficient between 0 and 12000 kg.

The average temperature coefficient between 0° and 100° listed above, 0.003968, is for the coil which had been subjected to pressure. Another coil of the same material, not subjected to pressure, gave 0.004009. The curves of resistance of each of these specimens against temperature were equally smooth, the departure from smoothness at any point not being more 1/7000 of the total change between 0° and 100°. It would seem, therefore, that 0.004009 should be taken as the best value for pure gold under normal conditions. This figure is to be compared with 0.00368 given by Jaeger and Diesselhorst[8] for "pure" gold. But the temperature coefficient of gold is unusually sensitive to impurity; Jaeger and Diesselhorst found that gold with 0.1% Fe, and 0.1% Cu and a trace of Ag had a coefficient of only 0.00203. Undoubtedly the lower value of Jaeger and Diesselhorst for "pure" gold is to be explained by slight impurities in their sample.

The initial pressure coefficient at 0° found graphically from the deviation curves above is −0.03117, which is to be compared with $-0.0_527$ of Beckman.[6] He had trouble with change of zero in his measurements, and states that the error may be over 3%. The mean temperature coefficient of Beckman's gold between 0° and 100° was 0.00390, nearly as high as the above.

The general character of the results is much the same as for silver. When the resistance-pressure curves are scaled to the same initial resistance, the curves at higher temperatures become slightly steeper but the curvature becomes less at an accelerated rate. There is also the same march toward higher pressures of the pressure of maximum deviation, but here it is more pronounced than for silver.

*Copper.* I am indebted to the Bureau of Standards for this material; it was supplied from the stock of materials which had been used there in an extensive series of tests on the conductivity and temperature coefficient of copper from different sources. This was furnished by them in the form of wire 0.06 inch diameter. It had been forged from a bar of electrolytic copper without melting after the electrolytic refining. They were not in a position to supply the chemical analysis, but stated that it had proved to be of very high conductivity, and was in all probability purer than another sample which they offered me which showed by analysis 99.995% Cu, trace of S, and no Ag, $Cu_2O$, As, or Sb. From 0.06 inch I reduced the wire to 0.046 inch by etching with nitric acid, to remove any contamination of iron introduced in the drawing. From this size it was drawn to 0.003 inch through diamond dies, annealed, and single silk covered by the New England Electrical Works. For the high pressure measurements it was wound into a coreless toroid and seasoned for temperature and pressure at the same time as cadmium. Its initial resistance at 0° was 35.5 ohms. Connections were made with silver solder.

The smoothed results are collected into Table X and the experimental values of average coefficient and maximum deviation from linearity are shown in Figure 12. This substance was one of the first investigated, before all the details of manipulation had been perfected. Pressure was transmitted by kerosene at all temperatures instead of by petroleum ether at 0° and 25°, so that the error from viscosity is larger than necessary. In spite of this, however, the maximum departure from a smooth curve, except the one most viscous point at 0°, was only 0.3% of the total effect, and the numerical average was 0.05%. The zero drift at 75° and 100° was larger than usual, amounting to 2%, because the thermostatic control, which is more difficult at higher temperature, had not been perfected. At 50° the drift was 0.8%, at 25° 0.07%, and at 0° 0.00%. The curves of deviation from linearity are distinctly not symmetrical, and the pressure of maximum deviation progresses regularly with rising temperature. The curves are reproduced in Figure 12.

## TABLE X.

### COPPER.

| Temp. °C. | Resistance | Pressure Coefficient At 0 kg. | Pressure Coefficient At 12000 kg. | Pressure Coefficient Average 0–12000 kg. | Maximum Deviation from Linearity | Pressure of Maximum Deviation |
|---|---|---|---|---|---|---|
| 0° | 1.0000 | $-.0_5201$ | $-.0_5175$ | $-.0_51832$ | −.00045 | 4200 |
| 25 | 1.1073 | 196 | 174 | 1812 | 42 | 4650 |
| 50 | 1.2146 | 192 | 173 | 1796 | 38 | 5100 |
| 75 | 1.3219 | 188 | 172 | 1782 | 35 | 5550 |
| 100 | 1.4293 | 184 | 171 | 1770 | 31 | 6000 |

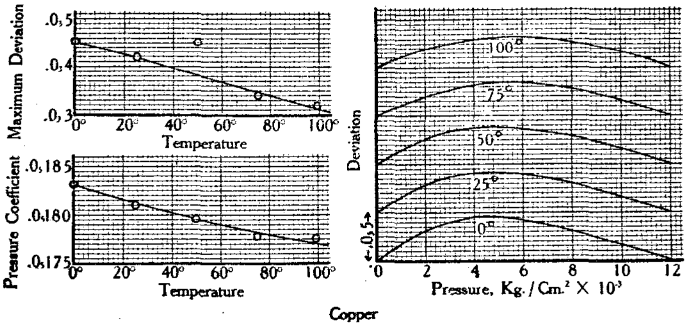


FIGURE 12. Copper, results for the measured resistance. The deviations from linearity are given as fractions of the resistance at 0 kg. and 0°C. The pressure coefficient is the average coefficient between 0 and 12000 kg.

A run at 25°, made with the same sample before the apparatus was running as regularly as finally, gave a mean coefficient to 12000 kg. 0.16% higher than that found in the final run.

The temperature coefficient listed in the table, 0.004293, was obtained from a coil of the same wire, subjected to the same seasoning but which had not been subjected to the pressure runs. There were somewhat large irregularities in the coil which was subjected to pressure. The value 0.004293 is a trifle higher than Jaeger and Diesselhorst's [8] highest value, 0.00428 for copper with less than 0.05% of Zn or Fe. The initial value of the pressure coefficient at 0°, found graphically from the deviation curves was $-0.0_52008$. This is to be compared with $-0.0_5187$ of Lisell.[2] He does not give the temperature coefficient of his Cu, but states that chemical analysis showed no trace of any foreign metal. The difference in our values cannot be due to difference of thermal or mechanical treatment, because Lisell found no perceptible difference between the pressure coefficient of the same material when in the annealed or hard drawn state.

If the resistance-pressure curves are scaled to the same initial resistance, the curves for higher temperatures are less steep, and of much less curvature. There is the same abnormal march of the pressure of maximum deviation shown by silver and gold.

*Nickel.* This was obtained from the Electrical Alloys Co. of Morristown, N. J.; and was stated by them to be of high purity, although they gave no analysis. The purity was not as high as it should be, as will be seen later from the temperature coefficient. It was provided by them in the form of wire 0.003 inch diameter, double silk covered. For the measurements under pressure it was wound into a coreless toroid of 176 ohms resistance at 0°. The initial resistance was chosen so high because the pressure effect is very small. It was seasoned by many times heating and cooling between 0° and 140°, and by a single exposure to 12000 kg. at 25°. Connections were made with silver solder.

A few readings were made on another sample of nickel, kindly furnished by Leeds and Northrup. It is of the grade used by them in resistance thermometers, and is of high purity. It was probably not so pure as that used in the final readings, however, because its temperature coefficient was 3% lower. The results obtained with it are only of orienting value, because the temperature during annealing was accidentally allowed to get so high as to slightly discolor the silk insulation. At 25° the average pressure coefficient was 4% less than that of the purer sample.

The smoothed results are collected in Table XI and the experimental values of mean coefficient and deviation from linearity are shown in Figure 13. The results for nickel are not quite as regular as for many other metals, because the pressure coefficient is unusually small compared with the temperature coefficient. The zero drifts were not large, however, being respectively 1.1%, 0.16%, 0.2%, 0.1%, and 0.3% of the total pressure effect at 0°, 25°, 50°, 75°, and 100°.

The maximum departure of any point from a smooth curve was 0.6% of the total effect, and the average numerical departure was 0.15%.

## TABLE XI.

### NICKEL.

| Temp. °C. | Resistance | Pressure Coefficient At 0 kg. | Pressure Coefficient At 12000 kg. | Pressure Coefficient Average 0–12000 kg. | Maximum Deviation from Linearity | Pressure of Maximum Deviation |
|---|---|---|---|---|---|---|
| 0° | 1.0000 | $-.0_51581$ | $-.0_51393$ | $-.0_51473$ | −.000262 | 5240 |
| 25 | 1.1108 | 1578 | 1428 | 1498 | 242 | 5680 |
| 50 | 1.2288 | 1586 | 1464 | 1524 | 244 | 6020 |
| 75 | 1.3542 | 1600 | 1499 | 1549 | 263 | 6300 |
| 100 | 1.4873 | 1631 | 1535 | 1575 | 305 | 6540 |

The deviations from linearity are distinctly not symmetrical; they are also reproduced in Figure 13.

The average temperature coefficient between 0° and 100° was 0.00487. This would indicate a metal of fairly high commercial

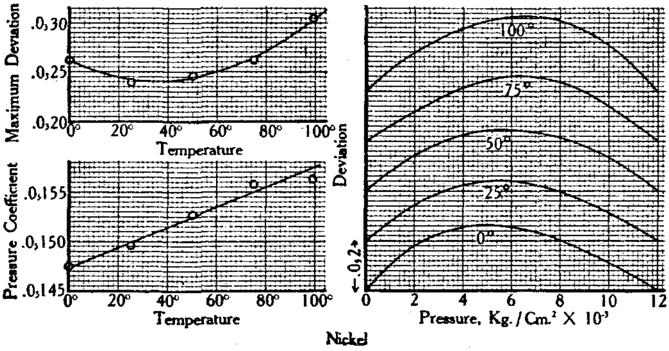


FIGURE 13. Nickel, results for the measured resistance. The deviations from linearity are given as fractions of the resistance at 0 kg. and 0°C. The pressure coefficient is the average coefficient between 0 and 12000 kg.

purity, but not at all of the purity attainable by electrolytic means. Thus Jaeger and Diesselhorst[8] give for their nickel, with an analysis of 97.0 Ni, 1.4 Co, 0.4 Fe, 1.0 Mn, 0.1 Cu, and 0.1 Si, 0.00438 as the temperature coefficient, but Fleming[14] gives for electrolytic nickel 0.00618. It is probable that the temperature coefficient of nickel, like that of iron, is excessively sensitive to slight impurities. The initial value of the pressure coefficient at 0° as given graphically by the deviation curve was $-0.0_5158$, against $-0.0_5138$ of Lisell[2]. Lisell does not give the temperature coefficient of his sample, but states that chemical analysis showed no trace of impurity.

The distinctive features of the behavior of nickel are shown when the pressure-resistance curves are scaled so that the initial resistances at different temperatures are the same. The curves at higher temperatures are steeper, but with less curvature, the rate of decrease of curvature being retarded at higher temperatures. The march of the pressure of the maximum is very pronounced. Too much significance should not be attached to any unusual features because of the lack of perfect purity of this sample.

*Cobalt.* I owe this material to the kindness of Dr. Herbert T. Kalmus, who had prepared some samples of colbalt wire under the direction of the Canadian government. His analysis showed Co 98.71, Ni none, Fe 1.15, Si 0.14, Ca none, S 0.012, C 0.039, and P 0.010. The number of the sample which he sent me was H 214; under this number some of its other physical properties have been described in a publication of the Canadian government entitled "The Physical Properties of the Metal Cobalt". It was furnished in the form of wire 0.04 inch diameter. From this size it was most kindly swaged and drawn down to 0.003 inch under the direction of Dr. W. D. Coolidge of the General Electric Company. During the process of reduction it was annealed at 540°, and after its final reduction I further annealed it by heating to a cherry red for a few seconds in a muffler, exposed to the air. These elaborate processes were necessary because the wire cannot be drawn through dies like most metals. For the measurements under pressure it was wound bare on a bone core; at 0° its initial resistance was 71.9 ohms. Connections were made by silver soldering. After winding it was further seasoned by heating with the core to 135° for 2 hours. After the first application of pressure there was a permanent increase of resistance of 0.2% of the total resistance, but after the first application of pressure the zero drift was no greater than for most substances.

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[14] J. A. Fleming, Proc. Roy. Soc. **66**, 50–58 (1899–1900).

Runs were made at only three temperatures, 0°, 50°, and 100°. I hoped at the time that a purer sample would be available and intended this run for orienting purposes, but the purer sample has not yet been successfully prepared.

The smoothed results are collected in Table XII, and the experimental values of the mean coefficient and maximum departure from linearity in Figure 14. The pressure coefficient of cobalt is smaller than that of any other substance which I have found; one would, there-

TABLE XII.

COBALT.

| Temp. °C | Resistance | Pressure Coefficient At 0 kg. | Pressure Coefficient At 12000 kg. | Pressure Coefficient Average 0–12000 kg. | Maximum Deviation from Linearity | Pressure of Maximum Deviation |
|---|---|---|---|---|---|---|
| 0° | 1.0000 | $-.0_6941$ | $-.0_6805$ | $-.0_6873$ | −.000204 | 6000 |
| 50 | 1.1825 | 845 | 755 | 800 | 161 | 6000 |
| 100 | 1.3651 | 755 | 697 | 726 | 118 | 6000 |

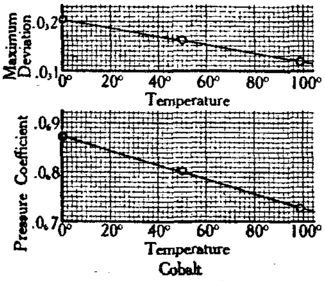

FIGURE 14. Cobalt, results for the measured resistance. The deviations from linearity are given as fractions of the resistance at 0 kg. and 0°C. Since the deviations are symmetrical about the mean pressure, there is no need of giving the detailed deviation curves. The pressure coefficient is the average coefficient between 0 and 12000 kg.

fore, expect the results to be more irregular. The greatest departure of any point from the smooth curve is 0.8% of the total pressure effect and the numerical average is 0.35%. The deviation from linearity is symmetrical and parabolic within the limits of accuracy; there is no need of exhibiting the deviations graphically.

The mean temperature coefficient found above is 0.00365. Kaye and Laby give 0.0033. It is probable that this value is quoted from Reichardt[15], who gives 0.00326 for cobalt 99.8% pure. There seem to be no measurements of pressure coefficient for comparison.

A distinctive feature of the behavior of cobalt is the abnormally low pressure coefficient. When the resistance-pressure curves are scaled to the same initial resistance, the steepness is less by an unusual amount at the higher temperatures, as is also the curvature.

*Iron.* This material was American Ingot Iron, obtained from the American Rolling Mill Co. of Middletown, Ohio. It is of exceptional purity, showing less than 0.03% total impurity. Under microscopic analysis, done under the direction of Professor Sauveur, it appeared to consist entirely of ferrite. It is much purer than most samples which have been prepared in small amounts by the most refined methods in the laboratory. It was not furnished by the manufacturers in the form of wire, but this particular sample was provided in a sheet $\frac{1}{16}$ inch thick. From this a square wire was cut with the shears, the corners rounded with the file, and it was drawn to 0.04 inch through steel dies. It was then heavily etched and drawn to 0.005 inch through diamond dies and wound with single silk insulation by the New England Electrical Works. It was annealed by heating to redness after the final drawing. For the pressure measurements it was wound into a coreless toroid of 70 ohms resistance at 0°. Connections were made with silver solder. It was seasoned by heating repeatedly to 140°, and by several applications of 12000 kg.

The smoothed values are shown in Table XIII and the experimental values for mean coefficient and departure from linearity in Figure 15. The maximum zero drift was 0.03% of the total pressure effect. This was one of the earliest substances investigated, and pressure was transmitted at all temperatures by kerosene. The results are somewhat more irregular than usual. Except for four bad points, the maximum departure of any individual reading from a smooth curve was 0.54% of the total pressure effect, and the numerical average was 0.12%. Within the limits of error the deviation from linearity was

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[15] G. Reichardt, Ann. Phys. **6**, 832–855 (1901).

## TABLE XIII.

### IRON.

| Temp. °C. | Resistance | Pressure Coefficient | | | Maximum Deviation from Linearity | Pressure of Maximum Deviation |
|---|---|---|---|---|---|---|
| | | At 0 kg. | At 12000 kg. | Average 0–12000 kg. | | |
| 0° | 1.0000 | $-.0_52405$ | $-.0_52119$ | $-.0_52262$ | −.00043 | 6000 |
| 25 | 1.1416 | 2420 | 2150 | 2285 | 465 | 6000 |
| 50 | 1.2918 | 2436 | 2180 | 2308 | 495 | 6000 |
| 75 | 1.4519 | 2451 | 2209 | 2330 | 525 | 6000 |
| 100 | 1.6206 | 2468 | 2238 | 2353 | 56 | 6000 |

symmetrical and parabolic, and is therefore sufficiently characterized by the mean value in the table without graphical representation.

Iron was the first substance on which I made measurements, and many runs were made before the difficulties of temperature control were overcome, or the best methods of manipulation discovered. All the earlier results agreed within their larger limits of error with the final results.

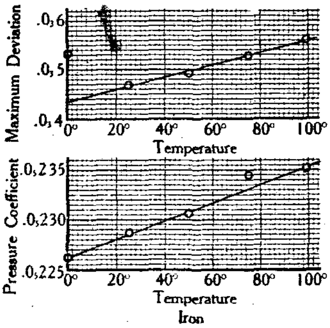

FIGURE 15. Iron, results for the measured resistance. The deviations from linearity are given as fractions of the resistance at 0 kg. and 0°C. Since the deviations are symmetrical about the mean pressure, there is no need of giving the detailed curves. The pressure coefficient is the average coefficient between 0 and 12000 kg.

The mean temperature coefficient between 0° and 100° given above is 0.006206. This value was found from a coil of the same iron as that above, seasoned for temperature in the same way, but not subjected to pressure. The mean temperature coefficient of the coil used for pressure measurements was 0.006201. The agreement is unusually good, and shows very little internal change produced in this material by the application of pressure. I have adopted the value 0.006206 as more probably that of the pure metal in a state of complete ease. The unusually large departure of the relation between temperature and resistance from linearity is to be noticed. The value found above for the temperature coefficient of iron is exceptionally high, as is to be expected from its exceptional purity. Kaye and Laby give for pure iron 0.0062, essentially the same as the above. Dewar and Fleming [11] give 0.00625, for an iron said to be very pure, but without analysis. The large effect of impurity is to be inferred from data of Jaeger and Diesselhorst,[8] who give for a sample composed of 99.55 Fe, 0.1 C, 0.2 Si; 0.1 Mn, trace of P, S, Cu, the value 0.00461, and for a purer iron with 0.1% C, other metals not determined, 0.00539.

The initial pressure coefficient at 0° has been found to be $-0.0_5246$ by Beckman,[6] against $0.0_52405$ found graphically from the deviation curves above. The discrepancy is not as great as usual between our results, and is in the opposite direction.

When the resistance-pressure curves are scaled to the same initial resistance at different temperatures the curves for higher temperature become slightly steeper, but are less curved.

*Palladium.* This was furnished by Baker and Co. in the form of wire 0.04 inch diameter. It was not etched, but was from this size drawn down to 0.003 inch in diamond dies, annealed to redness, and single covered with silk by the New England Electrical Works. It was wound into a coreless toroid of 68.8 ohms resistance at 0°, and seasoned at the same time as the cadmium. Connections were made by silver soldering.

The smoothed values ore collected in Table XIV, and the experimental values of mean coefficient and maximum deviation in Figure 16. The zero drifts during the runs were respectively 0.3%, 0.07%, 0.24%, 0.03%, and 0.9% of the total pressure effect at 0°, 25°, 50°, 75°, and 100°. Pressure was transmitted at all temperatures by kerosene; it would be possible to increase the regularity of the results by using petroleum ether at 0° and 25°. The effect of viscosity seemed to be greater than usual. Except at 0°, the greatest departure of any single reading from a smooth curve was 0.2% of the total pressure

## TABLE XIV.

### PALLADIUM.

| Temp. °C. | Resistance | Pressure Coefficient At 0 kg. | Pressure Coefficient At. 12000 kg. | Pressure Coefficient Average 0–12000 kg. | Maximum Deviation from Linearity | Pressure of Maximum Deviation |
|---|---|---|---|---|---|---|
| 0° | 1.0000 | $-.0_5198$ | $-.0_51855$ | $-.0_51895$ | −.000215 | 5160 |
| 25 | 1.0810 | 1965 | 184 | 1887 | 245 | 5430 |
| 50 | 1.1609 | 1945 | 183 | 1879 | 24 | 5700 |
| 75 | 1.2388 | 192 | 1825 | 1871 | 21 | 5970 |
| 100 | 1.3178 | 1895 | 183 | 1863 | 16 | 6240 |

effect, and the average numerical departure was 0.075%. At 0° the maximum departure was 0.7%, and the numerical average 0.31%. The deviation from linearity is not parabolic or symmetrical; the curves are shown in Figure 16.

The average temperature coefficient listed above is 0.003178 between 0° and 100°. This was obtained from a coil of the same sort as that on which the pressure measurements were made, subjected to the same seasoning, but used as the comparison coil of the Carey Foster

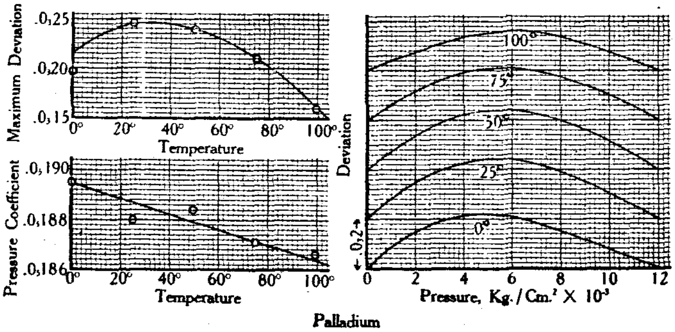

FIGURE 16. Palladium, results for the measured resistance. The deviation from linearity are given as fractions of the resistance at 0kg. and 0°C. The pressure coefficient is the average coefficient between 0 and 12000 kg.

bridge in the bath external to the cylinder. The coil on which the pressure measurements were made showed essentially the same value, 0.003175, but the results were a trifle less regular. The abnormal deviation of the resistance of palladium from linearity with the temperature is to be noticed; the slope of the resistance-temperature curve is greater at 0° than at 100°. This behavior is also shown by platinum, but is the opposite of that of all the other metals previously listed in this paper. This palladium cannot be very pure, because its temperature coefficient is low. Kaye and Laby give 0.0037, and Waidner and Burgess from the Bureau of Standards give .00333 to .00337. The latter value is probably as good as that of Kaye and Laby.

For the initial pressure coefficient at 0° Beckman[6] gives $0.0_5219$ against $0.0_51983$ found graphically from the deviation curves above. Beckman does not give the temperature coefficient of his palladium.

When the resistance-pressure curves are scaled to the same initial resistance, the curves at higher temperatures are less steep. At first the curvature becomes greater with increasing temperature but passes through a maximum and becomes rapidly less.

*Platinum.* This was the best Heraeus platinum, used in thermometry. I am indebted for it to Professor H. N. Davis, who before letting me use it had made a number of measurements on its temperature coefficient. The wire was wound bare on a bone core, was of 0.0038 inch diameter, and had an initial resistance at 0° of 21 ohms. It would have been better if I had been able to obtain a longer piece, but all the usual sources of supply were closed, and I was fortunate to obtain this. It had been annealed to a red heat a number of times by Professor Davis; after winding on the bone core it was still further annealed by carrying back and forth between 20° and 130° a number of times over an interval of four hours. It was seasoned for pressure by one application of 12000 kg. at 0°; a decrease of resistance of 2% of the total pressure effect was produced by the preliminary application. It was soldered with ordinary soft solder to copper leads.

The smoothed results are collected in Table XV, and the observed values of mean coefficient and deviation from linearity are shown in Figure 17. Because of the too low resistance the results are more irregular than usual. There was an unexplained systematic discrepancy, amounting to not more than 3%, at the maximum pressure at 0°, 25°, and 50°. Except for these irregularities at the maximum the greatest departure of any single point from a smooth curve was 0.5% of the pressure effect, and the average numerial departure was 0.14%.

Within the limits of error the departure from linearity is parabolic and symmetrical, and is sufficiently characterized by the numerical values in Table XV.

The mean coefficient between 0° and 100° listed above, 0.003868, was obtained during the pressure measurements. This was somewhat lower than the value found by Professor Davis for the same piece, 0.003905, before it had been subjected to pressure. His value is high and indicates the highest purity. The same abnormal deviation from linearity with temperature was found as for palladium. The deviation was too small to be sure of quantitatively, and in the table the relation is given as linear.

TABLE XV.

PLATINUM.

| Temp. °C. | Resistance | Pressure Coefficient At. 0 kg. | Pressure Coefficient At 12000 kg. | Pressure Coefficient Average 0–12000 kg. | Maximum Deviation from Linearity | Pressure of Maximum Deviation |
|---|---|---|---|---|---|---|
| 0° | 1.0000 | $-.0_51975$ | $-.0_51765$ | $-.0_51870$ | −.000315 | 6000 |
| 25 | 1.0967 | 195 | 1771 | 1862 | 30 | 6000 |
| 50 | 1.1934 | 1935 | 1774 | 1854 | 283 | 6000 |
| 75 | 1.2901 | 1915 | 1776 | 1846 | 27 | 6000 |
| 100 | 1.3868 | 190 | 1777 | 1838 | 255 | 6000 |

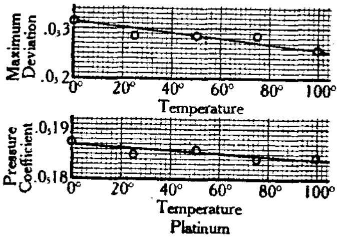

FIGURE 17. Platinum, results for the measured resistance. The deviations from linearity are given as fractions of the resistance at 0 kg. and 0°C. The pressure coefficient is the average coefficient between 0 and 12000 kg.

For the initial pressure coefficient at 0° the above deviation curves give by graphical construction $-0.0_5198$, against $-0.0_5177$ found by Lisell.[2] He does not give the temperature coefficient of his platinum, but chemical analysis showed an impurity of 0.28% iridium. We will see later that the discrepancy is very probably due to impurity. Lafay [3] gives for "pure" platinum the pressure coefficient $0.0_5186$; he found the relation between pressure and resistance linear up to 4000 kg.

Before this run with Heraeus platinum, a complete series of runs was made with a specimen from Baker and Co., their purest. This was subjected to preliminary treatment in all details like that of palladium, and the results were of the same regularity as those for palladium. The pressure coefficient was found to decrease linearly with temperature from $0.0_51766$ at 0° to $0.0_51742$ at 100°. The deviation curves were not symmetrical, and have the same distinct progression with temperature of the pressure of maximum deviation as have those of nickel. But a low value for the mean temperature coefficient was found, 0.003466, showing distinctly large impurities. It is to be noticed that the initial pressure coefficient at 0° is almost exactly the same as that found by Lisell. This makes it almost certain that the lowness of his pressure coefficient is due to the 0.28% of iridium. The fact that the deviation curves for impure platinum show a marked progression of the maximum, whereas those of pure platinum do not, suggests that the progression of the maximum found for nickel may be a spurious effect due to impurity.

The general character of the results is exactly the reverse of what we should be tempted to call normal. If the resistance-pressure curves are scaled to the same initial resistance for all temperatures, the slope and curvature are both less at higher temperatures.

*Molybdenum.* This was obtained from the General Electric Co. through the kindness of Dr. W. D. Coolidge. It was in the form of bare wire 0.0011 inch diameter. It was seasoned by a number of excursions between 0° and 130°, and was then wound bare on a bone core. The resistance at 0° was 83.8 ohms. Some little trouble was found in making a suitable connection; silver solder will not stick to it. It may be readily attached to platinum by arcing in hydrogen, which is the method employed by the General Electric Co. But this does not make a perfectly sharp contact, there being a small region of imperfect contact where the fine molybdenum wire leaves tangentially the surface of the larger platinum wire. It is very difficult to make a butt joint without the use of special devices. It seemed to me that this region of imperfect contact was an undesirable feature, because it might give rise to slight changes of resistance under pressure. Very good connections were finally made by using pure gold as a solder. A wire of platinum and molybdenum were laid side by side and wrapped with pure gold wire, 0.004 inch diameter. By arcing in hydrogen from a graphite point to the end of the wire the gold melts and runs back over the surface, wetting both platinum and molybdenum, and making a very good contact.

Before the final runs on molybdenum, a complete set of runs was made with the molybdenum wound on a core of hard rubber instead of bone. There were large initial irregularities due to the expansion of the rubber, which disappeared above 1000 kg. Between 1000 and 12000 kg. the results of the first run agreed within the limits of error with those of the final runs.

The smoothed results are collected in Table XVI and the experimental values of mean coefficient and deviation from linearity in Figure 18. With the exception of three bad points at 100°, the largest departure of any single point from the smooth curve was 0.4% of the total pressure effect, and the average numerical departure was 0.075%. The deviations from linearity are within the limits of error symmetrical and parabolic; there is no need for graphical representation.

TABLE XVI.

MOLYBDENUM.

| Temp. °C. | Resistance | Pressure Coefficient | | | Maximum Deviation from Linearity | Pressure of Maximum Deviation |
|---|---|---|---|---|---|---|
| | | At 0 kg. | At. 12000 kg. | Average 0–12000 kg. | | |
| 0° | 1.0000 | $-.0_5133$ | $-.0_51245$ | $-.0_51286$ | −.000124 | 6000 |
| 25 | 1.1071 | 132 | 124 | 1281 | 128 | 6000 |
| 50 | 1.2150 | 131 | 124 | 1275 | 132 | 6000 |
| 75 | 1.3238 | 1305 | 1235 | 1270 | 136 | 6000 |
| 100 | 1.4336 | 130 | 123 | 1265 | 140 | 6000 |

The average temperature coefficient between 0° and 100° is 0.004336. This is to be compared with the value 0.0034 given by Somerville.[16] In the tables of Kaye and Laby, the value is given as 0.0050, with no reference except the year 1910. Since I have been able to find no other reference for the temperature coefficient of molybdenum, and the work of Somerville was published in 1910, it seems plausible to suppose the value of Kaye and Laby is a misprint. The amount of impurity in my specimen was stated by Dr. Coolidge to be of the order of 0.2%. There are no previous measurements of the pressure coefficient for comparison over any range.

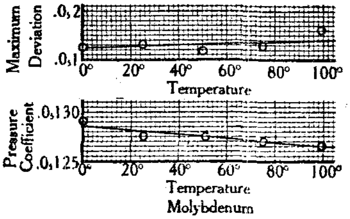

FIGURE 18. Molybdenum, results for the measured resistance. The deviations from linearity are given as fractions of the resistance at 0 kg. and 0°C. Since the deviations are symmetrical about the mean pressure, there is no need of giving the detailed curves. The pressure coefficient is the average coefficient between 0 and 12000 kg.

When the resistance-pressure curves are scaled to the same initial resistance for all temperatures the curves for higher temperatures are found less steep and less curved. This is like platinum.

*Tantalum.* This was obtained from the General Electric Co. through the kindness of Mr. MacKay. It is bare, 0.0022 inch diameter. It was subjected to the same seasoning as the molybdenum, and was wound bare on a core of bone with an initial resistance at 0° of 82.1 ohms. A perfectly satisfactory method of connection was not found. Neither gold nor silver solder will stick to the surface. Connection may be made to platinum by arcing in hydrogen, but the tantalum becomes excessively brittle in places where it has been subjected to a high heat, and the connection is very likely to drop off. Connection was finally made with a simple spring clip, consisting of a tightly wound helix of small piano wire. The spring is stretched, and the tantalum wire dropped between the extended spires. The results are somewhat more irregular than usual, but no consistent error seems introduced by this sort of a connection.

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[16] A. A. Somerville, Phys. Rev. 31, 261–277 (1910).

The smoothed results are shown in Table XVII, and the experimental values of mean coefficient and deviation from linearity in Figure 19. The zero creep during a run amounted on one occasion to as much as 2% of the total effect, but was otherwise less than 1%. The maximum departure of any single point from a smooth curve was 0.5% of the total effect, and the average departure was 0.1%.

## TABLE XVII.

### TANTALUM.

| Temp. °C. | Resistance | Pressure Coefficient At 0 kg. | Pressure Coefficient At 12000 kg. | Pressure Coefficient Average 0–12000 kg. | Maximum Deviation from Linearity | Pressure of Maximum Deviation |
|---|---|---|---|---|---|---|
| 0° | 1.0000 | $-.0_5 1487$ | $-.0_5 1373$ | $-.0_5 1430$ | −.00017 | 6000 |
| 25 | 1.0743 | 1497 | 1391 | 1444 | 17 | 6000 |
| 50 | 1.1486 | 1507 | 1409 | 1458 | 17 | 6000 |
| 75 | 1.2229 | 1518 | 1426 | 1472 | 17 | 6000 |
| 100 | 1.2973 | 1530 | 1442 | 1486 | 17 | 6000 |

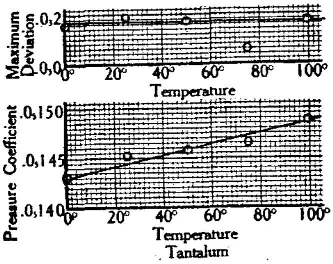

FIGURE 19. Tantalum, results for the measured resistance. The deviations from linearity are given as fractions of the resistance at 0 kg. and 0°C. Since the deviations are symmetrical about the mean pressure, there is no need of giving the detailed curves. The pressure coefficient is the average coefficient between 0 and 12000 kg.

Within the limits of error the deviation from linearity is symmetrical and parabolic; the numerical values are given in Table XVII.

The temperature coefficient found above between 0° and 100° was 0.00297. Kaye and Laby give 0.0033, evidently taken from W. von Bolton. On consulting the original paper,[17] however, it will be found that the value is given with only one significant figure, 0.003. The agreement of our results is, therefore, within the limits of error. The General Electric Co. was not able to give any information about the purity apart from that afforded by the temperature coefficient.

There seem to be no previous measurements of the pressure coefficient for comparison.

When scaled to the same initial resistance, the resistance-pressure curves are found to be steeper at the higher temperatures, but less curved.

*Tungsten.* This was obtained from the General Electric Co. through the kindness of Dr. W. D. Coolidge. It is bare and 0.0004 inch diameter. It was seasoned for temperature at the same time as Mo and Ta and for pressure by a single application of 12000 kg. at 0°. There was no perceptible change of zero after the very first application of pressure. It was wound loosely in two or three turns on a smooth bone cylinder, and held loosely in place with silk thread. Its initial resistance at 0° was 79.0 ohms. Connections were made with gold solder to platinum by arcing in hydrogen.

The smoothed results are collected in Table XVIII, and the experimental values of mean coefficient and departure from linearity are shown in Figure 20. The zero drift during a run was not large, being respectively 0.25%, 0.0%, 0.56%, 0.08%, and 0.054% of the total effects at 0°, 25°, 50°, 75°, and 100°. The individual results were irregular, however. At 75° and 100° in particular, there are large irregularities in a direction opposite from that of hysteresis. The maximum departure of any point from a smooth curve was 0.7% of the total effect, and the numerical average 0.15%. The deviation from linearity is slight; within the limits of error it is symmetrical and parabolic. The characteristic numerical values are given in Table XVIII.

The average temperature coefficient found above is 0.00322. This is low compared with the value 0.0050 of Somerville.[16] He found a rather unusually large departure from linearity with temperature. The source of Somerville's tungsten was the same as mine, the General

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[17] W. von Bolton, ZS. Elektrochem. **11**, 45 (1905).

## TABLE XVIII.

### TUNGSTEN.

| Temp. °C. | Resistance | Pressure Coefficient |  |  | Maximum Deviation from Linearity | Pressure of Maximum Deviation |
|---|---|---|---|---|---|---|
|  |  | At 0 kg. | At 12000 kg. | Average 0–12000 kg. |  |  |
| 0° | 1.0000 | $-.0_5 1279$ | $-.0_5 1189$ | $-.0_5 1234$ | −.000135 | 6000 |
| 25 | 1.0795 | 1283 | 1197 | 1240 | 14 | 6000 |
| 50 | 1.1595 | 1288 | 1204 | 1246 | 145 | 6000 |
| 75 | 1.2402 | 1292 | 1212 | 1252 | 15 | 6000 |
| 100 | 1.3219 | 1297 | 1219 | 1258 | 155 | 6000 |

Electric Co., but his sample was obtained in 1910. There would seem to be no reason to suspect the purity of his sample to be greater than mine. Dr. Coolidge says that the impurity is of the order of 0.2%.

There seem to be no previous measurements of the pressure coefficient for comparison.

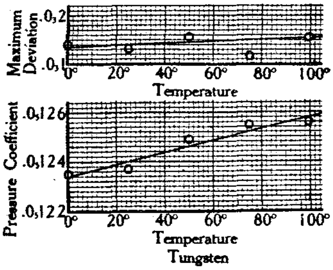

FIGURE 20. Tungsten, results for the measured resistance. The deviations from linearity are given as fractions of the resistance at 0 kg. and 0°C. Since the deviations are symmetrical about the mean pressure, there is no need for giving the detailed curves. The pressure coefficient is the average coefficient between 0 and 12000 kg.

The general behavior is like that of Tantalum. When scaled to the same initial resistance, the resistance-pressure curves become steeper at the higher temperatures, but have less curvature.

*Antimony.* This was obtained from Eimer and Amend. They were so kind as to make a special analysis for me; this showed Sb 99.45%, As trace, Fe trace, other foreign metals none. It was formed into wire 0.013 inch diameter by hot extrusion. It is necessary to heat to 350° or 400° C, and apply a pressure of 10000 or 15000 kg. A little trick is necessary to successfully extrude it; this matter is described at length in a forthcoming number of The Physical Review. The wire is excessively brittle. It cannot be wound into any sort of a spiral, but must be used in straight lengths, in a sort of grid. The grid used was composed of two pieces laid together in the form of a V in grooves cut on a bone cylinder, attached together by soldering with "fine" solder at the bottom of the V. Connections to the leads was made by soft soldering at the top to flexible copper conductors made of many strands of wire 0.002 inch diameter. The soldering must be performed by a rapid touch, because the fine wire alloys very rapidly to form an alloy of much lower melting point than the solder applied. The initial resistance at 0° was 0.90 ohms. The sensitiveness is not as great as would have been desirable, but the difficulties of manipulation in making a larger grid would have been very great.

Six runs were successfully made, at 0°, 25°, 50°, 75°, 100°, and at 0° again. It was a great gratification that such high pressures could be applied without rupturing this excessively brittle substance. At the low temperatures pressure was transmitted by petroleum ether, and at higher temperatures by kerosene. The behavior of antimony is abnormal, like that of bismuth, since it has a large positive pressure coefficient. There is rather large hysteresis, but within the limits of error the mean of points with increasing and decreasing pressure are linear. At 0° there was a permanent increase of resistance of 5% of the total resistance after the first application of pressure. After the first application of pressure the permanent changes of resistance remained much smaller.

The smoothed results are shown in Table XIX, and the experimental values of the mean coefficient in Figure 21. The change of pressure coefficient with temperature is abnormally large.

The temperature coefficient of this antimony wire between 0° and 100° was 0.00473. This may be compared with the value 0.00418 for cast antimony given by Matthiesen and von Bose.[18] Doubtless part of the difference at least can be ascribed to difference of mechanical treatment.

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[18] A. Matthiesen und M. von Bose, Pogg. Ann. **115**, 353–396 (1862).

## TABLE XIX.

### ANTIMONY.

Range 12000 kg.

| Temperature | Resistance at 0 kg. | Average Pressure Coefficient |
|---|---|---|
| 0° | 1.000 | $+.0_41220$ |
| 25 | 1.113 | 1107 |
| 50 | 1.229 | 994 |
| 75 | 1.349 | 881 |
| 100 | 1.473 | 768 |

There are no previous measurements of the pressure coefficient. The temperature measurements were continued from 100° to 170°, because a polymorphic transition has been suspected above 100°. In fact I did find a discontinuity in resistance at about 145°, thus verifying the approximate location of a transition as found by Jaenecke.[19]

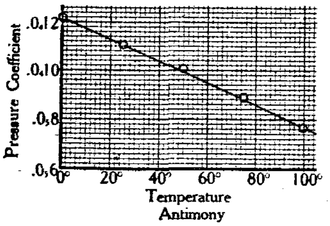

FIGURE 21. Antimony, the pressure coefficient at different temperatures. The coefficient is constant between 0 and 12000 kg.

No trace of the transition was found at any pressure in the temperature range of the measurements above, however, and the results are unaffected by the existence of the transition. The study of this transition is a separate question which must be taken up later. It must first of all be established that it is a true polymorphic transition in the sense that the crystals pass from one crystalline system to another.

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[19] E. Jänecke, ZS. f. phys. Chem. **90**, 313–339 (1915).

*Tellurium.* I owe this to the kindness of Professor G. W. Pierce, who obtained it from Mr. Samuel Wien of New York City. Beyond the statement that it had been especially refined by Mr. Wien, I have no direct knowledge of its purity. It was formed into wire 0.013 inch diameter by hot extrusion. When the right temperature has been found by trial, pieces one or two feet long may be obtained. Like antimony it is excessively brittle, and must be used in the shape of a grid of straight pieces.

Tellurium is only semi-metallic in its properties, and can not be soldered in the ordinary way. When fused it will stick to platinum, however. Connections were made by fusing to fine platinum wire. The end of the tellurium wire was allowed to rest by its own weight on a piece of fine platinum, and current was passed through the platinum sufficient to melt the tellurium and flow it over the surface. This connection was not entirely satisfactory, however, but apparently cracked above 10000 kg. because of the unequal compressibility of tellurium and platinum. There was a very large permanent increase of resistance after the run.

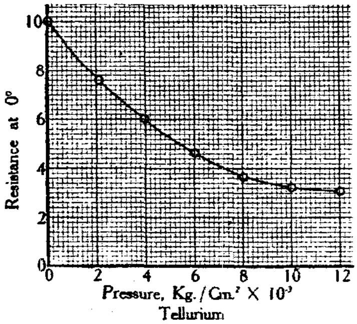

FIGURE 22. Tellurium, resistance as a function of pressure at 0°C. This curve was obtained with increasing pressure; on releasing pressure the curve is not retraced, and there is a large permanent increase of resistance.

Only one run was made on tellurium, at 0°. The points with increasing pressure were regular, and are shown in Figure 22. The resistance decreases under pressure, and the decrease is abnormally large, initially ten times more rapid than that of lead.

All of the electrical properties of tellurium are unusual, and are very variable; it is seldom that two observers obtain the same numerical value for any of its properties. The explanation for this seems to be that tellurium under ordinary conditions contains two modifications in unstable equilibrium, and the proportions of the two varieties changes greatly with the manner of treatment. The temperature coefficient of the same piece may be positive or negative according to the thermal treatment. The pressure coefficient of tellurium has been very recently measured by Beckman [6] who used small rods of tellurium cast in glass. His specimens were subjected to various thermal treatments, and it was possible to find a connection between the pressure coefficient and the manner of treatment. The specific resistance of his samples varied from 0.0493 to 0.617 ohms per cm. cube. The pressure coefficient varied linearly with the resistance from $-9.1\times 10^{-5}$. to $-26.6\times 10^{-5}$.

The specific resistance of the specimen used above was much lower than any found by Beckman, being only 0.00645 ohms per cm. cube. The initial pressure coefficient was $-0.00012$, which is included in Beckmann's range. The average temperature coefficient between 24° and 0° was $-0.0063$. This is larger numerically than any of the negative coefficients found by Beckman. It is evident therefore, that the properties of the extruded metal are quite different from those of the cast metal.

*Bismuth.* Runs were made on three different samples of varying purity. It is known that the specific resistance and temperature coefficient of bismuth are very sensitive to small impurities; a fraction of a per cent of lead or tin may change the temperature coefficient from positive to negative and increase the specific resistance several fold. Minute impurities also introduce complicated hysteresis effects. The samples used in this work were Kahlbaum's ordinary grade, Kahlbaum's "K" grade, and electrolytic bismuth of my own preparation. The purification of bismuth is a matter of some difficulty; it is known that no chemical method will suffice. I used the method of electrolysis from the solution of the fluo-silicate described by Foerster and Schwabe.[20] It is essential to keep the current density low and to continually stir the bath.

The wire, which was 0.013 inch in diameter, was formed in each case by extrusion when warm. It was wound loosely on a bone core and connected to the leads with "fine" soft solder. At 100° the purest sample alloyed with the solder and the leads dropped off, so that the run at 100° was not made; the connections to the two impurer samples were undamaged at 100°. The initial resistances at 0° were 3.8, 4.0, and 6.9 ohms respectively with decreasing purity; the lengths of the wires were in all cases about the same.

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[20] F. Foerster und E. Schwabe, ZS. Elektrochem. **16**, 279 (1910).

The effect of pressure on bismuth is abnormal, as it is for antimony, the pressure coefficient being large and positive instead of negative as usual. The positive coefficient was shown by all three samples. The most marked effect of impurity is that after a cycle of changes of temperature or pressure resistance does not return to its original value. Furthermore, the runs at constant temperature show great pressure hysteresis; this hysteresis becomes less as the impurity decreases. At 0° the most impure specimen showed a hysteresis of 12% of the total effect, and a permanent change of resistance after the run of 8% of the effect. The hysteresis is without doubt due to internal viscosity, which prevents the constitution of the mixed crystals keeping exact pace with the changes of pressure. This is suggested strongly by the fact that after every change of pressure an abnormally long time elapsed before the resistance became approximately constant. This view is also consistent with the observation that at higher temperatures the hysteresis rapidly became less. On making a run at 0° again after the run at 100° the original hysteresis had returned to almost its original value; the decreasing hysteresis at higher temperatures cannot, therefore, be an effect of accommodation. The average pressure coefficient between 0 and 12000 kg. decreased from $+0.0_4281$ at 0° to $+0.0_4201$ at 100°. The sample of grade "K" showed much less hysteresis, and did not require an appreciably longer time than any other metal to reach equilibrium after a change of pressure. The maximum permanent change of zero was 4% of the pressure effect at 0°, and the maximum hysteresis was also at 0° and was 3.5% of the total effect. The average pressure coefficient between 0 and 12000 kg. decreased from $+0.0_4269$ at 0° to $+0.0_4205$ at 100°.

The electrolytic bismuth was seasoned before the runs by several excursions between 0° and 130° at atmospheric pressure and by one application of 12000 kg. at 0°. It showed no peculiarity of behavior suggesting impurity except a hysteresis amounting at the most to 3% of the total effect at 0°. The permanent change of zero after a run was not over 0.3% of the total effect, which is not larger than found for several other metals, and no longer time was required for the attainment of steady conditions than was necessary for dissipation of the heat of compression. The curves of actual resistance against pressure are shown in Figure 23 at 0°, 25°, 50°, and 75°, and the numerical values of resistance are given in Table XX. It has already been mentioned that the run at 100° was omitted because the solder alloyed with the bismuth and melted it off. The resistance curves are convex toward the pressure axis; furthermore the steepness increases with pressure faster than the resistance itself, so that the instantaneous

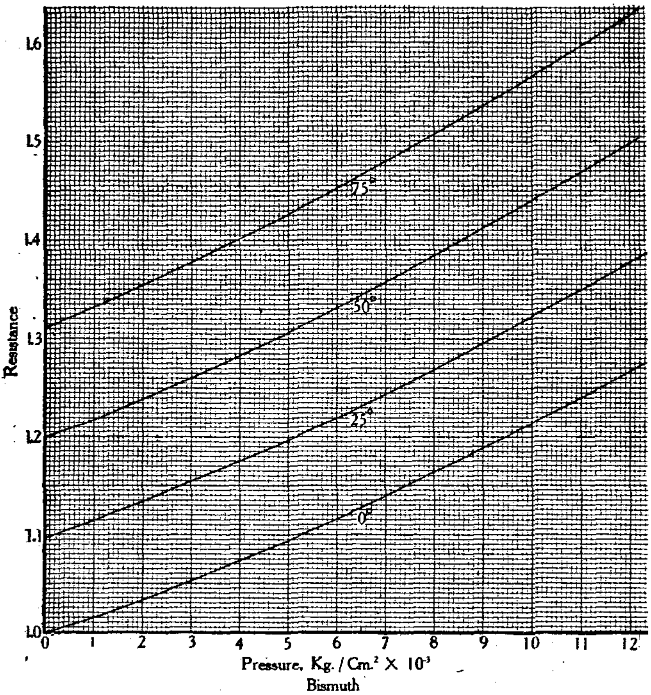

FIGURE 23. Bismuth, resistance as a function of pressure at several temperatures.

pressure coefficient increases also with pressure. The same abnormal curvature was shown by the two impurer samples. The numerical values of the average and instantaneous coefficients are shown in Table XXI. The behavior here is exactly the reverse of that of normal metals, for which the pressure coefficient becomes numerically less at higher pressures. The experimental values of the average coefficient to 12000 kg. are shown in Figure 24.

## TABLE XX.

### Resistance of Bismuth.

| Pressure kg/cm² | Resistance | | | |
|---|---|---|---|---|
| | 0° | 25° | 50° | 75° |
| 0 | 1.0000 | 1.0969 | 1.1996 | 1.3115 |
| 2000 | 1.0336 | 1.1333 | 1.2389 | 1.3540 |
| 4000 | 1.0726 | 1.1748 | 1.2830 | 1.4013 |
| 6000 | 1.1163 | 1.2204 | 1.3316 | 1.4527 |
| 8000 | 1.1638 | 1.2698 | 1.3841 | 1.5078 |
| 10000 | 1.2143 | 1.3225 | 1.4498 | 1.5664 |
| 12000 | 1.2672 | 1.3779 | 1.4980 | 1.6279 |

It is evident from a comparison of the data for the different grades of bismuth that the effect of impurity on the temperature coefficient is much greater than on the pressure coefficient. The temperature coefficient of the impurest bismuth above was on the average negative (= − 0.00039) over the range 0° to 100°; its resistance passed through a minimum at 75°. The coefficient of grade "K" was 0.00332 between 0° and 100°, and the instantaneous coefficient at 0°, was about 0.0027. The average temperature coefficient of the electrolytic bismuth above may be found from an extrapolation of the data of Table XX to be 0.00438, and the instantaneous coefficient at 0° 0.00381. Another sample from the same piece of wire, which had not been subjected to pressure at all, gave 0.00441 between 0° and 100°, and 0.00389 at 0°. These latter values may be accepted as more probably correct for this material in a state of ease.

The mean temperature coefficient of pure bismuth between 0° and 100° is given by Jaeger and Diesselhorst [8] as 0.00454, but as already explained, their value was obtained by an extrapolation, assuming linearity. If we assume that their deviation from linearity was the same as that found above, their temperature coefficient between 0° and 100° corrects to 0.00438, slightly less than the best value above.

## TABLE XXI.

PRESSURE COEFFICIENT OF BISMUTH.

| Temp. | Pressure Coefficient | | | |
|---|---|---|---|---|
| | Average | Instantaneous Coefficient | | |
| | 0–12000 kg. | 0 kg. | 6000 kg. | 12000 kg. |
| 0° | $+.0_42227$ | $+.0_4153$ | $+.0_4195$ | $+.0_4212$ |
| 25 | 2142 | 155 | 198 | 206 |
| 50 | 2076 | 155 | 192 | 199 |
| 75 | 2023 | 154 | 185 | 194 |

Other values for the temperature coefficient between 0° and 100° are 0.00429 by v. Aubel,[21] and 0.00458 by Lenard.[22] This last is considerably higher than any of the above. It is, however, difficult to estimate the accuracy which Lenard and v. Aubel ascribed to their own measurements. Lenard and Howard [23] give without comment as

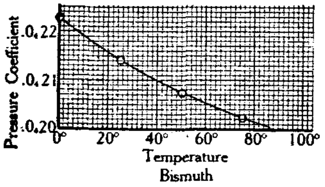

FIGURE 24. Bismuth, the average pressure coefficient between 0 and 12000 kg. as a function of temperature.

the coefficient of "analytically" pure bismuth (i. e. bismuth prepared chemically and known to contain some impurity) the impossibly high value 0.0052. Lenard, discussing his value 0.00458 for electrolytic bismuth says that he found, "as did v. Aubel," that the temperature coefficient was constant between 0° and 100°. Lenard gives none of the details of measurement; in view of his remark it is possible that he extrapolated as did Jaeger and Diesselhorst. On referring to v. Aubel it is found that he does indeed emphasize the constancy of the temperature coefficient, but that his numerical values show a consistent increase from 0.00412 between 0° and 19.5° to 0.00450 between 0° and 99.7°. It would seem, therefore, that the temperature coefficient of perfectly pure bismuth has not yet been definitely established, but that there is no reason to suspect that the electrolytic bismuth measured above contains enough impurity to sensibly affect the result under pressure.

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[21] E. v. Aubel, Phil. Mag. **28**, 332 (1889).
[22] Ph. Lenard, Wied. Ann. **39**, 619 (1889).
[23] Ph. Lenard und J. L. Howard, Electrot. ZS. **9**, 340 (1880).

For the pressure coefficient there is only one other determination, by Williams,[13] over a pressure range of 300 kg. at 0°. Within the limits of error he found the relation between pressure and resistance to be linear, and the coefficient to be $+0.0_4191$. He does not give the temperature coefficient of his specimen, but states that it was a spiral of electrolytic bismuth from Hartmann and Braun. The initial value which I found above was $0.0_4153$. It is significant that the impurer grades gave a higher initial coefficient; that of the Kahlbaum "K" specimen was $0.0_421$.

The distinctive features of the behavior of bismuth are as follows; the average pressure coefficient is positive, increasing in numerical value with increasing pressure and decreasing with increasing temperature, and the instantaneous coefficient at 0 kg. is nearly independent of the temperature, but at higher pressures it decreases at the higher temperatures. This last point means that as pressure increases the temperature coefficient of resistance decreases.

## General Survey of Results.

In Figure 25 are collected curves for all the metals measured, except Te, Bi, Sb, and Mg, giving the average pressure coefficient to 12000 kg. as a function of temperature. The most obvious and striking feature is the slight variation of coefficient with temperature; the variation is in all cases much less than the change of resistance itself. To see the significance of this, let us for the moment suppose that the coefficient is strictly constant with temperature. If this is true, the curve of resistance against pressure at any temperature may be obtained from that at any other temperature merely by changing the scale of the ordinates by the proper factor. It would therefore follow that the temperature coefficient of resistance would be strictly independent of

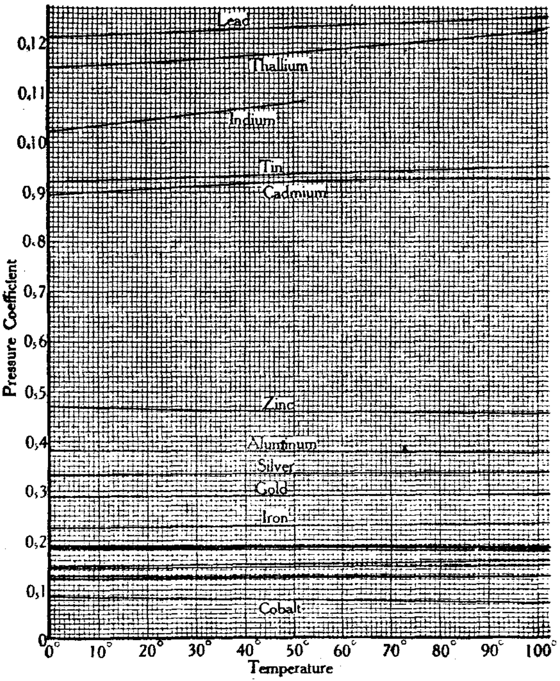

FIGURE 25. Collection of results, the average pressure coefficient between 0 and 12000 kg. as a function of temperature. The seven curves in the lower part of the diagram without labels are, reading upwards, for tungsten, molybdenum, tantalum, nickel, copper, platinum, and palladium.

pressure. As a matter of fact, the temperature coefficient of all substances is only slightly different at 12000 kg. from its value at 0 kg. Now, dropping the condition of invariability, it is easy to see that if the pressure coefficient of resistance increases at higher temperatures the temperature coefficient will be less at higher pressures, and vice-versa. In Table XXII are collected the average temperature coefficients between 0° and 100° at 0 and 12000 kg. for all the substances for which the data have been obtained. Except for the abnormal metals, the slight change in temperature coefficient is striking.

The meaning of a temperature coefficient independent of temperature may be stated in another way. Let us compare at 0° and 100° respectively the slope of the lines on which resistance is constant with those on which volume is constant. Since it is a matter of experiment that the compressibility and thermal expansion of most metals change relatively little between 0° and 100°, the slope of the constant volume line will be nearly independent of temperature. Since furthermore the relation between temperature and resistance is nearly linear for

## TABLE XXII.

AVERAGE TEMPERATURE COEFFICIENTS BETWEEN 0° AND 100° AT 0 KG. AND 12000 KG.

| Metal | Average Temp. Coefficient At 0 kg. | Average Temp. Coefficient At 12000 kg. |
|---|---|---|
| In | .00406 | .00383 |
| Sn | 447 | 441 |
| Tl | 517 | 499 |
| Cd | 424 | 418 |
| Pb | 421 | 412 |
| Zn | 416 | 420 |
| Al | 434 | 435 |
| Ag | 4074 | 4069 |
| Au | 3968 | 3964 |
| Cu | 4293 | 4303 |
| Ni | 4873 | 4855 |
| Co | 3657 | 3676 |
| Fe | 6206 | 6184 |
| Pd | 3178 | 3185 |
| Pt | 3868 | 3873 |
| Mo | 4336 | 4340 |
| Ta | 2973 | 2967 |
| W | 3219 | 3216 |
| Sb | 473 | 403 |
| Bi | 438 | 395 |

most metals between 0° and 100°, the actual change in resistance in ohms of a given piece of wire will be the same for 1° rise of temperature at 100° as at 0°. But since the pressure coefficient is approximately constant, the change of resistance for 1 kg. will therefore, be greater at 100° than it is at 0° in the ratio of the resistance at 100° to that at 0°. For most metals this is of the order of 40% difference. Therefore the slope of the line at constant resistance is greater at 100° than at 0°.

Since ordinarily the line of constant volume has a greater slope than that of constant resistance, it will result that at higher temperatures the two lines approach each other. If this tendency persists, it means that at high enough temperatures the resistance of a solid will *decrease* instead of *increase* along a line of constant volume with increasing temperature. This is an important point for theoretical considerations.

To a closer degree of approximation the pressure coefficient is not independent of temperature. The manner of dependence is plainly obvious from Figure 25; the coefficient may rise or fall with temperature. Furthermore, the relation between coefficient and temperature need not be linear; there are six examples of non-linear relation, Tl, Cd, Zn, Al, Ag, and Cu. The departure from linearity is so slight that it is not obvious on the scale of Figure 25. As a general rule, the coefficient increases with temperature for metals of low melting point, and decreases for those with higher melting points, although there are several exceptions. Except for this, there seems no obvious connection between the manner of departure from constancy and other physical properties.

At any constant temperature the relation between pressure and resistance is not linear, but the slope of the resistance-pressure curve becomes less at the higher pressures. This is true without exception for all the metals with a negative pressure coefficient, and is only what one would expect. The manner of departure from linearity varies from metal to metal, however. The variation is not regular, so the simple types of formula hitherto proposed to represent the dependence of resistance on pressure cannot be valid. There is a general tendency, however, for the maximum departure from linearity to be greater for those metals with the larger coefficient, as one would expect. Furthermore, the ratio of the maximum departure from linearity to the pressure coefficient is also greater for the greater coefficients. This means that as the effect of pressure on resistance increases from metal to metal, the *relative* curvature of the resistance-pressure curves increases also. This ratio is roughly proportional to the value of the pressure coefficient, as is shown by Figure 26, but there are several well marked exceptions, particularly at the smaller coefficients.

For any one substance, the variation with temperature of the ratio to the pressure coefficient of the maximum departure from linearity with pressure is of interest. This has already been mentioned under the individual substances: a curve with greater curvature at the higher temperature means a larger value for the ratio. The facts are a trifle surprising here; for Pb, Tl, In, and Sn, the substances with the largest average pressure coefficient, the curvature is greatest at the highest temperature, but for all other normal substances it is less. For some substances, such as cobalt, the magnitude of the variation is surprisingly large; half the relative curvature disappears between 0° and 100°. This effect, although small, cannot be explained by errors in the pressure measurements, because it will be remembered that the manganin measuring coil was always at the same temperature.

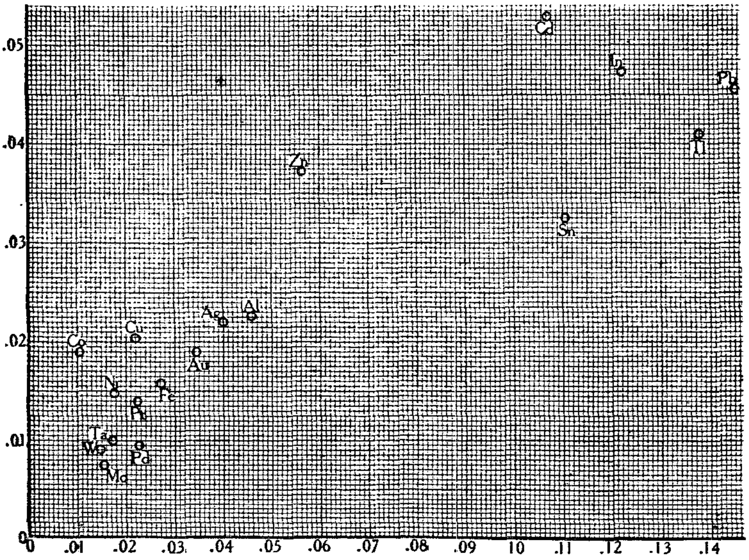

FIGURE 26. The ratio, at 0°, of the maximum departure from linearity to the change of resistance under 12000 kg. plotted against the change of resistance under 12000 kg. The diagram shows that in general the relative curvature of the resistance-pressure curves increases as the absolute value of the pressure coefficient increases.

The position of the pressure of maximum deviation from linearity is another characteristic datum; this has already been tabulated under the individual substances. This pressure shows a distinct tendency to fall below the mean pressure, that is, below, 6000 kg. This is as one would expect if the instantaneous pressure coefficient tends toward constancy, that is, if the equation of resistance against pressure can be written $R=R_0e^{-ap}$. As a matter of fact, if the equation could be written in this form, the pressure of maximum deviation would be approximately 3000 instead of 6000 kg. The closest approach to it is by Au and Cu at 0°, for which the pressures of maximum deviation are 4440 and 4200 respectively. The fact that the pressure of maximum deviation is so much above 3000 kg. for most metals means that the instantaneous coefficient becomes markedly less at the higher pressures. This is brought out in Table XXIII, in which the instantaneous coefficients at 0° are compared at 0 kg. and 12000 kg. The instantaneous coefficient at 12000 kg. was obtained by dividing the value already given in the tables by the resistance at 12000. Such a decrease of pressure coefficient with pressure is perhaps surprising after finding the constancy under pressure of the temperature coefficient. The displacement of the pressure of maximum deviation towards higher

### TABLE XXIII.

Comparison of Instantaneous Pressure Coefficient at 0° and 0 kg with that at 0° and 12000 kg.

| Metal | Instantaneous Pressure Coefficient at 0°. | |
|---|---|---|
| | At 0 kg. | At 12000 kg. |
| In | $-.0_4123$ | $-.0_4102$ |
| Sn | $0_4104$ | $0_5936$ |
| Tl | $0_4132$ | $0_4118$ |
| Cd | $0_4106$ | $0_5837$ |
| Pb | $0_4144$ | $0_4122$ |
| Zn | $0_5540$ | $0_5425$ |
| Al | $0_5416$ | $0_5365$ |
| Ag | $0_5358$ | $0_5321$ |
| An | $0_5312$ | $0_5286$ |
| Cu | $0_5201$ | $0_5179$ |
| Ni | $0_5158$ | $0_5142$ |
| Co | $0_6941$ | $0_6814$ |
| Fe | $0_5241$ | $0_5218$ |
| Pd | $0_5198$ | $0_5190$ |
| Pt | $0_51975$ | $0_5181$ |
| Mo | $0_5133$ | $0_5126$ |
| Ta | $0_5149$ | $0_5139$ |
| W | $0_5128$ | $0_5121$ |

pressure at the higher temperatures means that at higher temperatures the instantaneous pressure coefficient decreases more rapidly with increasing pressure than at lower temperatures. This comes to much the same thing as the statement that the curvature is less at the higher temperature, although the two statements are not entirely coextensive. The rate at which the pressure of maximum deviation is displaced at higher temperatures is very different for various substances and seems to have no obvious connection with other properties.

To sum up: different metals show minor irregularities in behavior, but they are alike in several general features which must be the first task of any theory to explain. These general features are the approximate constancy of pressure coefficient with temperature, and the accompanying constancy of temperature coefficient with pressure; contrasted with this the pronounced decrease of instantaneous pressure coefficient with rising pressure, and, of less compelling importance, the decrease in the curvature of most of the resistance-pressure curves at higher temperatures.

## Bearing of the Results on the Question of the Metastability of the Metals.

In the last few years a great deal of work has been done by Cohen[24] and his pupils on the variation in the behavior of metals after different kinds of heat treatment. They have found very small discontinuities in various physical properties which have been interpreted as indicating that a number of the metals occur in several polymorphic forms. Similar discontinuities have also recently been found by Jänecke,[19] but almost always at higher temperatures. The existence of such modifications is important both from the practical point of view and because of the intimate relation to the theory of allotropy of Smits.[25] Cohen's result for copper has recently been called in question by Burgess and Kellberg,[26] who could find no discontinuity in the electrical resistance in the expected place.

The results above throw light on the same question. If there are polymorphic transitions, there should be discontinuities in the resistance, and the discontinuity should be at different pressures at different temperatures. The metals for which such discontinuities should be expected between 0° and 100° are Cd, Pb, Cu, Zn, and Bi, according to Cohen. No such effects were found. The sensitiveness of the measurements may be estimated from the data already given; the accuracy is in most cases great enough so that a discontinuity of the order of 1/100% of the total resistance could have been detected. According to Jänecke, however, the transitions all occur above 100°, and none should have been found under pressure, if the phase stable at the higher temperature has the greater volume. I did find a discontinuity at 140° for antimony, which is much nearer the value of Jänecke than of Cohen. This has already been discussed. One of the metals examined is certainly known to have a transition in the temperature range 0° to 100°, tin at 20°. But the transition never starts under ordinary conditions, and one need not expect to find it under pressure. This has already been made the subject of a special investigation.[27]

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[24] E. Cohen, Numerous papers in Proc. Amst. Acad. since 1913.
[25] A Smits, Proc. Amst. Acad., numerous papers 1910–15.
[26] G. K. Burgess and I. N. Kellberg, Jour. Wash. Acad. Sci. **5**, 657–662(1915).

In criticism of the results, one may well admire the skill and care which Cohen and his pupils have bestowed on the measurement of these very minute effects; the existence of the discontinuities which they have found may doubtless be accepted. But it seems to me that their interpretation of the results may well be questioned; the existence of a discontinuity, more or less indefinite, need not of itself be an indication of true polymorphism. It seems that there are many possibilities in the rearrangement of crystalline grains or growth of the larger crystals at the expense of smaller ones (such as have been found by Ewing and Rosenhain [28] to be stimulated by strains), and that these possibilities of explanation should first be exhausted. The facts that different observers find different transition points and that it is in almost every case necessary to assume more than two modifications to explain the results lend color to suspicion. Furthermore, if the discontinuities are truly polymorphic in character, there was the best possible chance to detect them under pressure, but none were found. Before the interpretation assigned by Cohen to the results can carry conviction, it would seem to me that we have a right to ask for reproducible results with large individual crystals. It would be worth much effort to prepare such crystals in order to settle this vexed question.

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[27] P. W. Bridgman, Proc. Amer. Acad. 52, 164 (1916).
[28] J. A. Ewing and W. Rosenhain, Phil. Trans. (A), 353 (1900).

## THEORETICAL BEARINGS.

Before discussing the bearing of these results on electron theories of metals, it will pay to emphasize two points. The first is that the coefficients tabulated are the actual observed coefficients, measured by the ordinary methods with electrodes permanently fixed to determinate parts of the surface. But in theoretical discussion we are more inclined to be interested in the variation of specific resistance. To get this, the observed results must be corrected by a factor equal to the change of linear dimensions. It is easy to see that for normal metals the temperature coefficient of observed resistance is numerically smaller than the temperature coefficient of specific resistance by the linear thermal dilatation, and the pressure coefficient of observed resistance is numerically less than the pressure coefficient of specific resistance by the linear compressibility. This factor is not important for the temperature coefficient, rising in the extreme case above (In) to 1%, but for the pressure coefficient it may amount to 10% in some cases. In making correction for the compressibility we are confronted by the difficulty that only one or two compressibilities have been measured over any extensive pressure range. For the initial compressibility we have the data of Richards;[29] at higher pressures the best that we can do is to neglect the change of compressibility with pressure. For the less compressible metals any such change is probably slight. A number of years ago I measured the compressibility of iron up to 10000 kg. and of aluminum to 6500 kg., and could find no variation over this range.[30] Of course in any discussion of the pressure coefficient of specific resistance at atmospheric pressure this source of uncertainty does not enter.

The second observation is concerning the magnitude of the effects. It has been obvious enough that the data have presented no spectacular features, and I must confess to a sense of disappointment that an extension of the pressure range to at least four fold that of previous measurements has brought out no striking new facts to reward the extra effort. It is true that as far as I am aware the independence of temperature coefficient and pressure was not previously known, or at least was never emphasized, but it might have been discovered by measurements to only 3000 kg. if one had been willing to take the trouble.

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[29] T. W. Richards, Jour. Amer. Chem. Soc. **37**, 1643–1656 (1915).
[30] P. W. Bridgman, Proc. Amer. Acad. **44**, 255–279 (1909) and **47**, 366 (1911).

When the magnitude of the change of volume produced by a pressure of 12000 kg. is considered, however, it does seem that the results acquire a physical significance great enough to justify the extension of the range. The volume of many of the metals at 0°C and 12000 kg. is less than the volume at atmospheric pressure at 0° Abs. The resistance of most metals tends towards zero at 0° Abs., but at 0° C at the same volume the resistance is only a few per cent less than under normal conditions. Any valid theory must explain the surprisingly little effect of the element of volume alone apart from the element of temperature. It is furthermore known that at very low temperatures the connection between resistance and temperature changes its character; the relation ceases to be linear, and the resistance curve approaches the origin tangentially to the temperature axis. Whether the abrupt discontinuity shown by several metals a few degrees above 0° Abs. is an effect of a polymorphic transition does not yet seem to be settled. It is significant that no trace of any such effect is to be found at room temperature as the volume is decreased toward and beyond its value at 0° Abs. The question whether there is a change in the character of the resistance curves as the volume approaches that at 0° Abs. could not, of course, have been answered by measurements over a small pressure range; it is perhaps some justification of the extension of range that this question can now be answered.

An estimation as to the comparative volumes at (12000 kg., 0°C) and (0 kg., 0° Abs.) is given in the accompanying Table XXIV. The values of compressibility used in the computations have been taken from Richards[29], assuming constancy over the pressure range, and the volume at 0° Abs. has been taken from the data of Ch. Lindemann[31] on linear expansion to 20° Abs. The Table includes all the metals measured by Lindemann to 20°. Linear extrapolation of data of Grüneisen[32] to liquid air temperature shows that tin and magnesium also have a smaller volume under 12000 kg. than at 0° Abs., and probably iron does also. The two metals antimony and bismuth which are abnormal with respect to pressure coefficient are also abnormal here; the decrease of volume under 12000 kg. is more than three times as great as that on cooling to 0° Abs.

Let us now consider the bearing of the facts at high pressures on various proposed theories of electronic conduction in metals. We discuss first their relation to the classical gas-free-electron theory of

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[31] Ch. L. Lindemann, Phys. ZS. **12**, 1197–1199 (1911).
[32] E. Grüneisen, Ann. Phys. **33**, 33–78 (1910).

## TABLE XXIV.

### COMPARISON OF CHANGES OF VOLUME PRODUCED BY TEMPERATURE AND PRESSURE.

| Metal | Change of volume between 0°C. and 0°Abs at at 0 kg. | Change of volume between 0 kg. and 12000 kg. at 0°C. |
|---|---|---|
| Pb | .0189 | .0275 |
| Zn | .0057 | .0200 |
| Al | .0096 | .0173 |
| Ag | .0108 | .0119 |
| Cu | .0078 | .0089 |

Riecke, Drude, and Lorentz. The specific resistance according to this theory is given by

$$w = Const \frac{u}{Nl} = Const' \frac{T^{\frac{1}{2}}}{Nl}$$

where $u$ is the velocity of the electrons, $l$ their mean free path, and $N$ the number per cm$^3$. The constant has different numerical values according to Drude and Lorentz. The second form of the equation is obtained by putting $u^2$ proportional to $T$.

Apart from specific heat difficulties, which it seems to me have been over-emphasized, this theory has always had difficulty in giving a plausible explanation of the variation of resistance with temperature, it being necessary to suppose that $l$ decreases as the volume increases with rising temperature. But since the mean free path is supposed entirely determined by the positions of the atoms, these being immobile compared with the free electrons, the hypothesis of decreasing $l$ is difficult. A suggestion as to a way out is by considering the collisions among the electrons themselves; there may be a more than proportional increase in the number of such collisions as the space between the atoms becomes larger. But this possibility is removed when we consider the changes at constant volume. It is a result of these experiments that if a metal is warmed from 0° Abs. and pressure is simultaneously applied so as to keep the volume constant, that the change of resistance is very nearly the same as if the metal were allowed to expand freely when heated. But under heating at constant volume, the mean free path according to the fundamental point of view must remain constant, and we can explain the facts only by supposing N to decrease proportionally as $\sqrt{T}$ increases. Such a hypothesis is to say the least improbable.

The classical theory also meets difficulties in explaining the negative pressure coefficient of resistance at constant temperature. As volume decreases with increasing pressure at constant temperature, the mean free path must decrease, and the decreased resistance can be accounted for only by an increase in N. But from work by Wagner[33] on the effect of pressure on thermo-electromotive force, it appears that N must decrease slightly as pressure increases.

Dismissing, then, the gas-free-electron theory, we consider several of the recent attempts to improve upon it. Perhaps the most radical of these is the recent revival by J. J. Thomson[34] of a theory of his dating back to 1888, in which he assumed the atoms of a metal to be electric doublets continually emitting and absorbing electrons along their axes. In an electric field there is a resultant orientation in the direction of the field which gives rise to the current. This theory has many formidable difficulties, in fact it seems in certain aspects almost grotesque, but as Lees[35] has remarked, it seems to offer possibility of at least qualitative solution of many of the problems before which the older theory was helpless, and therefore should not be cast lightly aside. It is interesting to see what account this gives of the variation of resistance with pressure. At not low temperatures the formula for conductivity is

$$C = \frac{1}{3}\frac{NepdM}{kT}$$

where N is the number of doubtlets per $cm^3$, e electronic charge, p the number of electrons emitted by an atom per second, d the distance between centers of adjacent atoms, M the moment of the doublet, and k the gas constant. If now pressure is increased at constant temperature, d and M must both decrease, if anything, because of the change of dimensions. N will increase for the same reason, but this increase is sufficient to account for not over 25% of the effect. Hence most of the increase of conductivity must be due to an increase in p. Now this is a most unfortunate member to call on to do the brunt of the work, because it is already sorely overburdened. In fact, one may calculate p from data in a recent paper of Richardson[36] for tungsten at 2000°. It turns out that p is about $1.7 \times 10^{16}$. Since the figures given by Richardson lead to a minimum value of p it seems evident that parts of the theory must be radically recast. I owe the idea that p is improbably large to a remark of Professor E. H. Hall.

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[33] E. Wagner, Ann. Phys. **27**, 955–1001 (1908).
[34] J. J. Thomson, Phil. Mag. **30**, 192–202 (1915).
[35] C. H. Lees, Nat. **95**, 675–677 (1915).

Another recently suggested theory which seems to have possibilities is that of F. A. Lindemann.[37] According to this theory the electrons are rigidly arranged in the nodes of a space lattice between the atoms. The state of the electrons is therefore that of a perfect solid rather than of a perfect gas. Since however this theory gives essentially the same account of pressure effects as does Wien's recent theory as modified by Grüneisen, we may omit special discussion of this and pass to the Wien-Grüneisen theory.

The only serious attempt that has been made to bring pressure effects within the range of an electron theory has been by Grüneisen.[1] His starting point is the theory of Wien,[38] who supposes that the electron velocity is independent of temperature. With this assumption, combined with assumption of the quantum distribution of energy, Wien finds a function proportional to the mean free path. Starting with Wien's value of the free path, which he "generalizes", and with the help of his theorem that along a line at constant entropy $\frac{\beta\nu}{T}$ remains constant, Grüneisen finds a value for the pressure coefficient in terms of quantities most of which are known. There is reason to except that the unknown quantities are not as important as the others, and by neglecting them a formula is obtained for pressure coefficient in terms of compressibility, thermal expansion, specific heat, and temperature coefficient of resistance. The formula follows;

$$\frac{1}{w}\left(\frac{\partial w}{\partial p}\right)_t = \frac{1}{u}\left(\frac{\partial u}{\partial p}\right)_s - \frac{1}{N}\left(\frac{\partial N}{\partial p}\right)_s - \frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_s - \frac{1}{Cp}\left(\frac{\partial v}{\partial T}\right)_p\left[1 + \frac{1}{w}\left(\frac{\partial w}{\partial T}\right)_p T\right].$$

The first two terms, which represent the change of electronic velocity with pressure at constant entropy, and the change in the number of free electrons, may be neglected. Grüneisen compares his formula with experiment with surprisingly good results; in many cases the agreement is within the limits of error, and the worst discrepancy is for lead, where the difference is 50%. More recently Beckman[6] has extended the range of experimental material, and has applied Grüneisen's formula to all the available data. He concludes that the formula cannot be considered exact, but must be regarded only as an approximation. Grüneisen himself certainly did not claim more.

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[36] O. W. Richardson, Phil. Mag. **30**, 295–299 (1915).
[37] F. A. Lindemann, Phil. Mag. **29**, 127–140 (1915).
[38] W. Wien, Columbia Lectures, (1913), p. 29.

The new material of this paper allow a comparison with Grüneisen's formula over a somewhat wider range. Furthermore, since the numerical values found in this paper often differ considerably from those of Beckman, I have thought it worth while to recompute all the data. The results are shown in Table XXV. In the recomputation I have used my own values for the pressure coefficient and also for the temperature coefficient, except in several cases where higher values have been reported by other observers. In the recomputation I have paid attention to several minor points. For the temperature coefficient of resistance I have used the coefficient of *specific* resistance (which is strictly correct) instead of the coefficient of observed resistance. The difference is only a fraction of a per cent. Furthermore, I have used the instantaneous coefficient at 0° where the data are available, instead of the average coefficient between 0° and 100°. This again is the strictly correct procedure; for most substances it makes little difference, but for iron the difference is 8.5%. It must be recognized, however, that there are much greater uncertainties in the fundamental data entering the equations than can be introduced by the nicer points just mentioned. I have taken the fundamental data, except pressure and temperature coefficient, from the last edition of Landolt and Bornstein, selecting the values that seemed most consistent. I have given the preference to the values of specific heat of Jaeger and Diesselhorst at 18°. For zinc I have used the value of the thermal expansion of the National Physical Labaratory, $0.0_478$. There is much discrepancy for this substance between different observers; Grüneisen's own experimental value, $0.0_451$, seems certainly too low. The expansion of cadmium also is variously given; I adopted $0.0_490$ as a mean between Dorsey, $0.0_4881$, and Fizeau, $0.0_493$. The expansion of molybdenum and tungsten were taken from the Tables of the French Physical Society. It is to be noticed that the $C_p$ in Table XXV is in kg. cm. per cm$^3$ of material. The compressibility of indium has been assumed to be $2.0 \times 10^{-6}$, in

## TABLE XXV.

### Pressure Coefficient of Specific Resistance Calculated by Grüneisen's formula.

| Metal | Compressibility $-\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_\tau$ in kg | Thermal Expansion $\frac{1}{v}\left(\frac{\partial v}{\partial \tau}\right)_p$ | Temp. Coeff. of Specific Resistance at 0°C | Specific Heat $C_p$ kg. cm. per cm³ | Pressure Coefficient of Specific Resistance | | |
|---|---|---|---|---|---|---|---|
| | | | | | Computed Grüneisen | Observed | Computed by Beckman with G's formula |
| In | $2.0(?)\times 10_6$ | $.0_313$ | .00404 | 17.5 | $-.0_4137$ ?) | $-.0_4129$ | |
| Sn | 1.86 | $.0_451$ | 433 | 17.2 | $.0_5464$ | $.0_41106$ | $\begin{cases} .0_5672 \\ .0_582 \end{cases}$ |
| Tl | 2.26 | $.0_492$ | 520 | 16.4 | $.0_4113$ | $.0_4139$ | |
| Cd | 2.06 | $.0_490$ | 407 | 20.3 | $.0_5728$ | $.0_41132$ | $\begin{cases} .0_571 \\ .0_4117 \end{cases}$ |
| Pb | 2.28 | $.0_4879$ | 409 | 14.9 | $.0_4102$ | $.0_4152$ | $\begin{cases} .0_598 \\ .0_4104 \end{cases}$ |
| Zn | 1.66 | $.0_478$ | 407 | 27.8 | $.0_5427$ | $.0_5595$ | $\begin{cases} .0_5488 \\ .0_5276 \end{cases}$ |
| Mg | 2.84 | $.0_478$ | 396 | 18.3 | $.0_5601$ | $.0_555$ | |
| Al | 1.44 | $.0_474$ | 433 | 24.1 | $.0_5536$ | $.0_5464$ | $.0_5423$ |
| Ag | .99 | $.0_455$ | 406 | 24.9 | $.0_5372$ | $.0_5383$ | $.0_5384$ |
| An | .63 | $.0_443$ | 392 | 25.6 | $.0_5292$ | $.0_5333$ | $.0_5279$ |
| Cu | .735 | $.0_451$ | 431 | 35.2 | $.0_5241$ | $.0_5245$ | $.0_5211$ |
| Ni | .42 | $.0_4375$ | 62 | 40.9 | $.0_5206$ | $.0_5172$ | $\begin{cases} .0_5193 \\ .0_5157 \end{cases}$ |
| Co | (.50)(?) | $.0_437$ | 366 | 37.8 | $.0_5146$(?) | $.0_5110$ | |
| Fe | .58 | $.0_4375$ | 562 | 35.3 | $.0_5210$ | $.0_5260$ | $.0_5177$ |
| Pd | .53 | $.0_4355$ | 322 | 30.0 | $.0_5171$ | $.0_5216$ | $.0_5202$ |
| Pt | .27 | $.0_4292$ | 388 | 29.3 | $.0_5179$ | $.0_5207$ | $.0_5150$ |
| Mo | .45 | $.0_4108$ | 435 | 24.8 | $.0_650$ | $.0_5148$ | |
| Ta | .52 | $.0_4237$ | 298 | 23.1 | $.0_5135$ | $.0_5166$ | |
| W | .265 | $.0_4101$ | 318 | 27.7 | $.0_642$ | $.0_5137$ | |

absence of measurements, as fitting well into the periodic table as shown by Richards.[29] Also the compressibility of cobalt has not been measured; I have assumed a mean between iron and nickel.

For comparison, the computed values of Beckman, reduced from atmospheres to kilograms, are reproduced in Table XXV. It appears that the revised values sometimes give better and sometimes poorer agreement with theory: Beckman's conclusion need not be altered, therefore.

A mere comparison of the approximate formula with experiment is not at present sufficient to show, however, the correctness of Grüneisen's fundamental assumption, because the formula is obtained by neglecting an unknown term, $\frac{1}{u}\left(\frac{\partial u}{\partial p}\right)_s$, whose presumptive magnitude might possibly be $\frac{1}{4}$ of the entire effect. It therefore is pertinent to consider the nature of the assumptions which Grüneisen has put into his theory. The general nature of the underlying idea is as follows. It was a cardinal point of Wien's theory that the length of the free path is determined by the motions of the atoms; at higher temperatures the amplitude of atomic vibration becomes greater, and so interferes more with the freedom of electronic motion and decreases the free path. The sign of the pressure effect is explained by showing that as pressure increases at constant temperature the decrease of amplitude of atomic vibration in virtue of the increased frequency more than counterbalances the decreased distance between atomic centers due to volume compression, so that free path, and therefore conductivity, increase.

The starting point of Wien's theory is the formula already given, $w = \frac{2mu}{e^2Nl}$. Wien assumes that u and N are both independent of temperature. It is to be noticed, however, that Wien was concerned only with temperature effects, and for these, as already mentioned, changes of volume may be neglected. Wien's hypothesis that N is constant must not therefore be understood as committing Wien to the statement that when changes of volume are considered the number of free electrons per cm$^3$ is constant. On the contrary, it is clearly suggested, although not explicitly stated, that Wien meant the number of electrons per gm. to remain unaltered. This would mean that Nv is constant. Certainly in the absence of any special examination of the effects of varying electronic dissociation, this is the only plausible hypothesis to make. Now in Grüneisen's deduction of the formula he has not assumed either that N or Nv is constant, but has left N in the equation and differentiates it, and arrives at a formula containing $\frac{1}{N}\left(\frac{\partial N}{\partial p}\right)_s$. This he assumes can be neglected in numerical magnitude on the basis of experiments of Wagner[33] on the effect of pressure on thermo-electromotive force. But the calculation of $\frac{1}{N}\left(\frac{\partial N}{\partial p}\right)_s$ from Wagner's data proceeds on the assumptions of the gas-free-electron theory. One cannot grant that it is permissible to mix in the same equation two theories so opposed in fundamentals. If Grüneisen had assumed Nv constant, which seems the only defensible course until an explanation of thermo-electromotive force is provided on the basis of Wien's theory, the compressibility terms would have disappeared from his formula, with a change in the numerical value of 25%.

A criticism of Wien's fundamental point of view is pertinent here. His calculation of the mean free path proceeds on the assumption that the quanta of energy are located in the individual atoms, instead of in the elastic waves, which at present seems to be the accepted conception. It is evident that this change will considerably modify the physical picture of the manner in which the vibrating atoms interfere with the motion of the electrons, and might be expected to modify the result.

Another consideration vital to Grüneisen's theory is the way in which he has "generalized" Wien's expression for the free path. Grüneisen's expression is

$$\frac{1}{l} = Const\,\frac{h}{M\nu_m v}f\left(\frac{T}{\beta\nu_m}\right).$$

The important feature about this generalization is the appearance of the atomic volume v. The volume does not enter in the considerations of Wien; its introduction must have been part of Grüneisen's process of generalization. It is unfortunate that he gives none of the argument by which he reaches the above expression, because the factor v is important, contributing 25% of the total effect, and its appearance in the place where it is involves definite hypotheses about the atomic mechanism. If the free path is proportional to the volume, other things being equal, and if the free path is determined entirely by the vibrations of the atoms, as is supposed in this theory, then the atoms must behave effectively like mathematical points, and not as if they had extension in space. Such a hypothesis is at least opposed to the present view of the nature of the atoms in a solid and would seem to require discussion.

As it now stands, therefore, the theoretical basis for Grüneisen's formula requires elaboration in several particulars. But it must not be forgotten that for a first attempt at an explanation of the pressure effect the formula works surprisingly well and must contain a considerable element of truth. It seems to me that the element of truth is to be found essentially in the broad change in the fundamental point of view introduced by Wien, namely in considering the phenomena of conduction to be primarily determined by the motions and properties of the atoms of a metal, and not by the properties of the swarm of electrons playing in the spaces between inert atoms. I hope to show in a future paper that a somewhat different physical account can be given of the phenomena, still retaining Wien's broad view point, and that the formulas so obtained are somewhat like those of Wien in general character and are in better agreement with the facts.

## SUMMARY.

In this paper data are given for the change of resistance of 22 metals between 0° and 100°C over a pressure range from 0 to 12000 kg. Three of the metals are abnormal; bismuth and antimony both have a positive pressure coefficient, and the pressure coefficient of tellurium has an abnormally large negative value. The other 19 metals have minute differences in their individual behavior, but in broad outline the behavior of all is alike. The pressure coefficient changes very little with temperature, and therefore also the average temperature coefficient changes very little with pressure; this holds over a range of pressure great enough in many cases to compress the metal to less than its volume at 0° Abs. under atmospheric pressure.

The instantaneous pressure coefficient of all the normal metals on the other hand decreases markedly with increasing pressure. By far the larger number of the normal metals show a decreased relative curvature in the resistance-pressure curve at the higher temperature.

It is shown that none of the hitherto proposed theories can satisfactorily account for all these facts. The recent attempt by Grüneisen, however, must be recognized as a promising beginning.

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CONTRIBUTIONS FROM THE CRYPTOGAMIC LABORATORY OF HARVARD UNIVERSITY. No. LXXIX.

# *NEW LABOULBENIALES, CHIEFLY DIPTEROPHILOUS AMERICAN SPECIES.*

By Roland Thaxter.

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OF HARVARD UNIVERSITY. No. LXXIX.

# NEW LABOULBENIALES, CHIEFLY DIPTEROPHILOUS AMERICAN SPECIES.

By Roland Thaxter.

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The present Contribution deals almost entirely with Laboulbeniales parasitic on flies of various families, although I have thought best to include as an addendum to my recent paper on species of *Rickia* (These Proceedings, **52**, 1, 1916), two additional species of this genus which were accidentally omitted from it. With this exception only forms from the Western Hemisphere are here included, numerous others which have been received from Africa and the East Indies being reserved for later publication. A majority of the new forms belong to the genus *Stigmatomyces*, which proves to be large and very difficult. Four or five species of this genus are already known to be parasitic on coleopterous hosts, but of the thirty-five new forms here included, only one is found on beetles, while one other is associated with a host new for this genus, being parasitic on a minute Anthocorid bug. The remaining forms of this, as well as of the other genera here considered, all occur on Diptera.

The hosts from which this material has been obtained were collected partly by myself in the West Indies; while various interesting forms were very kindly collected for me by Mrs. J. B. Rorer, at Bocas del Toro, Panama; by Messrs. Bruce and Allen and by Mr. H. Phillip in the Island of Grenada, B. W. I.; by Mr. Carriker in Venezuela; by Mr. W. H. Mann in Mexico; by the late Professor Kellerman in Guatemala and by Mr. Philip Calvert in Costa Rica. The Jamaican and Arkansan forms were procured through local collectors. I desire also to acknowledge my obligations for host determinations to Mr. E. T. Cresson Jr., and to Professor A. L. Melander to whom I am further indebted for the very peculiar *Stigmatomyces Clinocerae*.

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### **Rickia flagellifera** nov. sp.

Hyaline. Body of the receptacle subtriangular, short and stout, triseriate; the basal cell longer than broad, its pointed distal end somewhat intruded between the subequal basal cells of the anterior and posterior series: the anterior series consisting of four successively larger cells, all but the basal one cutting off an appendiculate cell distally and externally, the uppermost in oblique contact with the base of the perithecium: the middle series consisting of four or five successively smaller flattened cells, and extending from the basal cells of the marginal series above the base of the perithecium, to the inner margin of which its two or three upper cells are united: posterior series consisting of five superposed cells, the third and fourth each cutting off two appendiculate cells, the others one; the four lower subequal, the uppermost small, displaced somewhat toward the perithecium, and forming the base of an elongate, slender, flagellate continuation of the posterior series, which consists of a single row of superposed elongate cells, some of them cutting off small appendiculate cells distally; the series ending in the primary appendage and its two basal cells. Perithecium relatively rather large and stout, about three quarters free, slightly bent away from the flagellum, the tip hardly distinguished; the apex broad and truncate, or broadly rounded. Spores about 20 × 2 μ. Perithecium 30–36 × 12–14 μ. Receptacle to base of flagellum 35–55 × 18–20 μ. Total length to tip of perithecium 55–80 μ. Longest flagellum 225 × 5 μ.

On *Leptaulax dentatus* F. No. 2393, Mindanao, Philippines.

A species quite unlike any other known form, and easily distinguished from other flagellate types by its triseriate receptacle.

### **Rickia pinnata** nov. sp.

Receptacle becoming somewhat broader distally, its outline more or less even; normally simple; or sometimes producing adventitious axes, usually when injured; the primary axis triseriate above the stout distally rounded basal cell, the secondary axes biseriate: hyaline, except that the median series is tinged with brown at the base. The cells of the anterior and posterior series similar, very numerous, mostly somewhat longer than broad: the posterior series extending to the perithecium, with the base of which it is in oblique contact; three or four of its distal cells, and sometimes a few near its base, cutting off distally and externally, without definite sequence single small cells which bear antheridia the antheridial cells of which become eventually free, or normal appendages: the median series present only in the primary axis and consisting of smaller more flattened cells, beginning above the lowest pair of the marginal series, and extending to the tip of the perithecium, with the curved inner margin of which a series of about ten of its distal cells is united; these cells, except the lowest which are rounded, are broader than long, the uppermost externally free, triangular, its pointed end reaching almost to the apex of the perithecium: the posterior series similar to the anterior, three or four of its lower cells cutting off appendiculate cells distally and externally, but without order; the series otherwise without appendages and extending to the penultimate cell of the middle series to which its terminal, triangular, distally pointed cell is united. Appendages of the usual type, evanescent, the primary appendage apparently lateral near the base, and hardly distinguished. Perithecia rich contrasting brown, the outer margin free, nearly straight, or slightly concave; the inner strongly convex; the broad short tip hardly distinguished, slightly bent outward, the apex truncate or flat-papillate, hyaline-edged. Spores 45 × 5 $\mu$. Perithecia 65–75 × 23–27 $\mu$. Total length to tip of perithecium, longer individuals, 500 $\mu$, by 20 $\mu$ near base and 28 $\mu$ distally.

On *Leptaulax dentatus* F. No. 2393, Mindanao, Philippines.

A peculiar and very distinct species most nearly resembling *R. Berlesiana;* its most striking peculiarity being the remarkable curved, crest-like or fin-like marginal series of cells which are united to the strongly curved posterior margin of the perithecium, this series being double if the perithecium terminates a primary axis, or single if the latter is secondary; the individual cells in the latter case being several times broader than long. Although the general structure is normal and typical, the antheridial cells appear to become separated as in the type formerly distinguished as *Distichomyces.*

## **Nycteromyces** nov. gen.

*Male Individual* consisting of a single series of superposed cells; a foot and well developed basal cell; an indeterminate number of small cells bearing compound antheridia, and two terminal, superposed, sterile cells, the upper peculiarly modified.

*Female Individual* consisting of a well developed basal cell, a small subbasal cell which bears a sterile appendage consisting of a stalk-cell and peculiarly modified terminal cell. The stalk-cells and basal cells of the perithecium not distinguishable at maturity, all partly surrounding the lower ascigerous region. Spores hyalodidymous, ascogenic cell single in the type.

In this very distinct type the stalk- and basal cells of the perithecium are apparently combined to form the wall of the lower portion of the ascigerous cavity, so that what appears to be its main body comprises these cells, as well as the usual four tiers of wall-cells: but since the types are all very young or fully matured, it has not been possible to determine how closely the cell-structure of this region corresponds to that which I have described in detail in connection with the genus Laboulbenia. The condition described is approached in *Stigmatomyces virescens,* among hermaphrodite forms, and is very similar to that which occurs in species of *Dimeromyces,* in which the cell-boundaries are quite obliterated at maturity. The peculiar sterile appendage-cell, which occurs in both sexes, recalls the somewhat similar sterile appendage in the female of species of *Dioicomyces.* The genus, however, is evidently most nearly allied to *Dimeromyces.*

### **Nycteromyces Streblidinus** nov. sp.

*Male Individual.* Basal cell hyaline, long and slender, tapering continuously to the base, where it is slightly enlarged in relation to the small irregularly formed blackened foot, extending as a pointed prolongation slightly beyond the latter. The two to five small short hyaline cells superposed above it, bearing single antheridia in a unilateral series, their venters in close contact, their short broad truncate-conical necks free, diverging upward; the antheridial cells about six in number: the subterminal sterile cell flattened, smoky or dull purplish brown, the terminal one bullet-shaped, or tending to subconical, similarly colored, deeper at the septum, distally apiculate. Total length 86–110 $\mu$. Basal cell 40–52 × 10 $\mu$. Antheridia 18 × 12 $\mu$. Terminal cell 18 × 10–14 $\mu$, the subterminal 4 × 15 $\mu$ broad.

*Female Individual.* Basal cell hyaline, long and comparatively slender, broader distally, its lower three fourths nearly uniform in width to the small irregular black foot, where it is slightly enlarged and extends into a brownish tooth-like prolongation resting on the host beside the latter. Subbasal cell very slightly longer than broad, pushed to one side by the base of the perithecial stalk, which occupies more than half the distal surface of the basal cell. Stalk-cell of the appendage smoky or purplish brown, much flattened, horizontal, separated by a thick more deeply colored septum from the concolorous, nearly symmetrical, bullet-shaped, abruptly apiculate sterile appendage-cell, which is similar to that of the male. Stalk- and basal cells of the perithecium not distinguishable as separate cells, the position of the latter indicated by one outer and two inner flattish elevations which lie some distance above the ascigerous cell; the bases of the lower (venter) wall-cells indicated by somewhat more distinct rounded elevations; similar, somewhat less prominent protuberances distinguishing the regions of the neck and tip; the region of the venter broad, of uniform width or slightly inflated, the region below it to the base of the stalk, which includes about half the total length, tapering nearly to its insertion, the neck-region slightly tapering, the tip narrower, clearly distinguished, hardly tapering, bent slightly inward; the apex well defined, slightly shorter than the tip, somewhat asymmetrical, the base slightly and abruptly spreading, as are the lips also, so that the margins are slightly concave, the two inner lips forming rather prominent blunt tooth like projections, the outer shorter and flattened; the whole perithecium, from the insertion of its stalk upward, clavate-fusiform, elongate, the ascigerous and sporigerous regions tinged with dull purplish brown. Spores 42–45 × 4 μ. Perithecia, including stalk 210–280 × 36–45 μ, apex and tip 35–40 μ. Basal cell 70–90 × 21 μ; the subbasal 10 × 18 μ. Stalk-cell of the appendage 4 × 16 μ broad; the appendage-cell 18 × 14 μ. Total length to tip of perithecium 290–360 μ.

On the superior abdomen and legs of *Strebla vespertilionis* Fabr. No. 2073a, M. C. Z., taken on bats in Venezuela (Carriker).

## STIGMATOMYCES.

With the exception of the three species, *S. Anoplischii*, *S. australis* and *S. Stilici*, no further additions appear to have been made to this genus since the publication of my second Monograph. The form described as *S. Italicus* by Spegazzini in his second contribution on Italian Laboulbeniales (Ann. d. Mus. Nac. d. Hist. Nat. d. Buenos Aires, **27**, 71, Fig. 37) I am quite unable to distinguish from *S. Papuanus*, with which he compares it, a widely distributed and variable species.

As material has accumulated the difficulties of the genus have become increasingly apparent, and the variability of many forms is such that anyone who did not have access to large series of specimens would undoubtedly have little hesitation in separating specifically many more forms than are recognized in the present treatment. The variability of many forms is remarkable and confusing. The difficulty is further increased by the fact that some of the species are not only widely distributed, but inhabit varied hosts which do not always belong to the same genus or even family, and it is thus quite unsafe to describe isolated forms without a considerable knowledge of the genus and its variations. *S. constrictus*, for example, is a case in point, being widely distributed and very variable; and, although it is found only on genera of Oscinidae, inhabits very diverse forms in this family. *S. Scaptomyzae* and *S. Limnophorae* are also examples of widely distributed and very variable types inhabiting diverse hosts, in different families.

The character of the appendage is evidently the most reliable means of distinguishing species, but even this may vary in some instances, so that the number and arrangement of the antheridia and androphorous cells is not always a safe guide.

In preparing the following descriptions I have found it desirable to distinguish the four regions of the perithecium, which correspond to the four successive tiers of wall-cells, as venter, neck, tip and apex, these being more or less clearly differentiated in a majority of species by differences in diameter, the presence of subtending elevations or depressions, superficial granulation, verrucosity and the like. The cells immediately below the perithecium are similar to those the arrangement and nomenclature of which in Laboulbenia were given in my first Monograph: consisting of three 'basal' cells, immediately below the ascigerous cavity or partly surrounding it; an external cell, the secondary stalk-cell; and the primary stalk-cell, corresponding to cell VI in Laboulbenia. The term receptacle is restricted to the basal and subbasal cell, and the stalk-cell of the appendage is that which lies immediately below the insertion of the appendage and is variously related to the cells about it.

It may here be mentioned that *S. Anoplischii*, originally described from Argentina, has again been obtained from Trinidad, B. W. I., and from Mexico, on genera of Elateridae, and that a peculiar form, *S. Lasiochili*, the first in the genus which has been recorded on a hemipterous host, is included in the following enumeration.

*On Sarcophagidae, Muscidae and Anthomyidae.*

### STIGMATOMYCES LIMNOPHORAE Thaxter.

Syn. *St. Sarcophagae* Thaxter.

A very large series of this form has been obtained on a variety of hosts, and from widely separated regions, and shows clearly that the two species above indicated cannot be distinguished specifically. The type of *S. Limnophorae* from California owes its apparent differences to the fact that none of the individuals are fully matured, while a majority are quite immature. Older individuals of this species often attain a length of 700 μ, although on certain hosts they may be constantly smaller, even when fully developed. The twist of the wall-cells is usually hardly distinguishable as a slight obliquity in the venter; while those of the neck may make a complete turn from base to apex, the neck showing corresponding ridges in some cases, and occasionally becoming rough-granular. The number of cells and antheridia in the appendage varies somewhat, and the latter are often more or less remote, owing to its considerable elongation; but the type is a clearly defined one, which is found among other species only in *S. verruculosus*. The form of the venter of the perithecium also varies from that given in my figures, in that the outline of its upper half is often characteristically and symmetrically concave on either side. This is especially true of individuals which occur on species of *Leucomelina*, in which, also, the perithecium as a whole may be somewhat sigmoid.

Additional material has been examined as follows. On *Lucilia dux*, No. 1763, Philippine Is., (Banks), and on what appears to be the same host from Sumatra (Jacobson). On species of *Leucomelina*, Nos. 1734 and 1743, Mandeville, and No. 1860, Balaclava, Jamaica, W. I. On *Limnophora* sp., No. 1644, Los Amates, Guatemala (Kellerman). On *Onesia* sp., Orizaba, Mexico (Mann). On undetermined genera of *Anthomyidae:* No. 2811, St. George, Grenada, W. I.; No. 1881, Troy, Jamaica, W. I.; No. 1817, Fayetteville, Arkansas; Nos. 2639, 2640, and 2646, Kamerun, West Africa.

*On Empididae.*

### **Stigmatomyces Drapetis** nov. sp.

Rather pale, becoming more or less suffused with dirty yellowish brown, straight or somewhat curved. Basal cell of the receptacle not differing greatly from the subbasal in length, sometimes considerably longer, tapering continuously to its narrow base and pointed foot, becoming somewhat suffused with the walls greatly thickened by secondary layers, so that the lumen of the lower half may be nearly obliterated; distally slightly rounded or inflated, usually slightly or distinctly broader than the often hyaline subbasal cell, which is narrower below, the margins straight or slightly concave. Stalk-cell of the appendage more deeply suffused, relatively short, the upper half broad, its outer wall greatly thickened, the distal margin broad and rounded and half or more free outside the very narrow insertion of the usually deciduous appendage. Appendage curved, broad in the middle and tapering to its base and apex; the axis consisting of three cells; the antheridia eight in number, sometimes less, turned sidewise or outward; with rather stout, prominent, slightly curved necks; the basal cell short, broad and pointed above, tapering to its narrow hyaline base which is separated from the very narrow insertion by a slight constriction; the three antheridia associated with it superposed and borne on a distinct small androphorous cell, which is separated from it externally; the subbasal cell bearing two antheridia which diverge more or less right and left; while the third bears one, followed by one or by two, superposed and terminal. Stalk-cell and secondary stalk-cell of the perithecium overlapping laterally, more or less similar, or the former larger; the basal cells similar, the external margins slightly convex and more or less conspicuously thickened; all the cells of this region suffused, and concolorous with the stalk-cell of the appendage and the venter of the perithecium; which is relatively rather short and stout, usually slightly shorter or hardly longer than the distal portion from which it is abruptly distinguished by well marked distal prominences corresponding to the terminations of its wall-cells; the neck paler, its margins straight or slightly concave, its base and apex abruptly spreading; distinguished from the stout, slightly tapering tip and apex by an abrupt and conspicuous inflation; the apex shorter than the tip, its blunt, often asymmetrically rounded termination at length subtended by a minute papilla on either side. Spores 38–40 × 3.5 $\mu$. Perithecia 120–150 $\mu$, the venter 50–60 × 35–50 $\mu$. Appendage 50–70 × 20 $\mu$. Receptacle 70–90 × 17–20 $\mu$. Total length 280–200 $\mu$ or less.

On species of *Drapetis*, Nos. 1707 and 1724, Mandeville, No. 1928, Battersea and No. 1870, Balaclava, Jamaica, W. I.; No. 2521, Bocas del Toro, Panama (Rorer). Also from Sangre Grande, Trinidad, B. W. I.

This species is especially well distinguished by its deciduous antheridium which seldom persists in older individuals, breaking readily from its very narrow insertion, even in less fully matured specimens. It is of a type quite unlike that of *S. Clinocerae*, which is the only other species as yet observed on flies of this family.

### **Stigmatomyces Clinocerae** nov. sp.

Perithecium with its three basal cells and the basal cell of the appendage suffused with amber-brown; the neck-portion paler, the remaining cells hyaline or nearly so. Receptacle relatively slender, of nearly uniform diameter, or somewhat tapering, the basal cell as long as the subbasal, or not more than half as long. Stalk-cell of the appendage hardly broader distally, about twice as long as broad, its base oblique, occupying somewhat less than half the distal end of the subbasal cell; its distal end but slightly broader than the insertion. Appendage straight, or slightly curved inward; consisting of as many as eighteen cells, more often of about fourteen, obliquely superposed, nearly equal in size, broader than long, externally strongly convex, and thus separated by prominent indentations; the basal cell slightly larger and deeply suffused, the rest hyaline, each bearing a single antheridium; the antheridia obliquely superposed in a single series, usually turned inward, the necks appressed, the venters about as large as the cells which bear them; the series terminated by a single antheridium, often abortive. Stalk-cell of the perithecium vertically elongated, parallel to the stalk-cell of the appendage and similar, though somewhat broader; the cells above it somewhat smaller, subequal; the secondary stalk-cell very obliquely separated from the primary stalk-cell, tapering downward to a pointed base which ends slightly below the middle of the latter. Venter of the perithecium much broader than the basal cell region, symmetrically and considerably inflated below, tapering considerably distally to the hardly distinguished neck; which is slightly shorter than the venter, slightly curved and tapering; the tip and apex not distinguished, slightly geniculate, relatively long, tapering to a blunt hyaline unmodified termination. Spores 18 × 2.8 μ. Perithecia 140–170 × 30–36 μ. Appendage, longest 74 × 8–10 μ. Receptacle 60–88 × 14–18 μ. Total length to tip of perithecium 260 μ, the longest 300 μ.

On the inferior abdomen of *Clinocera binotata* Loew. No. 2503, Tecoma, Washington.

This very distinct species, which was obtained from a host very kindly communicated to me by Professor Melander, is most nearly allied to the species which occur on flies of the genus *Limosina*, its indeterminate appendage corresponding closely to that of *S. Limosinae*, and similarly developed through the activity of a terminal cell, which separates successive basal segments to form the axis. Its type is quite unlike that of *S. Drapetis*, the only other form which has thus far been observed on Empididae.

*On Borboridae.*

### **Stigmatomyces longicollis** nov. sp.

Nearly hyaline, the venter and its basal cells, together with the basal cell of the appendage, only, suffused with pale yellowish brown. Basal cell of the short receptacle slightly longer than the subbasal, which is hardly longer than broad. Stalk-cell of the appendage somewhat prominently rounded outward below the broad insertion of the appendage. Appendage consisting of not more than eighteen or nineteen very obliquely superposed cells, long and slender, tapering; the basal cell flat and disc-shaped, the distal septum hardly oblique, narrower than the subbasal; which projects beyond it and, like those above, is flattened-elliptical in outline, with strongly convex outer margin; the lower cells bearing two, the upper single antheridia on their inner sides; the long necks appressed, the uppermost more or less abortive. Stalk-cell of the perithecium, in size and shape, not unlike that of the appendage inverted, and lying beside it; the secondary stalk-cell somewhat smaller and shorter, its inner margin in oblique contact with it; the basal cells more or less similar and uniform, the region usually very slightly broader than the base of the venter: the venter slightly and symmetrically inflated, passing without differentiation to the broad base of the neck, which is enormously elongated, relatively slender and of nearly uniform diameter above its base: the tip distinguished by a slight but distinct depression of the outline on either side; the very short apex narrowing to a small, blunt, symmetrical or slightly oblique, obscurely papillate extremity. Spores 18–20 × 2.5 μ. Perithecia 245–450 μ; the venter 56 × 36–35 × 26–30 μ; the neck 190–380 × 14 μ; the tip and apex 22 μ. Appendage 90–120 μ. Receptacle 40–45 × 18–20 μ. Total length to tip of perithecium 300–525 μ.

On the posterior legs and abdomen of species of *Limosina*. No.

1864, a (Type), b and c, Balaclava; No. 2049, Clarkstown, Jamaica, W. I.

This species is very closely allied to *S. crassicollis*, with which it agrees in general structure. It differs in its more slender habit, very elongate slender appendage, which is composed of twice as many cells, and in its enormously elongated slender neck and relatively smaller venter. About a dozen examples have been examined from five individual hosts.

### **Stigmatomyces crassicollis** nov. sp.

Pale yellowish or nearly hyaline, the venter, basal cell region and the base of the appendage more or less suffused with yellow brown. Receptacle tapering to a pointed base, usually straight, or very slightly curved, usually very short; the cells subequal, separated by a slightly oblique septum and forming, with the stalk-cells of the perithecium and that of the appendage, a compact triangular cell-group with even outline; rarely somewhat more slender and irregular. Stalk-cell of the appendage but slightly broader distally, its whole upper surface occupied by the very broad insertion of the flat, colored basal cell, the narrower outer end of which protrudes externally. Appendage curved toward the venter; the axis consisting of not more than nine or ten cells, the lower producing two, the upper single antheridia directed inward, those above the basal cell flattened elliptical, obliquely superposed, externally separated by well defined indentations. Stalk-cell of the perithecium about as long and large as that of the appendage, not reaching quite as high as the smaller secondary stalk-cell which overlaps externally its upper four fifths, and is similar in general to the basal cells above it. Perithecium usually straight, or the distal portion slightly curved; the venter sometimes very slightly narrower and indented above the basal cell region, giving it a slight cup and ball effect; short, stout, symmetrical, broadly ovoid, or sometimes almost spherical; its outline evenly continuous with that of the spreading base of the very stout and elongate neck, which tapers very slightly distally: the tip and apex very short, but slightly distinguished by its more rapid tapering; the apex nearly symmetrical bearing four similar minute papillae. Spores 20–22 × 2 μ. Perithecia 140–325 μ longest; the venter 35–60 × 38–64 μ; the neck 350 μ and less × about 25 μ toward base; the tip and apex 18 μ. Appendage 35–70 μ. Receptacle 50–70 × 22–35 μ. Total length to tip of perithecium, small 200 μ, largest 430 μ, average 275 μ.

On the abdomen and legs of species of *Limosina*. No. 1863 (Type), and 1865, Balaclava; No. 1747, Mandeville, Jamaica, W. I.

This species is most nearly related to *S. Papuanus* which occurs on similar hosts in the Western Hemisphere, and is distinguished from it by the different relations of its perithecial stalk- and basal cell, and by its symmetrical papillate apex which is never modified by the presence of the terminal prolongation so characteristic of the papuan form. Abundant material has been examined, the individuals varying for the most part only in size, although those numbered 1865 are somewhat more slender in habit, with less compact receptacles.

### **Stigmatomyces Grenadinus** nov. sp.

Rather slender and irregularly formed; basal region of the appendage and perithecium, as well as the venter, becoming tinged with pale brownish yellow. Receptacle hyaline, of nearly uniform diameter, with somewhat uneven outline; the basal cell shorter, and distally slightly broader, than the subbasal, which is symmetrically distinguished from the parts above by a slight indentation. Stalk-cell of the appendage stout, slightly longer than broad, not overlapping the subbasal cell; its outer margin convex, more strongly so distally where it bulges somewhat below the rather broad insertion. Appendage relatively small, hyaline except the faintly suffused basal cell; its axis consisting of not more than three or four successively smaller cells; the basal slightly broader than long, and faintly suffused; the second and third slightly oblique, longer than broad, their thick outer walls strongly and asymmetrically convex; the two lower cells bearing each two antheridia lying side by side, one much higher than the other: the third cell bearing a single antheridium which subtends one or sometimes two superposed terminal ones, both or all of these latter often abnormally developed: the antheridia rather large, almost wholly free, directed obliquely sidewise. Stalk-cell and secondary stalk-cell of the perithecium similar in size, irregularly four-sided, slightly longer than broad, the latter extending to the subbasal cell and similar to the basal cell above it, the outer margins of both individually rather strongly and evenly convex. Venter of the perithecium subpiriform, its base distinguished externally by a slight indentation, evenly inflated below, tapering somewhat distally where it is continuous with the slightly spreading base of the hardly differentiated neck: neck relatively stout, nearly uniform above its base, slightly longer than the venter, straight or slightly curved, its distal end slightly broader; the tip stout, but somewhat narrower, often slightly geniculate; the apex not distinguished, its termination broad and bluntly rounded, with slightly irregular outline. Spores 18 × 3 μ. Perithecia 85–100 μ; the venter 35–38 × 24–28 μ; the neck 35–38 × 11 μ; the tip 18 μ. Appendage about 35 μ. Receptacle 50–70 × 14 μ. Total length to tip of perithecium 150–185 μ.

On legs of *Limosina ferruginea* St., No. 2528, Grand Etang, Grenada, B. W. I.

This species was found in abundance on hosts flying over dung on the Grand Etang road. Although nondescript in appearance, it is of interest for the reason that it belongs to a type transitional between that of the more ordinary forms, and that of the group represented by *S. Limosinae* and its allies, to which it is very evidently nearly related.

### **Stigmatomyces pentandrus** nov. sp.

Receptacle relatively long and stout, subhyaline, of nearly uniform diameter throughout, slightly broader in the region of the horizontal septum; the walls unusually thick, the basal cell usually shorter than the subbasal. Stalk-cell of the appendage somewhat obliquely inserted on the protruding outer angle of the subbasal cell, rather deeply suffused with amber-brown, concolorous with the venter of the perithecium, narrow, its outer margin strongly concave, slightly prominent and rounded below the insertion of the short, straight, slightly tapering appendage. Appendage consisting of two cells, the basal hardly longer than broad and bearing two antheridia; while the subbasal bears but one, and is succeeded by two superposed antheridia, the terminal one erect and bearing a stout spine. Stalk-cell of the perithecium somewhat larger than the cells above it, and less deeply colored than the flattened basal cells; the whole region, including the stalk-cell of the appendage, relatively short, abruptly narrower than the distal end of the receptacle, and forming a slight constriction, thence expanding almost symmetrically upward to the venter of the perithecium; which is relatively very short and broad, symmetrically inflated; its surface, as well as that of the neck and at maturity even the tip, indistinctly and evenly granular; distally narrower and at once abruptly broader at its junction with the broadly, but not abruptly, spreading base of the neck; which is distinctly longer, stout, slightly paler, its distal half of uniform diameter, but abruptly broader below its junction with the tip, which is thus clearly distinguished by rounded elevations on either side; the tip paler, but becoming slightly granular at full maturity, abruptly and considerably narrower, short, stout, its margins more or less distinctly convex: the apex much shorter, abruptly narrower, hyaline, the lips nearly equal, rather prominent, the outer slightly longer. Spores 36 × 4 μ. Perithecia 124 μ; venter 42–51 × 42–44 μ; neck 50–54 μ; tip and apex 22–25 μ. Appendage 34 × 10 μ. Receptacle 110–170 × 22 μ. Total length to tip of perithecium 245–315 μ.

At the base of the posterior legs of an undetermined fly belonging to the Borboridae. No. 2037, Near Cartago, Costa Rica (Calvert).

This species recalls *S. indentatus* in the form and position of the stalk-cell of its appendage, while the latter is unlike that of most species known to me; the third cell being completely transformed to an antheridium. It is difficult to determine in the three specimens examined whether the basal cell of the appendage always bears two antheridia, owing to the position of the individuals. The form of the perithecium, although the venter is relatively shorter, recalls that of *S. constrictus*. I am greatly indebted to Mr. Calvert for communicating the specimen on which this species occurred.

*On Sapromyzidae.*

### **Stigmatomyces inflatus** nov. sp.

Form rather long and slender. Basal cell of the receptacle tapering to the pointed base, the upper half hyaline, the lower tinged with brownish yellow and more or less distinctly banded transversely; subbasal cell usually somewhat more than twice as long, hyaline, becoming but slightly broader distally. Stalk-cell of the appendage rather long and narrow, extending to the subbasal cell, but rarely overlapping it slightly, distally hardly prominent: appendage consisting of four superposed cells; the basal thick-walled, nearly twice as long as broad, persistent, amber-brown, concolorous with the venter of the perithecium; the subbasal occupying the whole width of the appendage, slightly tinged, its lateral walls persistent, bearing distally and externally two superposed antheridia; the rest of the appendage hyaline, thin-walled, and soon shriveled; the third and fourth cells bearing each two superposed antheridia which occupy the whole outer margin; the series ending in two obliquely superposed antheridia; the series of antheridial necks hardly diverging from one another. Stalk-cell of the perithecium relatively large and long, its outer half free, the pointed base of the secondary stalk-cell extending down beside its outer upper half, irregularly triangular, and much smaller; of the three basal cells, the inner extends lower than the others beside the distal end of the secondary, to the upper margin of the primary stalk-cell. Venter of the perithecium more deeply suffused, nearly symmetrical, its outline abruptly convex above the relatively narrow base, then somewhat concave, then abruptly convex and broader; so that the distal half, or less, appears to be rather abruptly inflated; the terminations of the wall-cells swollen, and separated by depressions; the neck abruptly distinguished, long, its base slightly broader and spreading, otherwise of about the same diameter throughout, or slightly broader distally; the tip slightly bent, distinguished by an abrupt depression and elevation of its outline, not quite symmetrical on either side; the apex much shorter, not distinguished, ending bluntly with minute hardly distinguishable papillate elevations about the pore. Spores about 18 × 3.5 $\mu$. Perithecia; stalk and basal cell region 54–62 × 18–23 $\mu$; venter 55–65 × 38–45 $\mu$; neck 90–100 × 15 $\mu$; tip and apex 26–30 × 15 $\mu$. Appendage 55–62 $\mu$. Receptacle 100–120 × 18 $\mu$. Total length to tip of perithecium 310–390 $\mu$.

On the superior surface of the abdomen of *Sapromyza* sp. Nos. 2495 and 2494, Orizaba, Mexico (Mann).

This species is perhaps more nearly allied to *S. Scaptomyzae* but is clearly distinguished by its inflated venter and other details of structure.

*On Trypetidae.*

### **Stigmatomyces Ensinae** nov. sp.

Nearly hyaline to the basal cells of the perithecium, the rest dull reddish amber-brown. Basal cell usually curved, tapering below, often somewhat inflated distally, and becoming narrower below the base of the subbasal cell; which is distinctly and usually very abruptly broader, stout, seldom more than twice as long as the basal cell, often inflated in the mid-region, and slightly contracted above and especially below. Stalk-cell of the appendage long and narrow, its attenuated base reaching to the subbasal cell, its distal end hardly broader than the basal cell of the appendage, from which it is separated by a slight abrupt constriction. Appendage consisting of four cells; the basal hardly broader distally, where it is separated by an almost horizontal septum from the subbasal cell, without antheridia; subbasal cell somewhat smaller than the cell above, both bearing distally and externally two antheridia which diverge irregularly laterally; the fourth cell bearing one antheridium which is followed by two others, superposed, bent inward, and terminating the appendage; the antheridia stout, rounded, closely appressed, so that the outline of the appendage is more or less even, without distinctly projecting necks, the spinous process from the terminal antheridium apparently not persistent; the antheridia and all the cells of the appendage persistent and rather thick-walled. Stalk-cell of the perithecium very large, somewhat broader than the subbasal cell, more than twice as long as the secondary stalk-cell; which is subtriangular in outline, and obliquely separated from it distally and externally; the basal cells triangular in outline, but slightly smaller than the secondary stalk-cell: venter nearly symmetrically long-elliptical, or narrower distally; the wall-cells becoming twisted in a spiral of two strong curves which may make somewhat more than half a turn, the edges forming broad wing-like elevations from base to apex; the somewhat spreading base of the neck clearly, but not abruptly distinguished; the neck distinctly shorter than the venter, its diameter nearly uniform above the base; the tip subtended by a slight and abrupt constriction, abruptly inflated, short, subsymmetrical; the apex of about the same length, truncate conical, or distally very slightly oblique when viewed sidewise; the lip-cells paired, two lower and two higher, but hardly projecting, the lips not at all distinguished or prominent. Spores $27 \times 3.6\,\mu$. Perithecia; stalk-cell $35–60 \times 25–30\,\mu$, the stalk-cell and basal cell region about $60–70 \times 35\,\mu$: venter $75–100 \times 50–55\,\mu$; neck $45–60 \times 18–20\,\mu$; tip $15 \times 18–20\,\mu$; apex $14–16\,\mu$. Appendage $50–55 \times 12.5\,\mu$; its basal cell $12–14 \times 7.5\,\mu$; its stalk-cell $35–76 \times 7.5\,\mu$. Receptacle $80–150 \times 20\,\mu$. Total length to tip of perithecium $300–400\,\mu$.

On various parts of *Ensina* spp. Mandeville, Jamaica, Nos. 1711, 1712.

This form, although it has been examined in considerable numbers, appears to be subject to the attack of a species of *Cladosporium* which destroys it. It is well distinguished by its peculiar appendage, the spiral, winged wall-cells of its venter, the large perithecial stalk-cell, the usually abruptly broader base of the subbasal cell of the receptacle, and the long and very narrow stalk-cell of the appendage.

### **Stigmatomyces verruculosus** nov. sp.

Habit rather long and slender. Receptacle elongate, the basal cell quite hyaline, about four fifths as long as the subbasal, which is more or less evidently suffused with pale brownish yellow; a more or less distinct enlargement in the region of the septum, involving both cells; the receptacle otherwise of nearly uniform diameter, or expanding slightly distally. Stalk-cell of the appendage relatively short and broad, its broad obliquely rounded base in contact with the subbasal cell, more deeply suffused with dull reddish amber-brown, concolorous with the perithecium and its basal and stalk-cells. Appendage slightly divergent and curved outward distally, consisting of three cells, a small fourth cell sometimes distinguished; the basal cell concolorous with the stalk-cell and abruptly narrower, more than twice as long as broad, and bearing distally two antheridia; the second and third cells successively smaller, bearing two and one respectively; the strongly curved termination of the appendage terminated by two, the uppermost often undeveloped. Stalk-cell and secondary stalk-cell of the perithecium of about the same size, broader than long, the basal cells above them somewhat smaller, the outer one and the secondary stalk-cell abruptly convex externally; venter darker, granular-verruculose, rather short and stout, strongly and symmetrically inflated, sometimes but slightly longer than broad; the wall-cells separated by a clean cut shallow groove, forming a little more than one quarter of a turn; neck granular roughened, but not very abruptly distinguished, tapering very slightly or not at all from its slightly spreading base, distally abruptly slightly enlarged below the tip; which is somewhat narrower, short and stout, symmetrical, not distinguished from the still shorter apex, the margins of which curve abruptly and symmetrically to the rather broad, nearly truncate termination; the lips hardly distinguished. Spores about 28 × 4 $\mu$. Perithecia; stalk- and basal cell region about 35 × 32 $\mu$; venter 48–55 × 40 $\mu$; neck 62–75 × 18 $\mu$; the tip and apex 19–21 × 14 $\mu$. Appendage about 60 $\mu$, its basal cell 21 × 7.5 $\mu$, its stalk-cell 27 × 10.5 $\mu$. Total length to tip of perithecium 280–312 $\mu$.

On the abdomen of *Ensina* sp. Mandeville, Jamaica, Nos. 1711 and 1712. St. George, Grenada, W. I. No. 2061.

This species is most nearly related to *S. Limnophorae*, and is in some respects intermediate between *S. Ensinae* and *S. Aciurae*. It is distinguished from the first mentioned species by its verrucose, not merely granular venter, relatively shorter, stouter neck, and shorter stouter appendage which is less highly developed, having usually but three cells. The wall-cells of the venter are distinctly spiral in the present species, while those of the neck are straight: but in *S. Limnophorae* the converse is true, the wall-cells of the venter in this species having merely a slight obliquity.

## **Stigmatomyces Aciurae** nov. sp.

Form elongate, the receptacle hyaline, the rest more or less tinged with dull amber-brown. Basal cell usually bent and broadly rounded at base, stout, hardly tapering; subbasal cell usually abruptly narrower, and separated from the basal by a more or less pronounced enlargement in the region of the septum; usually much elongated and of about the same diameter throughout, sometimes shorter and distally broader. Stalk-cell of the appendage reaching to the subbasal cell, but hardly if at all over-lapping it, relatively short and distally broad, its base blunt; the basal cell of the appendage large, narrow at the base, its margins slightly concave, broader distally, darker, bearing distally on the inner side two antheridia diverging irregularly laterally; the rest of the appendage consisting of two axis-cells, the lower larger, producing two suberect antheridia; the upper smaller producing one antheridium lying beside a single antheridium which terminates the appendage, and is furnished with a large stout spine at the base on its inner or outer side; these four antheridia erect, or but slightly divergent, forming a more or less regular terminal crown; the whole appendage somewhat thick-walled, suffused and permanent, except the necks of the antheridia which soon collapse. Stalk and basal cells of the perithecium relatively small, and forming a compact base, distally concave, the outer basal cell rather abruptly prominent below the narrower base of the venter; which is slightly inflated and nearly symmetrical; the margin subdistally concave below the distinctly prominent terminations of its wall-cells, which have a hardly perceptible twist, and an indistinctly and transversely striate or granular surface: neck abruptly distinguished, stout, longer than the venter, its margins concave, symmetrical, abruptly broader distally; the tip abruptly somewhat narrower, tapering slightly, relatively short; the apex distally oblique, not distinguished, its four cells forming coarse, paired prominences, the anterior higher, which surround the closely united lips; the latter combined to from a projection extending above the others. Spores 50–55 × 4.5 μ. Perithecium: stalk- and basal cell region 64–70 × 78 μ; neck 55–75 × 24 μ, its distal enlargement × 23–28 μ; tip and apex 32 μ. Appendage 55 × 18 μ its basal cell 20 × 12 μ distally; the stalk-cell 35 × 16 μ distally. Receptacle 150–350 × 18–27 μ. Total length to tip of perithecium 330–550 μ.

On the legs and abdomen of *Aciura* sp. No. 1714, and *Ensina* sp., Nos. 1711–12; Mandeville, Jamaica, W. I.

This species varies greatly in length, owing to the variable elongation of its subbasal cell. Its characters are otherwise in general very constant, except that the narrowing of the subbasal cell, and the enlargement of the distal end of the basal, are variable in their degree and abruptness. The species seems more nearly allied to *S. constrictus* in its general characters, although the appearance of its appendage suggests that of the somewhat anomalous *S. Nycteribiidarum* owing to the close grouping of the four terminal antheridia and the early disappearance of their necks. Abundant material has been examined.

*On Ephydridae.*

### **Stigmatomyces Notiphilae** nov. sp.

Quite hyaline, or nearly so, below the basal cells of the perithecium. Foot minute and abruptly distinguished. Basal cell of the receptacle stout, sometimes but slightly longer than broad; the subbasal cell longer, its width usually two to three times its length; the three cells immediately above vertically elongated and parallel to one another, lying side by side, the group as a whole usually rather abruptly broader than the subbasal cell, and more prominently so on the perithecial side; the posterior of these three cells, which is the stalk-cell of the appendage, is narrower than the others and usually extends slightly higher and lower, often just overlapping the end of the subbasal cell; the middle one (stalk-cell of the perithecium), usually slightly smaller and shorter than the secondary stalk-cell which extends somewhat higher. Stalk-cell of the appendage narrower below its distal margin, often rather abruptly horizontal and twice, or more than twice as broad as the constricted narrow dark insertion of the appendage, which lies close against the base of the inner basal cell of the perithecium. Appendage straight, erect or slightly divergent, four-celled; the basal cell small, faintly colored, narrower below, its distal margin usually slightly oblique, bearing no antheridia; the subbasal usually five sided, but slightly longer than broad, bearing a single antheridium on the inner side; the third and fourth subtriangular, flattened, the latter often slightly larger, each bearing a single antheridium; the series of three superposed, their curved necks turned toward the perithecium, and terminated by a fourth which is conspicuously spinose on its inner side just below the middle. Inner basal cell of the perithecium more than twice as long as broad, extending somewhat lower than the slightly stouter and shorter outer cells which lie parallel to it, all faintly yellowish: venter of the perithecium more distinctly suffused, of nearly the same diameter throughout, or slightly inflated, about as long as the distal portion; the wall-cell becoming distinguished at maturity by usually inconspicuous ridges, which acquire a spiral twist, making about a quarter turn, and each ending in a slight prominence below the rather abruptly distinguished neck, which tapers slightly: the tip clearly, but not very abruptly distinguished, slightly more than twice as long as broad, distally somewhat inflated, or even geniculate below the short rounded slightly inflated apex; the lip-cells hardly, or but slightly prominent. Spores 36 × 4.5 μ. Perithecia above basal cells, 185–200 × 30–35 μ, maximum 250 × 40 μ: the basal cell region 30–40 × 26–28 μ. Receptacle 70 × 25 μ, maximum 150 × 28 μ. Total length 325–400 μ, maximum 410 μ: to base of appendage 125–140 × 38 μ; maximum 235 × 38 μ. Appendage 65–75 × 10 μ, maximum 85 μ.

On the abdomen and legs of *Notiphila* spp., No. 1859 (Type), Balaclava Jamaica: Nos. 2808-2809, St. George, Grenada, W. I.

A species very clearly distinguished by its peculiar appendage, and the arrangement of its cells above the receptacle. It usually grows in tufts on the legs, but the shorter and more compact form, which is found on the abdomen, has been taken as the typical one. The species varies but slightly, except that individuals growing on the legs may be very much elongated. The four ridges which separate the wall-cells of the venter, are not at first distinguishable, but become more prominent as individuals become fully mature.

Of the forms which are parasitic on species of *Paralimna, P. ciliata* and *P. decipiens,* a large series has been examined, and it has proved very difficult satisfactorily to determine their specific limitations. The two forms which I have called *S. Jamaicensis* and *S. curvirostris* seem well defined, owing to the character of the appendage, as well as to their general form. Among the others, however, such numerous variations appear to exist, that I have even hesitated to separate among them the two most clearly distinguished types. This has been largely owing to the fact that, despite their considerable differences in form and cell relation, the appendage is in general very similar in all; ending in two superposed antheridia, the upper spinose; while of its three axis-cells the basal bears three, the subbasal two and the distal single antheridia, all of which are superposed in a single row with little if any of the usual right and left divergence. The basal cell, moreover, is somewhat colored and persists, keeping its form, while the rest of the appendage is quite hyaline and usually shrivels at an early stage, so that its exact structure is ascertained with difficulty. The form and relative development of the receptacle and perithecium, however, vary greatly; and although there are two well distinguished types, both of which may occur in the same position on the hosts abdomen, those which grow elsewhere often vary so considerably that it is difficult to determine whether they should be regarded as distinct species, and, if not, to which of the two primary species they should be referred, if they are treated as varieties. The following disposition of them must therefore be regarded as tentative.

### **Stigmatomyces curvirostris** nov. sp.

Pale yellowish, the venter of the perithecium becoming tinged with amber-brown, its surface at maturity transversely finely granular-punctate. Basal cell of the receptacle somewhat longer than broad, somewhat narrower below; the subbasal cell of the same width and about twice as long, separated by a horizontal septum, its upper half or two thirds overlapped by the long slender stalk-cell of the appendage; which tapers very slightly to its base, and is rather abruptly and prominently rounded below the basal cell of the appendage, where it is tinged with amber-brown. Appendage consisting of four cells; rather long, slender and distally attenuated; the insertion horizontal and on a level with the distal end of the secondary stalk-cell of the perithecium, the basal cell twice as long as broad, or less, tinged with amber-brown, bearing distally from a very oblique insertion, three superposed antheridia, the upper two of which lie closely against the margin of the subbasal cell; the latter, and also the cell above it, as large or longer than the basal cell, separating two superposed antheridia each; the fourth cell, separating a single antheridium, and surmounted by two superposed antheridia which terminate the appendage. Stalk-cell and secondary stalk-cell of the perithecium of about equal length, the two combined more than twice as long as broad, separated by a nearly vertical septum, the latter cell somewhat broader distally, and barely separated from the subbasal cell of the receptacle by a narrow external protrusion from the base of the former; the other three basal cells, lying above these two, relatively long, subequal and forming a short stout stalk about as broad as the lower part of the venter, the base of which lies far above the insertion, of the appendage. Perithecium bent outward from the base; its venter seldom broader than the latter, except just above the middle, where it may be rather abruptly swollen at maturity, its distal third or fourth rather abruptly narrower above this swelling; the neck abruptly somewhat narrower, more or less strongly curved outward, rather long and of the same diameter throughout; the tip slightly distinguished and very slightly narrower, hardly tapering; the blunt apex not at all distinguished; the lips broadly rounded and not prominent. Spores about 28 × 2.5 $\mu$. Perithecia above base, 195–225 × 30 $\mu$. Appendage 80–90 $\mu$, its stalk-cell 100–118 × 12 $\mu$. Receptacle 58–70 × 15 $\mu$. Length from subbasal cell to base of venter 68–78 × 22 $\mu$. Total length to tip of perithecium 300–390 × 30–40 $\mu$.

Growing at or near the tip of the abdomen of *Paralimna ciliata* Cress. Nos. 1871 (Type), 1733, 2052 from Balaclava, Mandeville and Clarkstown, Jamaica. Nos. 2805, 2810, from St. George, Grenada, W. I. On *Parydra* sp. No. 2042, Clarkstown, Jamaica, W. I.

This species is distinguished from allied forms on similar hosts by its rather slender, four-celled, tapering, persistent appendage, the narrow stalk-cell of which is greatly elongated, and may extend downward almost to the basal cell of the receptacle. The stalk-cell and secondary stalk-cell of the perithecium are also unusually elongated, and lie nearly parallel to one another below the three upper basal cells, which form a short stout perithecial stalk. Like *S. Paralimnae* and other related forms, its appendage is peculiar in that the antheridia, when more than one are produced from a single cell, are more or less exactly superposed, with little if any of the usual right and left divergence. The terminal antheridium appears to lack the persistent spine usually present in its allies.

### **Stigmatomyces rostratus** nov. sp.

Yellowish or more often tinged with brown or olivaceous above the relatively short pale receptacle. Basal cell similar to the subbasal, or usually somewhat longer, narrower below, hardly if at all overlapped by the stalk-cell of the appendage. The latter relatively short, its base bluntly rounded, its distal end broad, its outer margin slightly concave, distally rather abruptly convex. Region of the insertion rather conspicuously hyaline, nearly horizontal. The appendage consisting of three cells, the basal more deeply suffused, concolorous with the stalk-cell, and almost as long, bearing distally and externally three superposed antheridia; the subbasal cell often as long as the basal, narrow, bearing two superposed antheridia which lie almost wholly above it; the third cell half as long, bearing a single antheridium which is followed by two that are terminal and superposed, the eight superposed in a single row, or but faintly suffused, hyaline, concolorous with the subbasal and third cell. Stalk-cell of the perithecium subtriangular, the distal angle rounded or truncate, hardly longer than it is broad at its base, which occupies the whole distal margin of the subbasal cell of the receptacle; secondary stalk-cell slightly smaller, subtriangular, distally prominent externally, the prominence associated with a peculiar thickening similar to that of the outer basal cell above it, the margin of which is rather abruptly concave; the inner basal cell somewhat narrower, its middle opposite the insertion of the appendage. Venter of the perithecium relatively very long, yellowish, suffused with brown or olivaceous, concolorous with the cells below, but becoming darker; its lower half or more almost symmetrically inflated, tapering gradually to the slightly but abruptly distinguished distal portion of the perithecium, which may equal the venter in length, but is usually somewhat shorter; the whole of nearly the same diameter throughout, except that the tip is very slightly narrower, usually curved outward; the apex not distinguished blunt; the lips coarse and but slightly prominent. Spores about 28 × 3 μ. Perithecia: venter 100–115 × 35–45 μ; neck and tip 90–100 × 16–18 μ. Receptacle 80–95 × 22–26 μ. Appendage 75–85 μ, its stalk-cell 35 × 16 μ: its basal cell 28–30 μ. Total length to tip of perithecium 330–350 μ.

On *Paralimna decipiens* Lw., No. 1915, (Type), Porous, No. 1913, Williamsfield; No. 1741 and 1748, Mandeville; No. 1747; Clarkstown, Jamaica. No. 2529, Grand Etang, Grenada; W. I.: in all cases near the tip of the abdomen.

This species is not subject to any considerable variation except in color, the olivaceous shade being absent in some of the material. On the legs of the same host, however, occurs a shorter and stouter form, which is not unlike the less well developed types of *S. Paralimnae* which occur in a similar position, and I have been unable to decide whether it should be regarded as a variety of the present species or of the last mentioned form, or whether it should perhaps be regarded as specifically distinct. It occurs rather rarely, and is usually very crowded, but does not seem to vary definitely toward either type.

### **Stigmatomyces Paralimnae** nov. sp.

Form usually subsigmoid, rather short and stout; the perithecium, above its basal cells, comprising two thirds to three fourths of the total length; yellowish, becoming tinged with amber-brown, except at the base, sometimes with an olivaceous tinge. Basal cell of the receptacle short, stout, subtriangular; subbasal cell broader than long, overlapped more or less on its posterior side, sometimes as much as two thirds or more, by the stalk-cell of the appendage; the latter relatively large and prominently convex externally, about twice as long as broad, distally narrow, persistent, slightly colored: the rest of the appendage hyaline, rather broad, soon collapsing; the antheridia obliquely superposed in a single row, the necks with little if any right and left divergence: the basal cell usually producing three antheridia, the subbasal two, the uppermost one; the appendage terminated by two superposed antheridia, and lying flat against the perithecium, or with the antheridia directed outward. Stalk-cell and secondary stalk-cell of the perithecium in oblique contact; the former somewhat larger, its base in contact with the whole or nearly the whole of the upper surface of the subbasal cell; the two combined broader than long, obliquely separated: the three basal cells above but slightly smaller, forming a short, broad insertion, even slightly broader than the base of the venter; the latter granular, when fully mature, and more deeply suffused; about three times as long as broad, its margins becoming slightly convex; the four wall-cells slightly spiral, making less than a half-turn, variably prominent, typically separated by often indistinct furrows, and usually more or less abruptly individually prominent distally, thus accentuating the abrupt transition to the much narrower neck; distal portion of the perithecium very indistinctly, or not at all, distinguished into neck, tip and apex, except that the neck portion is usually nearly hyaline or paler, while the tip and apex are more distinctly suffused; the whole termination usually more or less strongly curved, slightly inflated near the middle, or tapering slightly from near the base; the lip-cells asymmetrically and variably prominent, sometimes conspicuously papillate, or even vesicular, hyaline, often forming a slightly bent, snout-like termination. Spores 35 × 4 μ. Perithecia, venter 60–75 × 22–28 μ; distal portion 50–58 × 15 μ. Length from foot to base of venter 50–70 × 22–30 μ. Appendage about 40–45 μ. Total length to tip of perithecium, 150–175 μ.

On the abdomen, legs and base of wing of *Paralimna ciliata* Cress., No. 1811 (Type), Fayetteville, Arkansas. No. 1731, Mandeville and No. 2052 and 2042 Clarkstown, Jamaica, W. I. No. 2805, St. George, Grenada, W. I.

The material from Fayetteville, which was obtained for the most part of the abdomens of several individual hosts, has been taken as the type of this species, while shorter straighter and less characteristic forms were taken from the legs. The material from the West Indies differs somewhat from the type, the distal portion of the perithecium tapering a little more distinctly to the apex, which is thus slightly narrower, and is more often slightly but abruptly bent, two of the lips-cells being more prominent than the others, and giving the extremity a more snout-like habit.

A further variation, in which the distal portion of the perithecium is more or less prominently recurved, has also been found on West Indian material of *P. ciliata:* No. 1916, Porous, Jamaica; Nos. 2805 and 2810, St. George, Grenada; also on a specimen collected by Mrs. J. B. Rorer at Bocas del Toro, Panama. This form always occurs near the base of the left wing, and is larger than the type, measuring up to 275 μ or more in length. The venter of the perithecium is often more distinctly narrowed distally and may measure 100 × 38 μ, the recurved distal portion measuring about 80 × 14 μ. The portion comprising the receptacle and the perithecial base, is also relatively longer, as compared with the type.

A still more striking departure from the short stout Type is seen in a variety which occurs near the base of the legs. This variation is characterized by its more slender and elongate habit, attaining a total length of nearly 300 μ, and being more or less evenly curved, as a rule, from base to apex. The venter of the perithecium is more or less symmetrically inflated, and the tip is usually distinguished by a rather abrupt, though slight, subterminal bend. The cells of the receptacle are much elongated, and almost all individuals are slightly twisted, so that one views the receptacle edgewise in most preparations. Material of this variety has been examined from Mandeville and Clarkstown, Jamaica, Nos. 1732 and 2055, the latter on *P. decipiens,* the former on *P. ciliata:* also from St. George, Grenada, Nos. 2805 and 2610, on *P. ciliata.*

### **Stigmatomyces Jamaicensis** nov. sp.

More or less suffused with brownish amber, paler below. Form as a rule relatively short and stout. Basal cell of the receptacle hyaline, somewhat longer than broad, obliquely separated from the subbasal, which is usually slightly longer, or may be similar or even shorter, its distal septum horizontal. Stalk-cell of the appendage relatively large, extending nearly to the basal cell of the receptacle, or even slightly overlapping it; long-triangular, distally broader than the base of the appendage, prominently rounded externally, more deeply suffused: axis of the appendage consisting of four cells, all bearing antheridia superposed in a single series, its outer margin becoming strongly convex; its insertion usually oblique, lying on a level with the base of the venter of the perithecium, its basal cell relatively small, short, deeply colored, bearing three superposed antheridia, and often considerably overlapped externally by the subbasal cell; which is considerably longer, externally convex, giving rise distally and inwardly to three, or sometimes two, antheridia: the third cell similar, but slightly smaller, and bearing two antheridia, while the uppermost bears one, and is followed by two which are superposed, the terminal one spinose externally. Stalk-cell of the perithecium but slightly larger than the nearly triangular secondary stalk-cell, from which it is obliquely separated by a strongly curved septum, and which it separates from the subbasal cell of the receptacle by an abrupt narrow external prolongation: the three basal cells of the perithecium hardly smaller, subtriangular in outline, their bases horizontal and nearly coincident. Perithecia asymmetrical rather stout, straight or usually bent inward, the wall-cells having a slight spiral twist from left to right which is more conspicuous in the short stout neck; the venter relatively larger, its margin nearly straight, its outer strongly convex; the terminal portion irregularly bluntly conical; the neck, tip and apex hardly distinguished by slight elevations and depressions, the lips slightly oblique, rather broad, and hardly prominent. Spores $30 \times 4\ \mu$. Perithecia 80–115 $\times$ 28 $\mu$. Appendage 60–75 $\times$ 12 $\mu$, its stalk-cell 30–50 $\times$ 10–12 $\mu$, its basal cell 8–10 $\mu$. Receptacle 38–46 $\times$ 15–18 $\mu$. Length to base of appendage 60–70 $\mu$. Total length to tip of perithecium 135–195 $\mu$.

On *Paralimna ciliata* Cress. Clarkstown, Jamaica, W. I. No. 2054 at base of wings; No. 2056 on posterior legs.

This somewhat nondescript type, of which the larger individuals occurred at the base of the left wing, seems sufficiently well distinguished from depauperate forms of other species on this host, by the terminal twist of its perithecium, and the character of its appendage, which is more like that of *S. curvirostris;* but is somewhat more highly developed, broader, with more cells and antheridia, the latter turned toward the perithecium, the insertion of which is just opposite the base of the ascigerous cavity. The locality mentioned is the only one from which this form has been obtained, although large numbers of the same host have been examined from other sources.

### **Stigmatomyces brevicollis** nov. sp.

Short and more or less distinctly curved throughout, rather uniformly suffused with dull amber-brown, except the paler receptacle and hyaline apex. Receptacle relatively short and slender, slightly and evenly suffused with dirty yellowish; the basal cell longer and distally somewhat broader than the subbasal cell, often distinctly narrower at its base. Stalk-cell of the appendage rather short and broad, its external margin usually straight, slightly prominent below the rather narrow insertion of the appendage. Appendage rather strongly curved, relatively long, slightly exceeding the venter, the basal cell twice as long as broad, the antheridia turned sidewise, large, prominent; the lower markedly divergent, with straight prominent necks; two arising from the basal and from the subbasal cells and one from the third; the series ending in a single terminal member. Stalk-cell of the perithecium distinctly larger than the cells above, the general region short and hardly broader than the distal end of the subbasal cell. Venter of the perithecium relatively long, hardly inflated; its diameter nearly the same throughout, longer than the distal portion; the neck short and relatively very broad, sometimes hardly distinguished, sometimes subtended by a slight prominence of the venter wall-cells: the tip and apex diverging inward somewhat abruptly, so as to form an external rounded hunch-like elevation, but not otherwise distinguished; shorter than the neck, truncate-conical: the apex hyaline but not otherwise distinguished; the lips not at all, or but very slightly, prominent. The wall-cells of the venter neck and tip forming a continuous spiral twist of less than one turn, which is often clearly defined, but is not accentuated by any ridge or furrow. Spores 35 × 3.5 $\mu$. Perithecia 85–95 $\mu$; venter 55–64 × 28–34 $\mu$; the neck × 18 $\mu$. Appendage 50–52 × 8 $\mu$. Receptacle 42–54 × 14–16 $\mu$. Total length to tip of perithecium 140–170 $\mu$.

On the head, thorax and legs of species of *Psilopa*. No. 1858 (Type) and 1874, Balaclava, Jamaica. Nos. 1812 and 1813, Fayetteville, Arkansas.

In general appearance this small and rather insignificant form resembles *S. Ochtheroideae* and *S. humilis*, but differs in its three celled appendage, spiral wall-cells and in other details. The material from the West Indies and from Arkansas is abundant, and does not differ essentially.

### **Stigmatomyces indentatus** nov. sp.

Receptacle uniformly hyaline or faintly yellowish, usually straight, and often tapering from its rather broad distal end to its narrow base, the septum very slightly oblique, and associated with a variably distinct slight indentation; the two cells nearly equal, or the subbasal much longer, in which case it is of nearly uniform width throughout. Stalk-cell of the appendage dark amber-brown and strongly and abruptly concave externally, yellow on its inner side, inserted on a shelf-like protrusion of the subbasal cell opposite the distal septum of the latter; its position, in connection with its concavity, resulting in a characteristic constriction or indentation of this region: its distal end but slightly broader, and inconspicuously prominent below the rather broad insertion. Appendage concolorous with the venter of the perithecium and reaching hardly beyond its upper third; consisting of four successively smaller cells; the basal hardly longer than broad; the three lower bearing each two antheridia, with necks diverging in a double series; the fourth bearing a single one, which is followed by one which is terminal and externally spinose; the appendage usually lying flat against the perithecium. Stalk-cell of the perithecium, and the four cells above it, relatively small, more or less similar; their external margins nearly even, concolorous with the venter of the perithecium. Venter usually somewhat longer than the distal portion, relatively large, straight; its axis bent slightly inward, its surface inconspicuously granular, more clearly so distally, regularly ovoid, or narrower distally, and then slightly broader at its junction with the base of the neck; which is straight or slightly curved, distinctly paler above its slightly spreading base; the tip more distinctly colored, hardly distinguished, one or both of its margins slightly concave, nearly twice as long as the apex; which is almost hyaline, often bent abruptly outward, the lips subsymmetrical, distinct, but not very prominent. Spores 24 × 4 $\mu$. Perithecia 120–147 $\mu$; the venter 70–77 × 35–42 $\mu$.

Receptacle 85–125 × 22 μ. Appendage 42 μ. Total length to tip of perithecium 210–315 μ.

On the superior surface of the abdomen, near the tip, of *Psilopa* sp., No. 1808b, (Type), Fayetteville, Arkansas. In the same position on Psilopa sp., No. 2496, Orizaba, Mexico, (Mann).

This species is clearly distinguished from other forms with four-celled antheridia by the shape and position of the stalk-cell of the appendage, the abrupt external concavity of which causes the individual to appear constricted in this region. The Mexican specimens agree in all respects with those from Arkansas, which are abundant and in good condition.

### **Stigmatomyces Ochtheroideae** nov. sp.

Rather strongly curved throughout and somewhat deeply suffused with dull amber-brown. Receptacle relatively short and slender, straight or slightly curved, the basal cell tapering somewhat below, sometimes finely transversely punctate, distally somewhat broader than the base of the much shorter subbasal cell. Stalk-cell of the appendage overlapping the subbasal cell for about one third of its length, rather short and broad, evenly convex externally, but otherwise not prominent below the insertion of the appendage, which occupies its whole distal surface. Appendage lying sidewise, or with the antheridia turned outward, relatively large, short and compact; consisting of four successively smaller cells; the basal relatively large, short, much broader distally; bearing, like the subbasal, two stout divergent antheridia; the two distal cells bearing one each, and followed by a seventh terminal one, which bears a small spine near its base. Stalk-cell of the perithecium and the cells above it, more or less uniform, with somewhat rounded outlines; the secondary stalk-cell and the external basal cell above it individually prominent; the whole region compact, becoming distally even broader than the base of the venter through abnormal thickening of the external walls. Venter of the perithecium transversely mottled or granular, hardly if at all inflated, about twice as long as broad, the base of the stout neck abruptly spreading, and not distinguished at its line of junction; the broad stout tip bent rather abruptly inward, subtended externally by a more or less abrupt distal external elevation of the neck; the apex short, stout, somewhat shorter than the tip, not at all distinguished from it, and tapering to the rather broad, asymmetrical, slightly sulcate termination; the lips rounded, the outer much broader and somewhat more prominent. Spores 36 × 4 μ. Perithecia 100–110 × 30–35 μ. Appendage 45–50 × 16 μ. Receptacle 55–65 × 14–16 μ just below the septum. Total length to tip of perithecium 150–175 μ.

On the superior surface of the thorax of *Ochtheroidea* spp., No. 2826, Port of Spain, Trinidad, B. W. I.: No. 2519, Bocas del Toro, Panama, (Rorer). No 2062, Grenada, B. W. I.

In general form this species resembles *S. humilis*, *S. borealis* and *S. brevicollis*, but differs in the character of its appendage and in other minor points. The mottling of the perithecium is clearly seen with higher magnifications, and in the Trinidad specimens the basal cell is transversely punctate, and less clearly so in those from Grenada, although in the single specimen from Panama this cell is unmodified.

### **Stigmatomyces compressus** nov. sp.

Erect, usually straight, rather deeply suffused with clear amber-brown, except the hyaline receptacle; which is often slightly bent, its anterior margin more or less strongly convex, its distal end sometimes conspicuously broader than the base of the subbasal cell; which is separated from it by a usually slightly oblique septum, and is distinctly shorter, its distal margin oblique. Stalk-cell of the appendage concolorous with the more deeply suffused venter of the perithecium, relatively large and stout, abruptly prominent above the subbasal cell, which it overlaps one third, or usually much less, and below the insertion of the appendage; which is rather broad and distinguished by a deeply suffused concolorous septum. Appendage rather stout; consisting of four cells successively slightly smaller; the basal and subbasal bearing two, while the two upper bear single antheridia, the series ending in a terminal one with an external spine. Stalk-cell of the perithecium flattened, five-sided, obliquely separated from the subbasal cell of the receptacle; the cells above it more or less uniform, slightly smaller, subtriangular, the two external ones with somewhat convex margins: venter straight, erect comprising about one half the total length of the perithecium, almost symmetrically inflated, its surface granular or irregularly mottled, sometimes with indistinct small elevated patches: the neck abruptly distinguished, stout, spreading slightly at the base, as well as distally below the tip; which is thus rather clearly distinguished, as broad as the mid-portion of the neck;

the apex more than half as long, not at all distinguished, distally abruptly compressed with rounded margins, the broad termination bearing four minute abruptly distinguished papillate lip-terminations, which are rather distant, and almost symmetrically placed around the pore. Spores 30 × 3.8 μ. Perithecia 120–140 × 38–42 μ, maximum 150 × 48 μ. Appendage about 55 μ. Receptacle 70–100 × 16 μ, maximum 120 × 20 μ. Total length to tip of perithecium 220–280 μ, maximum 310 μ.

On legs and wings of *Psilopa* spp., No. 1725, (Type), Mandeville; Nos. 1855 and 1875, Balaclava, Jamaica, W. I. On base of wing of *Ochtheroidea glaphropus* Loew., No. 1710, Mandeville, and No. 1920, Porus, Jamaica.

This species is not unlike *S. rugosus* in appearance, but differs in its four-celled appendage and abruptly compressed papillate apex. The surface of the perithecium is usually rather finely granular-mottled, sometimes with a tendency to produce slightly elevated mottled patches about as large as the verrucosities of *S. rugosus*. In older specimens the wall-cells of the venter may show a slight obliquity, and their distal ends may be somewhat prominent, forming an indistinct ridge below the spreading base of the neck.

### STIGMATOMYCES MICRANDUS Thaxter, var. **Atissae** nov. var.

I have separated under this varietal name a form which grows on very minute dark species of *Atissa* in the West Indies, occurring more often on the upper surface of the head, or the adjacent superior surfaces of the thorax, or on the bases of the antennae, usually forming a conspicuous group in these positions, while not infrequently it may be found growing on the legs. I have never seen it, however, on the abdomen which was the position occupied by the type-form on its undetermined papuan host. The general form of the variety and its peculiar three celled appendage correspond closely to that of the type; the venter of the perithecium is either smooth or finely rough-granular, never verrucose, and the granulation may involve the neck as in *S. Psilopae*. In many cases the junctions of the wall cells of the venter are clearly indicated by slightly twisted ridges which, however, cannot always be clearly distinguished. The conformation of the distal portion of the perithecium is similar in general to that of the type, although the peculiar modification of the apex about the pore is less well marked than is indicated in the original figures (Monograph, II, Plate XLVI, fig. 24). The variety is smaller than the type, the average length in well developed individuals being about 180 $\mu$, the maximum hardly more than 200 $\mu$. Abundant material has been examined from the following sources, in all cases from species of *Atissa*. Nos. 1737 (Type), 1717 and 1738, Mandeville; No. 1917B, Battersea, and No. 2042 from Clarkstown, Jamaica, W. I. Nos. 2803–2804, St. George, Grenada: these numbers representing numerous individual hosts.

### **Stigmatomyces Psilopae** nov. sp.

Variably elongate, sometimes rather short and stout, wholly suffused with brownish yellow or amber-brown, except the hyaline receptacle. Receptacle straight or slightly curved, but slightly broader distally, the two cells usually of about equal length. Stalk-cell of the appendage relatively short and broad, its base hardly overlapping the subbasal cell, distally prominently rounded outward below the insertion. Appendage rather stout and compact, the axis consisting of four cells and bearing eight antheridia, their short slightly curved stout necks directed outward, or slightly sidewise; the terminal one bearing a conspicuous spine on its inner side at the base of its neck; the fourth cell of the axis bearing a single antheridium; the three lower each two; the basal cell somewhat larger than the others, narrower below. Cells of the stalk- and basal cell region more or less uniform in size, the outer somewhat prominent: venter of the perithecium becoming normally verrucose, the verrucosities irregularly disposed, or more or less distinctly transverse, rarely inconspicuous; oval or elliptical in outline, sometimes tapering more distinctly distally, rather abruptly distinguished from the abruptly spreading base of the usually elongate and slender neck; which is typically verruculose, but may be smooth, or only somewhat roughened, of about the same diameter throughout, or tapering slightly distally; its junction with the tip occupied by an abrupt and conspicuous inflation, which is nearly symmetrical on either side: the apex distally very slightly oblique and bearing a crown of four clearly defined somewhat divergent blunt prominences, a fifth, somewhat broader and slightly more prominent, lying between the two outer ones. Spores 35 × 3.5 $\mu$. Perithecia: venter 85 × 42–62 $\mu$; neck 70–140 × 18 $\mu$; tip 28–32 $\mu$. Receptacle 78 $\mu$ to very rarely 260 × 20–25 $\mu$. Appendage 52 × 14 $\mu$. Total length to tip of perithecium 300–435 $\mu$, the longest 540 $\mu$.

On species of *Psilopa*. No. 1853 (Type) No. 1867, Balaclava;

No. 1730, Mandeville; No. 1908, and 1909, Williamsfield; No. 1925, Porous; No. 1926, Battersea and No. 2045, Clarkstown, Jamaica, W. I. No. 2806, St. George, Grenada, W. I.

In general form and in the verrucosity of the venter of its perithecium, this species is similar to *S. micrandrus*, to which it is nearly related. It differs chiefly in its constantly four-celled appendage, its usually verruculose neck, and in the conformation of its apex. An abundant series of specimens has been examined.

On species of *Ilythea*, from the West Indies, Kamerun, Borneo and New England, I have obtained a form which, although subject to great variation, does not seem satisfactorily divisible into more than one species. The variations, however, are such, except as regards the appendage, that it is almost impossible to give a composite diagnosis which would be satisfactory. These variations are apparently in part regional, in part due to differences in position of growth, and in all probability to some extent are owing to differences in the hosts; although the latter do not vary very greatly. I have therefore selected as the Type the most characteristic variation, which has been found on the thorax, only, of a new species of *Ilythea* from Jamaica, and have appended notes on the more important deviations from this type that have come under my notice.

### **Stigmatomyces Ilytheae** nov. sp.

Receptacle contrasting with the more or less uniform suffusion of the parts above, quite hyaline, or becoming very faintly yellowish, very stout and thick walled; of almost uniform width, or but slightly narrower at the rounded base; the minute foot sometimes lateral; the subbasal cell somewhat longer than the basal. Stalk-cell of the appendage somewhat obliquely related to the subbasal cell, and occupying more than one third of its width; stout, hardly longer than broad, externally very slightly convex or almost straight, but distally rounded and protruding conspicuously below the insertion. Appendage consisting of five cells, relatively short and broad, tapering more or less uniformly to the erect terminal antheridium, which bears a spine sublaterally at the base of its erect neck: the basal cell deeply tinged with amber-yellow, concolorous with the stalk-cell, the walls of both clear amber-brown, small, short, subtriangular; the rest of the appendage paler; the fifth cell producing a single antheridium, while all the rest produce two; the series directed somewhat sidewise and outward, the necks small, somewhat appressed in two rows, but slightly curved. Stalk-cell of the perithecium broad, flattened, subtriangular, paler and usually larger than the cells above it; which are subequal, somewhat rounded, the external basal cell somewhat prominent externally, with thickened wall. Perithecium smoky brown throughout; the venter more deeply, suffused, straight, erect, hardly inflated, longer than the distal portion, of more or less even diameter throughout or somewhat broader distally; the neck diverging or curved outward more or less distinctly, more or less abruptly distinguished, its abruptly spreading base slightly asymmetrical; the tip slightly narrower; the two outer lips slightly prominent, the inner forming a small but characteristic subtriangular prominence, the distal free margin of which is horizontal and nearly straight, on a level with, or slightly lower than the outer lip-edges, while its outer (posterior) margin curves abruptly inward to a minute hyaline papilla which subtends it. Spores 35 × 3.5 $\mu$. Perithecia 175–192 $\mu$; the venter 70–98 × 28–42 $\mu$. Appendage 56 × 16 $\mu$. Receptacle 100–110 × 30 $\mu$, the smallest 56 × 28 $\mu$. Total length to tip of perithecium 210–300 $\mu$.

On the superior surface of the thorax of *Ilythea* sp. No. 1907 (Type) Williamsfield, and No. 2043f, Clarkstown, Jamaica, W. I.

Variations from the type-form above described may be distinguished as follows.

Var. a. From Clarkstown, Jamaica, No. 2043e, on the upper, surface of the abdomen and thorax, measuring 280–350 $\mu$ in length, the perithecium straight or slightly curved, the distal portion much longer, even twice as long as the venter, and long-conical, the tip slightly distinguished on both sides, the lips forming a more prominent median short termination, symmetrically subtended on either side by shorter projections.

Var. b. No. 2043d, from the same locality, is similar except that the distal portion of the perithecium is relatively somewhat shorter and describes a sigmoid curve.

Var. c. No. 2064, from St. George, Grenada (Brues and Allen), growing on the upper surface of the thorax, has a receptacle about as broad as the venter of the perithecium; which is slightly and symmetrically inflated and broader distally, where it is abruptly distinguished from the very long and more slender distal portion, which terminates in a fashion similar to that of varieties a and b.

Var. d. No. 2043d on the anterior legs of an *Ilythea* from Clarkstown, Jamaica, is long and slender and evenly curved throughout; the receptacle relatively slender and tapering below, the perithecium long, the outer wall-cells more conspicuously shorter than the inner, the neck slender and longer than the venter, from which it is not very abruptly distinguished, characteristically and slightly geniculate at its junction with the tip; the apex not at all distinguished, the termination unlike either of the preceding forms, the outer lips broad and rounded, the inner forming a minute papilla placed somewhat lower. Unlike other individuals from the legs, this variety attains a considerable length, measuring up to 315 μ in length.

Var. e. A single specimen, No. 2643, from Kamerun, West Africa, is very long and slender, the venter short and inflated, with indications of distal elevations which alone serve to distinguish the elongate, slender distal portion, the base of which tapers gradually, becoming then nearly uniform in width to the very apex, which is modified much as in var. a and b.

Var. f. No. 2043b from Clarkstown, Jamaica and occurring on the legs, is a shorter stouter form, measuring about 260 × 45 μ; the perithecium and receptacle of about equal length, the latter but slightly smaller than the former, both tapering more or less uniformly to the pointed base and apex, the outer and inner lips equally broad, not prominent, the inner slightly lower.

Var. g. No. 2132, from Sarawak, Borneo, a still smaller and more compact form, the longest measuring 150 × 25 μ, growing at the tip of the abdomen, the perithecium tapering throughout to the rather blunt extremity, the lips slightly prominent and more or less similar; the appendage four-celled, a condition which is occasionally found, especially in more depauperate forms of the other varieties.

Var. h. No. 1305, on *Ilythea spilota* Curtis, collected at Kittery Point, Maine, is similar in general form to Var. f, measuring from 190–200 μ, the perithecium about 120 × 35 μ, and longer than the receptacle. The appendage is apparently always four-celled, and the color slightly different, but otherwise it possesses no distinctive characters.

### **Stigmatomyces Chilensis** nov. sp.

Receptacle rather stout, nearly uniform, hyaline, becoming faintly suffused; the basal cell very slightly inflated distally; the subbasal distinctly longer, slightly broader distally than the stalk-cell region of the perithecium. Stalk-cell of the appendage amber-brown, concolorous with the venter and basal cell region of the perithecium; relatively short and stout, its base rather broad and hardly oblique, its outer margin straight, its distal margin but slightly broader than the very broad insertion of the relatively large appendage. Appendage becoming more deeply suffused with a faint smoky brown tinge, its axis consisting of five short cells of about equal length, except the fifth which is somewhat smaller; bearing ten rather crowded antheridia, their large prominent, nearly straight necks directed obliquely outward, one terminal without visible spine, often reaching to the distal end of the venter, one from the fifth axis-cell and two from each of the others. Stalk-cell of the perithecium longer than broad, pale, subtriangular, its outer and upper inner angles broadly rounded; somewhat larger than the secondary stalk-cell, which is similar in form; the basal cells relatively large, the outer distally prominent beyond the base of the venter, its outer margin somewhat concave. Venter straight, rarely slightly inflated, the margins usually straight and diverging slightly, the distal end which is faintly striate or punctate, thus distinctly broader than the base, and marked by four conspicuous flat rounded elevations, variably prominent and corresponding to the four wall-cells: the slightly spreading base of the very stout neck thus abruptly distinguished, about equal to the venter in length, distally slightly enlarged, more distinctly so externally, so that the very slightly narrower tip and apex are more or less clearly distinguished from it: the hyaline and distally somewhat oblique apex not at all distinguished; the tip very slightly inflated distally, so that its lower margin is somewhat concave, especially externally; the rounded outer lips broader and more prominent than the inner. Spores 30 × 3 $\mu$. Perithecia 130–160 $\mu$, largest 192 $\mu$: venter 55–64 × 28–32 $\mu$ at base × 35–42 $\mu$ at apex. Appendage 52–60 × 18 $\mu$. Receptacle 90–140 $\mu$, the longest 160 $\mu$. Total length 250–300 $\mu$, the longest 400 $\mu$.

On the legs and abdomen of a species of *Discocerina*, No. 1464, vicinity of Concepcion, Chile, Nov. 1905.

Although the conformation at the tip of the perithecium in this species suggests that of the less well marked forms of *S. Discocerinae*, and its venter is not unlike that of *S. Caribbeus*, both of which occur on flies of the same genus, it is very clearly distinguished from either by its large coarse five-celled appendage. Abundant material has been examined, the individuals showing little variation except in size.

### **Stigmatomyces Discocerinae** nov. sp.

Typical form relatively short and stout, rather deeply suffused with dirty amber-brown, except the nearly hyaline or slightly yellowish receptacle. Receptacle more or less uniform and abruptly distinguished as a stalk-portion from the parts above; the basal cell usually longer, its lower third often curved, tapering to the pointed base; the subbasal cell often but slightly longer than broad, separated from the basal by a more or less evident constriction, its outer membrane usually, but not always, more or less conspicuously crinkled or corrugated, the surface sometimes appearing transversely striate. Stalk-cell of the appendage relatively short, its broad base slightly oblique, its distal end somewhat broader, but not conspicuously bulging outward below the broad insertion of the appendage. Appendage short, compact, bearing a terminal spinose antheridium and five or seven others, according as the axis consists of three or four small cells; the basal similar, or but slightly larger than the others, and concolorous; the terminal cell bearing one, the rest two, antheridia; their necks short and stout, and directed sidewise; the appendage lying flat against the perithecium. Stalk-cell of the perithecium broad, five-sided, larger than the cells above it; which are subequal, except the secondary stalk-cell, which is small and externally slightly prominent; the outer basal cell above it externally slightly concave, its curvature continuous with that of the inflated base of the venter. Venter oval or elliptical in outline, usually strongly and symmetrically inflated, darker, the surface inconspicuously finely granular-mottled, with a tendency to transverse striation; the broad spreading base of the short neck abruptly distinguished, the margins of the latter more or less concave, owing to a variably developed, but typically conspicuous, distal enlargement which subtends the tip; the margins of the tip also more or less distinctly concave, owing to a second variably prominent distal enlargement at its junction with the apex; the latter stout, its distal margin broad, slightly oblique when viewed sidewise, with relatively large but not very prominent papillate lips. Spores 30 × 3 μ. Perithecia 95–105 × 30–35 μ. Appendage 30 × 12 μ. Receptacle 35–55 × 12–15 μ. Total length to tip of perithecium 125–175 μ.

On species of *Discocerina* usually on the legs or thorax. Nos. 1848 (Type) and 1846, Balaclava; Nos. 1716 and 1727, Mandeville; No. 1923, Porous, and No. 2045, Clarkstown, Jamaica, W. I.; No. 2316, Port of Spain, Trinidad, B. W. I.; the material derived from numerous individual hosts.

A form growing on a somewhat larger species of *Discocerina* from Balaclava, Jamaica, No. 1847, is larger and more slender, the total length reaching 235 μ, the receptacle 85 μ. The distal portion of the perithecium lacks the characteristic double elevation, and the receptacle is of a nondescript type. It can however hardly be assigned to any other species. The present form is perhaps most nearly related to *S. pauperculus* from which it differs in the form of its perithecium and receptacle while it also lacks the deeply colored contrasting basal cell which distinguishes the appendage of the papuan form.

### **Stigmatomyces Caribbeus** nov. sp.

Pale, slightly sigmoid. Receptacle tapering slightly throughout, the basal cell shorter and more or less distinctly curved: stalk-cell of the appendage almost a pointed oval, its point slightly overlapping the subbasal cell, its protruding outer margin evenly curved throughout, the insertion broad. Appendage tapering, consisting of three cells, bearing six relatively large antheridia directed sidewise, a terminal one, two each from the basal and subbasal, and one from the upper of the three axis-cells; the subbasal cell relatively large, longer than the basal, its margin evenly convex, that of the smaller third cell somewhat less so. Cells of the basal cell region subequal, the stalk-cell somewhat larger and triangular, the outer basal cell and the secondary stalk-cell both protruding independently and conspicuously, especially the former, which bulges beyond the base of the venter. Venter straight, pale, the wall-cells ending in a corresponding number of clearly defined, distal, somewhat asymmetrically placed elevations, which serve abruptly to distinguish the abruptly spreading base of the neck; which is otherwise rather stout and nearly uniform, usually more or less strongly curved outward, hardly distinguished from the tip and apex; which are but slightly if at all narrower, or may be even slightly inflated distally, the distal margin of the almost equally broad apex often flat truncate, or slightly asymmetrical when viewed sidewise; the lip-cells forming inconspicuous flattish papillae. Spores about 30 × 3 μ. Perithecia 122–140 μ; the venter 70 × 32 μ; the neck 65–70 × 10–12 μ. Appendage 40–50 μ. Receptacle 70–100 × 15–20 μ. Total length to tip of perithecium 210–265 μ.

On the abdomen of species of *Discocerina*. No. 2518, (Type), Bocas del Toro, Panama, (Rorer); No. 2053, Williamsfield and No. 2044, Clarkstown, Jamaica, W. I.

This appears to be a very rare species, since it has been obtained in only three instances among the very large number of hosts examined from the West Indian region. The appearance of its venter recalls that of the simpler forms of *S. constrictus*, but its form is otherwise different and the appendage is of quite another type.

### **Stigmatomyces ambiguus** nov. sp.

Nearly straight to the distal end of the venter, above which the distal portion of the perithecium is more or less strongly curved or bent inward. Receptacle more or less uniform throughout, slightly broader at the horizontal septum; the basal cell nearly hyaline, slightly narrower below; the subbasal distinctly suffused, shorter or slightly longer than the basal. Stalk-cell of the appendage more deeply suffused with dull amber-brown, concolorus with the uniformly suffused perithecium, its base slightly overlapping the subbasal cell, rather short and broad, prominently rounded below the rather broad insertion. Appendage relatively large and stout, about as long as the venter consisting of four successively slightly smaller cells; the basal rather large, somewhat longer than broad, concolorous with the stalk-cell; the three lower cells bearing each two antheridia, the fourth a single one followed by a terminal one; the antheridia rather large with prominent nearly straight necks turned outward, or somewhat sidewise, and divergent. Stalk-cell of the perithecium rather small, rounded distally, its base straight and oblique; the cells above it but slightly smaller, more or less uniform, somewhat less deeply suffused than the rest of the perithecium. Venter straight, its surface indistinctly transversely irregularly granular-punctate, nearly symmetrical, but slightly inflated: neck about as long as the venter, curved or bent inward, concolorous, less conspicuously granular-punctate; stout, but clearly distinguished above its slightly but abruptly spreading base; somewhat broader distally below the slightly but abruptly distinguished short tip, which is not distinguished from the much shorter apex; the margins of the two slightly and nearly symmetrically convex; the apex broad, subtruncate and slightly oblique distally, the lips hardly distinguished. Spores about 35 × 3.5 $\mu$. Perithecia 140–155 $\mu$, the venter 60–70 × 30–32 $\mu$. Receptacle 88–105 × 21 $\mu$. Appendage 60 × 15 $\mu$. Total length to tip of perithecium 210–280 $\mu$.

On *Ochtheroidea* sp., No. 2062b (Type), St. George, Grenada, W. I., at base of right wing. On *Psilopa* sp. on wing, No. 1387, Island of Margarita, Venezuela (Blakeslee).

The specimens from the two localities above mentioned are absolutely identical, and seem well distinguished from other forms which occur on related hosts. Its nearest ally appears to be the rather variable *S. Discocerinae*, which is distinguished by its normally shorter more compact appendage, uniformly hyaline receptacle indented at the septum and differently developed perithecium.

## Stigmatomyces dubius Thaxter.

A form that seems to correspond so closely to this species that I am unwilling to separate it specifically, has been found growing on the legs of *Ochthera exsculpta* Loew., No. 2807, St. George, Grenada, B. W. I. It corresponds in general form with the New Guinea types from Ralum, New Pommerania, which were obtained from a fly allied to *Ochthera* and possibly belonging to this genus. In the West Indian form, the appendage consists normally of eight cells, of which the seventh is spinose. The tip of the perithecium also, is more abruptly distinguished from the neck, while the lips are hardly prominent and lack entirely the tongue-like prolongations of the type.

## Stigmatomyces gracilis Thaxter.

I refer to this species, also with some hesitation, a form which occurs on the abdomen and at the base of the posterior legs of *Ochthera mantis* De G., from Fayetteville, Arkansas, Nos. 1802 and 1803. The appendage is terminated by a spinose antheridium, and, although it consists of usually six cells, instead of five as in the New Guinea material, is otherwise identical. The general form of the individuals from the abdomen is stouter, curved, or with the axis of the perithecium directed inward at an angle to that of the receptacle, while the tip is much less abruptly distinguished from the neck. The apex may be exactly similar in both; or more or less well developed, slightly divergent, ear-like projections may be present at maturity from the outer lip-cells. These projections, however, may be quite undeveloped. The individuals growing at the base of the posterior legs, however, are longer more nearly straight and slender. Until an opportunity occurs to examine and compare material on *Ochtherae* from other localities, it seems undesirable to separate either of these Western Hemisphere forms from the Papuan types.

### **Stigmatomyces Ochtherae** nov. sp.

Relatively short and stout, curved or sometimes straight. Receptacle subhyaline, the cells thick-walled, the basal broadly rounded below, shorter than the subbasal, somewhat longer than broad. Stalk-cell of the appendage rather deeply colored, dull brownish, concolorous with the perithecium; relatively large, subtriangular, short and stout, as broad as long, very prominent, overlapping the upper fourth to half of the subbasal cell; its inner margin almost coincident with that of the receptacle; its nearly flat or slightly rounded distal surface more than twice as broad as the rather narrow insertion of the appendage. Appendage relatively large and broad, usually curved against the side of the perithecium, with the antheridia directed outward; consisting of five to seven cells, the upper bearing a single antheridium followed by a terminal one; the remainder two each, the necks small, slightly curved, not prominent; the basal cell deeply colored, short, cup-shaped. Stalk-cell of the perithecium broader than long, paler and larger than the cells above it, which are concolorous with the perithecium. Venter of the perithecium short, hardly inflated, the neck not at all or very slightly distinguished, considerably longer, very broad, its inner margin concave below, becoming straight, its outer somewhat convex, abruptly so where it joins the tip, and sometimes swollen in this region so that its diameter may equal that of the venter: the tip short, its outer margin bending inward almost at right angles, but otherwise hardly distinguished, very short, the hyaline lips forming an abruptly papillate termination close to the inner margin. Spores about 35 × 3.5 μ. Perithecia 100 × 31–35 μ. Appendage 70–85 × 15–20 μ. Receptacle 66–85 × 18–25 μ. Total length to tip of perithecium 160–180 μ.

On the superior surface of the abdomen, near the tip, of *Ochthera* sp., No. 1760, Balaclava, Jamaica, W. I.

This species is perhaps as nearly allied to *S. gracilis* in the character of its appendage, as to any other forms, but is so peculiar that it is sufficiently unmistakable. The swollen end of the perithecium gives it a characteristically hunched appearance quite unlike that of any known species.

**Stigmatomyces borealis** nov. sp.

Slightly bent in the middle, rather small and stout, the clear hyaline receptacle contrasting with the clear amber-brown or yellow uniform suffusion of the portions above; the basal cell becoming slightly suffused with brownish at maturity, the suffusion involving the base of the subbasal cell. Receptacle rather short, the posterior walls distinctly thicker, slightly broader at the septum, the basal cell tapering throughout to the foot, more than twice as long as the subbasal, which is usually hardly longer than broad, and separated by a slightly oblique septum. Stalk-cell of the appendage overlapping the subbasal cell slightly, short, its external margin slightly concave, distally somewhat broader than the insertion. Appendage lying flatwise against the perithecium, with the insertion somewhat above the base of the ascigerous cavity; erect, somewhat tapering, as long or nearly as long as the venter, consisting of three cells; the basal slightly longer than broad, the subbasal but slightly smaller and subtriangular, both bearing two closely associated antheridia with short somewhat divergent necks; the third smaller and bearing a single antheridium, followed by the terminal one which bears a small spine. Stalk-cell and secondary stalk-cell of the perithecium nearly equal and similar, flattened subtriangular and relatively small, the latter wholly separated from the subbasal cell; the basal cells of the perithecium smaller, uniform, hardly protruding. Perithecium stout, usually considerably larger than all the other portions combined, more or less strongly falcate, the inner margin slightly concave, the outer strongly convex; the venter relatively large and broad, tapering to the hardly differentiated distal portion; the neck distinguished by a scarcely perceptible contraction, half or less than half as long as the venter; the tip slightly distinguished, more clearly so externally, much shorter than the wholly undifferentiated apex which tapers to a blunt point, the outer lips forming a slight angular prominence. Spores 34 × 3.5 $\mu$. Perithecia 96–110 × 30–40 $\mu$. Appendage 48 $\mu$. Receptacle 60–85 × 22–24 $\mu$. Total length to tip of perithecium 150–200 $\mu$.

At the base of the posterior legs of *Parydra imitans* Loew., No. 1372, Kittery Point, Maine.

Of the sixteen individuals which have been examined and include fully matured conditions, one is much more slender and elongate than the others, which are very uniform, and measures 265 $\mu$ in total length. The species is of a mondescript type without well marked distinguishing peculiarities, yet I am unable to include it in any of the other species found on Ephydridae. In general form it resembles *S. humilis*, *S. Ochtheroideae* and *S. brevicollis* which, however, differ in the character of their appendages, as well as in minor points.

### **Stigmatomyces lingulatus** nov. sp.

General habit rather elongate with a slight median twist, arcuate, or even distally recurved. Receptacle pale yellowish, rather long, stout, more or less conspicuously wider in the middle in the region of the septum, the two cells of nearly equal length, the subbasal more often distinctly narrower distally. Stalk-cell of the appendage turned partly sidewise, more deeply suffused, rather short, barely overlapping the subbasal cell, somewhat prominently rounded below the insertion, which lies somewhat above the base of the ascigerous cavity. Appendage persistent, concolorous with the venter; its axis consisting of three cells, the basal twice or somewhat more than twice as long as broad, the subbasal flattened, two sided, both bearing two antheridia; the third smaller and bearing a single antheridium which is followed by the terminal one, the necks of the series of six having a distinct right and left divergence. Stalk-cell of the perithecium small, flattened, subtriangular, paler than the cells above it; which are more or less similar, concolorous with the venter, and not externally prominent. Venter of the perithecium very faintly granular, the inner margin nearly straight, or slightly concave, the outer strongly and evenly convex; the wall-cells terminating in rather inconspicuous individual prominences, which render the transition to the neck abrupt: distal portion of the perithecium considerably longer than the venter, strongly and rather evenly curved; the neck hyaline and narrower below, the distal third and the short undifferentiated tip becoming distally broader; the apex but slightly shorter than the tip, not distinguished, very slightly geniculate, two of the lip-cells combined and prolonged to form a symmetrical tongue-like prolongation tapering to a blunt extremity, and extending free beyond the broader blunt termination of the other two lip-cells. Spores 38 × 4.5 μ. Perithecia; venter 70–80 × 40 μ; the distal portion 90–100 × 14 μ near the base and × 18 μ distally. Appendage 75–80 × 12–14 μ; the basal cell 24 × 10 μ; the stalk-cell 30 μ. Receptacle 115–135 × 24–28 μ in the mid-region. Total length to tip of perithecium 300 μ or somewhat more.

On the superior surface of the abdomen of *Parydra humilis* Will. No. 1868, Balaclava, Jamaica.

A species most nearly allied to *S. protrudens* and *S. pinguis*, from which it is abundantly distinguished by the form of its receptacle and perithecium, as well as by the tongue-like development of two of the lip-cells.

### **Stigmatomyces pinguis** nov. sp.

Receptacle stout, subhyaline or pale yellowish; the basal cell tapering slightly below its anterior margin, nearly straight; the posterior more or less convex, distally slightly inflated and more or less distinctly broader than the base of the subbasal cell, which is separated from it by a more or less oblique septum, and is usually from one half to one third as long, becoming broader distally. Stalk-cell of the appendage rather short, darker dull amber-brown, concolorous with the venter and basal cell region of the perithecium; its external margin straight or somewhat concave, overlapping the subbasal cell very slightly, distally protruding somewhat below the rather broad insertion of the appendage, which is slightly higher than the base of the ascigerous cavity of the perithecium. Appendage permanent, uniformly suffused, straight or usually convex on the axis side, lying somewhat obliquely or with the antheridia turned outward; the axis consisting of three cells, the basal distinctly longer than broad, bearing two slightly divergent antheridia from which it is very obliquely separated; the two remaining cells flattened, the subbasal longer and bearing two antheridia; while the third bears only one, which is followed by the single terminal antheridium, the neck of which becomes abruptly curved toward the axis-side. Stalk-cell of the perithecium occupying the whole width of the receptacle, much flattened, subtriangular, paler than the deeply suffused cells above it; which are concolorous with the venter of the perithecium, rather small and irregularly triangular, the outer ones variably prominent. Venter of the perithecium relatively large and stout, coarsely granular, broadly ovoid, the inner margin distinctly more convex; about as long as the distal portion which is strongly and abruptly curved inward in the region of the tip, the neck not very abruptly distinguished, its wall-cells separated by more or less distinct furrows, its spreading base delimited by a horizontal line of separation from the venter; otherwise stout and nearly uniform, or slightly broader where it joins the short tip and apex, which are not otherwise distinguished; the termination broad, subtruncate, the outer lips more prominent. Spores 28 × 4 μ. Perithecia 120–135 μ; the venter 65–78 × 40–55 μ, the neck 16 μ in diameter. Appendage about 60 × 12 μ. Receptacle 65–98 × 22–26 μ. Total length to tip of perithecium 200–260 μ.

On the under surface of the left wing of *Parydra pinguis* Walk. No. 1805d, Fayetteville, Arkansas.

This species has been found on a single individual only, and is distinguished by its very stout venter and the evenly incurved neck of the perithecium; the appendage convex on the axis side and the neck of the terminal antheridium abruptly bent in the opposite direction, the wall-cells of the neck more or less distinctly prominent and thus separated by corresponding furrows. The species is evidently related to *S. protrudens* and *S. lingulatus*.

### **Stigmatomyces protrudens** nov. sp.

Receptacle subhyaline, relatively stout and of nearly uniform diameter throughout, the basal cell usually bent and slightly narrower just above the foot; longer, sometimes twice as long as the subbasal. Stalk-cell of the appendage relatively short and irregularly triangular, slightly overlapping the subbasal cell, becoming narrower distally below the insertion of the appendage with which it is concolorous, both being more deeply tinged with dull amber-brown than the perithecium and its basal cell region, which are uniformly pale yellowish with slight amber-brown suffusions. Appendage inserted nearly opposite the base of the ascigerous cavity, relatively large and sometimes nearly as long as the venter of the perithecium, lying flatwise against it, usually curved throughout, the axis side concave; its axis consisting of three cells; the basal somewhat longer than broad, shorter than the subbasal, both producing two rather large antheridia, the short stout curved necks of which diverge slightly right and left; the upper cell smaller bearing a single antheridium which is united to the terminal one. Stalk-cell of the perithecium subhyaline, small subtriangular, occupying the whole width of the subbasal cell from which it is somewhat obliquely separated. Secondary stalk-cell and basal cells small, nearly uniform, irregularly triangular, concolorous with the perithecium, the external cells slightly prominent. Venter of the perithecium straight, erect, subsymmetrical, but slightly inflated, somewhat longer, as a rule, than the distal portion, its base not abruptly distinguished from the basal cell region and hardly broader;

neck-portion slightly curved; stout but rather abruptly distinguished, of nearly uniform width the very short, abruptly tapering tip distinguished only by a combined external protrusion from the distal ends of its two outer wall-cells which may be hunch-like or rounded, or form an abrupt rather narrow divergent free projection; the apex very short, the lips not at all prominent, forming together a more or less evenly rounded papilla. Spores 30 × 4 μ. Perithecium 120–140 × 30–45 μ, its protrusion to 16 μ. Receptacle 70–100 × 26 μ. Appendage 65–70 × 15 μ, its stalk-cell 25–27 × 12–14 μ. Total length to tip of perithecium 225–275 μ.

On the thorax, wings and superior abdomen of *Parydra pinguis* Walk. No. 1805, Fayetteville, Arkansas.

This form is well distinguished by the peculiar subterminal projection from its perithecium. It is not usually in very good condition, having a somewhat shriveled look, but was obtained from several different individuals of its host, and does not vary to any great extent, except in the form and comparative prominence of the subterminal projection.

### **Stigmatomyces Parydrae** nov. sp.

Form short and stout, usually strongly curved, dirty yellowish brown, except the nearly hyaline, or but slightly suffused receptacle. Basal cell of the receptacle usually strongly curved, tapering slightly below, twice to several times as long as the subbasal cell; which is squarish or even broader than long, and similar to, or but slightly larger than, the five more or less similar irregularly subtriangular cells of the stalk and basal cell region. Stalk-cell of the appendage relatively short and stout, slightly narrower below and overlapping the subbasal cell slightly, or not at all; hardly or but slightly prominent below the basal cell of the appendage, which occupies its whole distal surface. Appendage relatively large, somewhat curved inward, lying somewhat obliquely sidewise against the perithecium, the necks of the antheridia very short, stout, bent outward, the axis consisting of normally five, sometimes four cells, all of which, except sometimes the uppermost, bear two somewhat divergent antheridia, the series thus partly double and ending in a single terminal one. Venter of perithecium not distinguished from the basal cell region, both more deeply suffused, its outer margin more strongly convex especially distally where it curves inward to the well distinguished but relatively stout neck-portion; which is much shorter than the venter, distally slightly broader where it joins the tip, from which it is not distinguished; the tip strongly convex externally, straight or slightly concave on its inner side; the apex very short, hyaline, abruptly distinguished, papilliform, slightly prominent externally above the usually persistent insertion of the trichogyne, its outline more or less evenly rounded, the lips not at all prominent. Spores about 30 × 4 μ. Perithecia: basal and stalk-cell portion about 18 × 27 μ; venter, average 60 × 35 μ, maximum 75 × 40 μ; distal portion about 45–50 × 11 μ. Receptacle 45 × 20–22 μ, maximum length 80 μ. Appendage typically 60–70 × 16 μ, sometimes smaller. Total length to tip of perithecium, average 180 μ, maximum 225 μ.

On legs, wings, and thorax of *Parydra quadrituberculata* Linn., No. 1804, Fayetteville, Arkansas.

Although this species has no very striking peculiarities, it is well distinguished by its stout form, short subbasal cell, large many celled appendage, and stout perithecium; the well distinguished stout neck-portion strongly curved and ending in the small, abruptly distinguished, button-like apex. It varies very slightly, individuals on the wing being somewhat longer.

## *On Oscinidae.*

### Stigmatomyces constrictus Thaxter.

Syn. *S. Elachipterae* Thaxter.

This species, the original host of which from the papuan region was not determined, proves to be characteristic of various genera of the Oscinidae, and an examination of a very large series of specimens leaves no doubt as to its identity with *S. Elachipterae*, which was found on a species of the oscinid genus *Elachiptera* in New Hampshire. It is one of the most variable of all the species of *Stigmatomyces*, and the peculiarly narrowed base of the subbasal cell, which suggested the specific name, proves to be quite as often wholly lacking, as it is present. The general habit may be short and stout in well developed individuals which measure only 150 μ in length; while long slender forms occur, which may reach a length of 450 μ, although such are not often met with. The type of appendage is always the same, and is very characteristic; but the number of antheridia is subject, as usual, to slight variations. The perithecium, especially its termination, is in general characteristic in form, but, although it is usually quite smooth, it may as in the case of *S. purpureus*, be modified by variably developed tubercular outgrowths, four vertical double rows of which may be developed on the venter, and which may also involve the neck. Such types are more often found on the wings of the host, and have been obtained on species of *Oscinis* from Mexico, Jamaica and Trinidad, W. I. The material examined includes twenty five numbers: from various localities in Jamaica on species of *Oscinis*, *Siphonella* and *Hippelates;* two numbers from Bocas del Toro Panama, on *Siphonella* and *Oscinis;* one number on *Siphonella* from the Grand Etang, Grenada, W. I.; five numbers on Oscinidae from Trinidad, W. I., and eight numbers from Kamerun on Oscinidae, of which one, only, the very beautiful and peculiar *Anatrichus erinaceus* has been kindly determined for me by Professor Aldrich.

*On Drosophilidae.*

### **Stigmatomyces Sigaloessae** nov. sp.

Receptacle subclavate, slightly broader at the septum, the subbasal cell sometimes much longer than the basal, stout and rather abruptly broader and convex distally, the region immediately above it abruptly somewhat narrower. Stalk-cell of the appendage short subtriangular, distally somewhat inflated. Appendage relatively small, somewhat curved, consisting of usually six or sometimes seven cells; the basal larger, distally oblique, somewhat suffused with amber-brown; the rest small, broader than long, each bearing a single antheridium; the series terminated by two; all somewhat irregularly superposed in a vertical series. Stalk-cell and secondary stalk-cell of the perithecium, as well as the basal cells and the stalk-cell of the appendage, not differing greatly in size, and forming a rather short, compact region somewhat suffused with amber-brown, narrower below. Venter of the perithecium relatively large, with broad base, subelliptical, or tapering more distinctly distally, more or less suffused with amber-brown; the wall-cells distinguished by a conspicuous ridge, or wing, which is strongly spiral, making a half turn with two strong curves: the neck and tip very slightly or not at all distinguished from one another, abruptly distinguished from the venter, hardly tapering, hyaline; the apex well distinguished, as long as the tip, the terminations of its cells forming four rounded, well defined prominences symmetrically disposed about a central, terminal, rounded, tongue-like median projection formed by the symmetrically appressed lips. Spores 28 × 4 μ. Perithecia; basal and stalk-cell region 18–20 × 32–35 μ, venter 75–82 × 42–47 μ; neck, tip and apex 66 × 46–54 μ. Appendage 45–50 × 8–9 μ, its stalk-cell 16–18 × 8 μ. Receptacle 72–156 × 23–28 μ. Total length to tip of receptacle 225–310 μ.

On the superior surface of the abdomen of *Sigaloessa* sp. No. 1713, Mandeville, Jamaica, W. I.

This species is well characterized by the prominent spiral elevations on the venter, by its peculiar termination and the arrangement of its antheridia which are superposed in a single series, and arise from relatively very small cells. The appendages in all of the seven specimens examined lie flatwise against the perithecium, so that its free side view is visible in no instance, and it has not been possible to determine whether the uppermost antheridium is spinose.

### **Stigmatomyces Leucophengae** nov. sp.

Short and stout, pale dirty yellowish, the perithecium somewhat darker. Basal cell of the receptacle stout, tapering slightly below, hardly twice as long as broad; the subbasal irregularly five-sided, owing to its oblique separation from the stalk-cell of the appendage, the anterior margin half as long as the posterior, which is intruded below; the distal margin oblique and more or less strongly convex. Stalk-cell of the appendage as long as or longer than the receptacle, the upper cell of which it slightly overlaps, abruptly prominent below the appendage, which is slightly and abruptly constricted at the base, and consists of four superposed subequal cells, each bearing a single rather large antheridium on the inner side; the series ending in an erect terminal antheridium, bearing a large brownish spine externally; the basal cell not differentiated in color from the rest, hardly larger, and bearing no antheridium. Stalk-cell and secondary stalk-cell of the perithecium parallel, of about equal length; the latter somewhat larger, their upper margins horizontal and irregularly continuous: the three basal cells above subequal; the outer irregularly concave externally: venter long, straight, of about the same diameter throughout, wall-cells separated by a more or less distinct furrow distally, and each ending in a broad prominence; the four prominences surrounding the base of the neck; which diverges usually at an angle from the venter, is short, hardly longer than broad, and indistinguishable from the tip, except that it is slightly inflated, and the region between the two is marked by a shallow depression; the tip about as long as the neck, and distally prominent on the inner side, sometimes less so on the outer below the abruptly narrower, short, bilobed, slightly sulcate apex, which is slightly inflated and about as long as broad. Spores 28 × 3.5 μ. Perithecia; stalk-cells and basal region 30 × 20 μ; venter 58–65 × 20–24 μ; neck and tip together 23–27 × 11.5 μ; apex 8 × 8 μ. Appendage 55–60 × 8 μ. Receptacle 26–30 × 24 μ. Total length to tip of perithecium 150–210 × 27 μ.

On the thorax and abdomen of *Leucophenga* sp. No. 1814, Fayetteville, Arkansas.

This species appears to be most nearly allied to *S. Notiphilae*, from which it differs, however, in numerous details. Abundant material has been examined.

*On Streblidae.*

### **Stigmatomyces Streblae** nov. sp.

Slightly curved, hyaline becoming faintly tinged with pale yellow, the base of the appendage becoming slightly brownish. Basal cell of the receptacle slightly curved, slightly broader distally, the base stout, rounded, with a small pointed black foot turned sidewise; subbasal cell slightly oblique and somewhat broader than long, irregularly triangular or four-sided. Stalk-cell of the appendage lying directly above the subbasal cell, smaller and abruptly slightly narrower, of somewhat irregular outline, its pointed external lower angle slightly overlapping the subbasal cell: externally very slightly convex below, the broad insertion occupying its whole distal surface. Axis of the appendage consisting of three large and one or two small, usually sterile cells, and bearing in all nine and ten antheridia; the basal cell tinged with yellowish brown, five-sided, much broader than long, the two distal sides meeting at a very obtuse angle, the outer united to the subbasal cell, the inner bearing a small somewhat flattened cell from which two smaller ones arise; one distal, the second at the right, both of which bear pairs of antheridia, both independent, and one placed somewhat lower than the other: the subbasal cell bearing one such small cell, on which a pair of antheridia are similarly borne: the third cell bearing a pair directly: the fourth cell rarely becoming an antheridium, usually sterile and associated with a second small terminal cell: the appendage divergent, bearing the antheridia on the upper (inner) surface: the necks of the latter thick and gelatinous, becoming more or less completely disorganized, their venters, together with the cells from which they arise, becoming so closely united that they appear as a compact cellular mass. Stalk-cell of the perithecium of somewhat rounded flattened outline, obliquely placed, smaller than the subbasal cell and in contact below with the basal cell of the receptacle; its outer margin short and straight; secondary stalk-cell larger, more rounded, externally strongly convex: basal cells small: venter very slightly inflated, somewhat longer than the distal portion, bearing distally four discrete, rather conspicuous, tubercle-like prominences, which serve abruptly to differentiate the venter from the distal portion of the perithecium; the latter somewhat bent inward, and geniculate at the junction of the tip with the neck; which is rather stout, hardly tapering, the base spreading slightly; the tip clearly distinguished by the abrupt convergence of the outer margin from its junction with the neck: the apex snout-like, small, bent slightly outward, its distal margin flat or slightly rounded, the lips hardly if at all distinguished. Spores 28 × 3.5 μ, the lower segment relatively very short and blunt. Perithecia 95–105 μ; the neck about 35 × 16 μ, the venter 50–56 × 24–28 μ. Appendage 28 × 18 μ. Receptacle 65–80 × 16–20 μ. Total length 170–200 μ.

On the legs and wings of *Strebla vespertilionis* Fabr., collected on bats in Venezuela (Carricker), No. 2073b, M. C. Z.

This species, as well as the following, is rather clearly distinguished from other members of the genus by the characters of the appendage, but I have thought it undesirable to erect a new genus for its reception in view of the fact that such a genus would practically be based on the fact that the antheridia, where they occur in pairs, are independent, and do not appear to arise, as is normally the case in this genus, through the transformation of the cell which bears the primary antheridium to a secondary antheridium, on which the primary one appears to be borne. The same variation in relation sometimes occurs in the genus *Corethromyces*, although here, also, the normal development and association of the antheridial groups is like that of the more highly developed typical species of *Stigmatomyces*. In the present type, as a result of the gelatification of the thick antheridial necks, the mature appendage has an unusual appearance, and would hardly be recognized as of the *Stimatomyces*-type; the cells closely cohering in a compact mass. The general structure of the appendage seems to be identical in both this and the following species, which also occurs on a dipterous parasite of bats; and is closely allied, although it is at once distinguished by its greater size, the presence of a penetrating rhizoid and the conformation of its perithecium.

*On Nycteribiidae.*

### **Stigmatomyces Nycteribiidarum** nov. sp.

Hyaline, becoming pale yellowish. Basal cell of the receptacle large and stout, about three times as long as broad and nearly uniform, developing no foot, and penetrating the host by means of a well developed rhizoid; subbasal cell relatively small, misplaced by the primary stalk-cell of the perithecium, which lies obliquely beside it. Stalk cell of the appendage hardly as long as broad, its basal septum somewhat oblique, its outer margin convex, slightly prominent below the broad insertion. Basal cell of the appendage somewhat brownish, especially below, slightly broader and evenly rounded distally, the margins slightly concave, the structure of the appendage as a whole like that of *S. Streblae.* Subbasal cell of the receptacle and the primary and secondary stalk-cells of the perithecium more or less similar in size, associated obliquely side by side, and forming a somewhat oblique series; the primary stalk-cell partly in contact with the basal cell of the receptacle, the secondary stalk-cell somewhat broader and larger; the region occupied by the two combined externally strongly convex, bulging outward below the concave outer margin of the outer basal cells, which are subequal and smaller: venter stout, straight, erect, but slightly inflated below the middle; the wall-cells having a twist of about one fourth of a turn, forming four corresponding terminal ridges which serve abruptly to distinguish the stout neck; the latter nearly uniform in width, or slightly spreading at the base; the tip abruptly distinguished, bent very slightly outward, slightly tapering to the large, blunt, somewhat asymmetrically rounded extremity. Spores about 25 × 3.5 $\mu$, the shorter basal segment tapering to a slender point. Perithecia 190–215 $\mu$; venter 110–120 × 45–54 $\mu$; neck 70 × 22 $\mu$. Appendage about 50 × 20–24 $\mu$. Basal cell of the receptacle 85–105 × 30–35 $\mu$. Total length to tip of perithecium 330–365 $\mu$.

On the inferior abdomen of a Nycteribid parasitic on *Antibaeus Grenadinus*, M. C. Z., (Brues), No. 2057.

A species closely allied to the preceding, from which it differs in its greater size, the form of its perithecium, and the basal cell of its appendage, in the possession of a penetrating rhizoid, and in minor points.

*On Anthocoridae* (Hemiptera).

**Stigmatomyces Lasiochili** nov. sp.

Very faintly tinged with greenish yellow, the basal cell and appendage, only, somewhat suffused with smoky or purplish brown. Receptacle relatively short and tapering to the pointed foot, the basal cell faintly tinged with purplish brown below, in contact distally with the secondary stalk-cell, and obliquely separated from the subtriangular subbasal cell, which overlaps two thirds to three fourths of its posterior margin. Stalk-cell of the appendage short, subtriangular, slightly longer than its distal width, very slightly prominent below the insertion. Appendage rather slender, its axis consisting of normally three cells, bearing three antheridia; the basal cell faintly brownish, elongate, about as long as the rest of the appendage, including the terminal spinose antheridium, somewhat inflated or nearly cylindrical, bearing no antheridium; the rest of the appendage, including the antheridia, more deeply colored, and separated by a horizontal dark septum; the second and third cells subequal, obliquely separated, externally slightly convex, each bearing a single antheridium distally: antheridia nearly free, the venter relatively small, the necks long, stout, slightly divergent, hardly curved, directed inward or sidewise. Stalk-cell of the perithecium similar to that of the appendage, or but slightly larger, lying beside and parallel to it, and extending to the insertion; secondary stalk-cell nearly twice as large as the primary, externally concave, lying beside it, and extending higher up: the basal cells about as large as the primary stalk-cell, and but slightly, if at all, enveloping the base of the ascigerous cavity. Perithecia, relatively large; the venter becoming very slightly inflated, two or three times as long as the short neck, which is but slightly narrower, its margins somewhat concave, subtended by four rather distinct terminal elevations of the wall cells: the tip broad, its margins concave below, more clearly distinguished than the venter, and subtended by more pronounced elevations; the apex minaret-shaped, subtended by four blunt short divergent outgrowths; its bluntly pointed apex apparently formed by the four similar, closely appressed lip-cell terminations. Spores 40–45 × 3.5 $\mu$. Perithecia 135–160 × 30–35 $\mu$.

Appendage 50–55 μ. Receptacle including foot 30–35 × 16 μ. Total length to tip of perithecium 175–210 μ.

On various parts, especially the legs, of *Lasiochilus pallidus* Reut., a small bug belonging to the Anthocoridae. No. 2771, Grand Etang, Grenada.

The host of this species, which was kindly determined for me by Mr. Van Duzee, is the first hemipterous host reported for this genus. The species is allied to *S. virescens* and *S. Coccinelloides*, but differs widely from either of them. Abundant material was obtained at the Grand Etang where the host is very common.

*On Coccinellidae* (Coleoptera).

### **Stigmatomyces Coccinelloides** nov. sp.

Hyaline or faintly yellowish. Basal cell of the receptacle straight, more than twice as long as broad, the subbasal very small, externally convex, twice as broad as long, obliquely separated from the basal, the narrower convex distal end of which is in contact with the stalk-cell of the appendage; which is about as broad as long, and in equal contact with the basal and subbasal cells, and with the stalk-cell of the perithecium; not at all prominent below the insertion of the appendage: which consists of three or four successively slightly shorter axis-cells of about equal width; the third, or all above the basal cell, bearing usually single antheridia, two to four or five in all, one of them terminal and spinose on its inner side; all the antheridia free, or nearly so, their long slender necks slightly curved outward. Stalk-cell of the perithecium short, somewhat shorter than the secondary stalk-cell, the two superposed and forming a short, constricted stalk to the perithecium; the basal cells forming the walls of the lower portion of the venter: venter, neck and tip not at all distinguished, and forming a rather long stout body, its inner margin nearly straight, its outer somewhat convex, the junctions of its wall-cells hardly indicated; the apex abruptly narrower, short, relatively broad and somewhat spreading when viewed anteriorly or posteriorly; the lip-cells growing out into tooth-like projections, the lateral ones shorter, similar, symmetrical, somewhat spreading, subtended on the inner side by a vesicular outgrowth; the outer and inner closely appressed, and forming together a somewhat longer tooth-like projection, median in position, and bent slightly inward, the outer member somewhat broader than the inner and pointed, the inner narrower distally, but truncate, the apex thus having a very different appearance from different points of view. Spores about 45 × 3.5 $\mu$. Perithecia 85–110 × 24–30 $\mu$, its apex 22 × 20–18 × 25 $\mu$. Appendage, to tip of terminal antheridium, 35–40 $\mu$, the antheridia about 18 $\mu$. Receptacle 20–25 × 10–12 $\mu$. Total length to tip of perithecium 120–140 $\mu$.

On the elytra of minute species of Coccinellidae: No. 2560 (Type) and 2059, Grenada (Brues); Nos. 1706, Mandeville, 1736 and 1752 Balaclava, Jamaica, W. I.; No. 2384, Mindanao, P. I.; No. 2175, Sarawak, Borneo.

The material of this species from Borneo and the Philippines varies somewhat from the West Indian type, the terminal cell of the appendage usually bearing two paired antheridia, while each cell above the basal may bear a single one. The projections formed by the lip-cells in this material are also relatively shorter and more blunt, but are otherwise entirely similar. In the type, and in No. 1706, all the appendages bear only two antheridia, but in the other West Indian specimens a third or fourth also arises from the subterminal cell, and the cell below it, in almost all the individuals examined. The species seems to be a rare one, and is not often found in a fully matured condition.

*Stigmatomyces virescens*, to which this species is most nearly allied, seems also to be widely distributed, and has been obtained from Brazil, Hayti, Jamaica and Sarawak, Borneo; individuals from the latter locality agreeing in all respects with the type.

## **Ilytheomyces** nov. gen.

Receptacle compact, two celled, the foot and basal cell usually indistinguishable; the subbasal cell bearing the stalk-cell of the perithecium terminally and the appendage laterally. Appendage consisting of an axis of superposed cells indeterminate in number, the subbasal cell cutting off a small androphorous cell distally on the inner side, which produces typically two large, simple, colored, paired antheridia, one of which may be lacking or may be replaced by, or associated with, a sterile branchlet: the terminal cells of the axis, all those, as a rule, above the subbasal cell, giving rise to variously complicated and modified branches, usually from both the inner and outer sides. Perithecium normal, the basal and stalk-cell region well developed, the apex more often bearing a variably developed 'trigger-appendage.'

More than a dozen forms of this type are known to me from the Eastern as well as the Western Hemisphere, all of which are parasitic on species of the dipterous genus *Ilythea*. The type appears to be so constant in the not inconsiderable number of forms already known, that I have concluded to separate it generically from any of the species of *Corethromyces*, which is its nearest ally, although some of the members of this genus may bear free single antheridia. In the present instance the antheridia arise in usually divergent pairs from a small special cell, androphorous cell, separated distally on the inner side, from the subbasal cell of the appendage. This condition which, with slight modifications, is characteristic of all the species, is the principal basis for this separation, but it is well to bear in mind that it may prove an insufficient one, when more species have been accumulated. It differs from the nearly allied *Rhizomyces*, a genus also characteristic of dipterous hosts, in the general type of its appendage, the position of its branches, and especially of the antheridia, both being external in relation to the axis of the appendage.

The identity of the axis-cells of the appendage in *Ilytheomyces*, especially of the terminal ones, is generally lost; owing to the deep suffusion of this region, as well as to the fact that the distal ones, as they are formed, appear to undergo a threefold proliferation; the distal proliferation forming the next axis-cell above; or, if it is the last of these cells, forming a terminal branch; while the other two proliferations form the superior and inferior branches, respectively. The terminal axis-cell is thus little more than the point of union of three branches, one terminal, one superior and one inferior. The blackened base of the appendage in all the species, appears to be in contact with three cells, the basal and subbasal cells of the receptacle and the stalk-cell of the perithecium. The ascogenic cell is solitary, where the number has been determined, and the four regions of the perithecium proper are usually not definitely differentiated, with the exception of the apex, which may be very clearly distinguished.

In a majority of the species the basal cell is very small, and so combined with the foot, that it is more or less indistinguishable from it, except for its hyaline upper margin.

Among the species herewith enumerated, *I. elegans* has the most highly developed appendage and is taken as the type. *I. anomalus* on the other hand, is the simplest, and is peculiar for several reasons, notably in bearing a single antheridium, only, from its androphorous cell. Many of the species are furnished with 'trigger-organs,' which arise as appendages from the apex of the perithecium, and evidently function, like those of *Ceratomyces* and other genera, as a means of effecting a sudden and copious discharge of spores when they come in contact with another host.

### **Ilytheomyces elegans** nov. sp.

Basal cell of the receptacle minute, almost wholly hyaline, and distinct from the foot, bulging outward below the insertion of the appendage: subbasal cell partly hyaline more or less deeply tinged with blackish brown on the side next the host, this suffusion associated with a more or less definite blackened protrusion or buffer; about twice as large as the basal cell, its lower half lying beside the latter, its upper half in contact, on its inner side, with the base of the appendage. Basal and subbasal cells of the somewhat divergent appendage opaque and indistinguishable, the androphorous cell small, flattened, hyaline, so obliquely separated from the subbasal cell as to occupy almost its whole inner margin; producing a large sessile antheridium on the right side, wholly dark brown, except the apex of the slightly outcurved neck, and on the left side a similar antheridium which may or may not be associated with, or replaced by, an erect variably developed sterile branch: the remaining cells of the nearly straight, or but slightly curved, axis usually five or six in number, seldom more: each giving rise externally to a peculiar branchlet, its base black, opaque, tooth-like, curved outward and bearing distally from the upper convex surface a perfectly hyaline vesicular, variably developed terminal portion; and also producing on the inner (upper) side single branches, more or less appressed against the axis, and branching; the lower branches more highly developed, the lowest sometimes repeating on a smaller scale the branching of the axis as a whole; the branches brown, the branchlets distally or wholly hyaline. Stalk-cell of the perithecium black-brown, becoming opaque, narrower below: hardly longer than broad, usually slightly pointed distally; the cells above hyaline, the region usually distinctly broader than the base of the venter, and lying wholly above the primary stalk-cell; the inner basal cell long and larger than the outer which protrudes more or less distinctly above the smaller secondary stalk-cell. Perithecium rather long and slender, slightly inflated near the middle, or tapering slightly almost from its base; distally somewhat curved and tapering slightly; the wall-cell regions not distinguished, the lips hyaline, slightly vesicular; the outer wall-cell of the apex developing, at the right, a slightly divergent, strongly outcurved, slender, elongate appendage, brown with hyaline tip, the base darker, partly concealing the blunt apex of the perithecium, marked by a depression nearly opposite the pore on the inner side, which may be subtended by a more or less well defined tooth-like projection. Spores about 25 × 2 $\mu$. Perithecia 52–75 × 16–18 $\mu$; its appendage 75–140 × 4–5 $\mu$. Appendage 50–80 $\mu$. Receptacle 10–14 × 12–14 $\mu$. Total length to tip of perithecium 90–145 $\mu$.

On the inferior surface of the abdomen of the right side of species of *Ilythea*. No. 2043 (Type) and No. 1856, Clarkstown and Balaclava, Jamaica, W. I. No. 2524, St. George, Grenada (Brues), No. 2515, Bocas del Toro, Panama (Rorer).

This striking species which may be regarded as the type of the genus, since it illustrates its highest development, is clearly distinguished from all other American forms which are known to me, by the row of characteristic outer branchlets from the axis of the appendage. Individuals from different sources show some variation. Nos. 2524 and 1856 differ from the others in that the appendage is usually shorter with fewer cells; the perithecium and its base decidedly smaller, while the appendage, or trigger-organ, is much longer, and lacks the tooth-like projection which is always present at the base in the shorter type. These differences, however, do not appear sufficient for even varietal separation.

### **Ilytheomyces manubriolatus** nov. sp.

Basal cell of the receptacle small, indistinguishable from the foot which thus appears relatively larger, its upper edge, only, hyaline; the subbasal cell broader than long, its outer margin strongly convex, lying above the basal cell, and beside the base of the appendage; which is also in contact with the basal cell of both the receptacle and the appendage. Axis of the latter blackish brown, distally curved or recurved, the basal and subbasal cells similar and distinguished by a slight indentation of the inner margin: the upper and lower margins of the subbasal cell free; the well defined androphorous cell arising from it distally, on the inner side, subhyaline, and bearing two large, stout, brown, nearly straight, hyaline-tipped antheridia side by side; one of the latter on the left being sometimes replaced by a short, simple, sterile branch: the two or three cells of the axis above it producing stout, upcurved, smoky brown branches above and below: the divergent bases of the lower deeply suffused externally, bent abruptly outward, the paler stout distal portion curved abruptly upward; those from the upper (inner) side stout, curved slightly inward, the two lower or the lowest only, once branched above the basal cell. Stalk-cell of the perithecium broader than long, brown, narrower than the hyaline region above; the secondary stalk-cell bearing a variously developed, straight, tooth-like, hyaline outgrowth which projects, usually somewhat obliquely, from the left side; the basal cells more or less similar, the region relatively short and compact, narrower than the base of the venter: the perithecium dull brown, darker along the inner side, of nearly the same diameter throughout, or slightly inflated below, somewhat narrower beneath the relatively very broad, nearly flat extremity, which may appear narrow if viewed edgewise; the inner lip-cell darker externally, and when viewed sidewise forming distally a slight rounded prominence; the outer lip-cell developing a long, nearly uniform, dark brown appendage, the short opaque base of which projects at right angles, thence curving in a bow abruptly upward and outward and slightly downward. Spores about 22 × 2 μ. Perithecia 40–50 × 13–16 μ, the apex × 12 μ; the appendage 40–60 × 3 μ; the projection for the secondary stalk-cell 8–14 μ. Appendage about 30 μ, its longest branches 45 × 5 μ. Receptacle and foot 15 μ. Total length to tip of perithecium 75–85 μ.

On species of *Ilythea*, occupying the upper surface of the inner angle of the left wing. No. 2064, (Type), St. George, Grenada; No. 1856, Balaclava, Jamaica; and also Port of Spain, Trinidad, B. W. I. No. 2514, Bocas del Toro, Panama, (Rorer).

This species is well distinguished by its perithecium, the broad flat apex of which bears an evenly curved trigger-appendage, which is as long or longer than the perithecium itself. It is closely allied to the following species, from which it is distinguished by the form of its perithecium, and the origin of the tooth-like outgrowth which, in the present instance, arises from the secondary stalk-cell. The perithecium is often somewhat twisted, so that it may be viewed in a preparation, either wholly or partly edgewise. Under these conditions the appearance of the tip necessarily varies greatly. As a rule, however, the lateral view, as above described, is the one which is seen.

**Ilytheomyces Panamensis** nov. sp.

Basal cell of the receptacle not distinguishable from the foot, subbasal cell small, hyaline, developing a relatively large hyaline outgrowth equaling it in diameter, projecting obliquely outward from its left side, of variable form and length and of somewhat irregular outline, slightly tapering, or usually very blunt. Appendage strongly divergent, curved outward, its axis blackened; its base in contact with the basal and subbasal cells of the receptacle, and with the stalk-cell of the perithecium: its basal cell slightly larger than the subbasal; the two somewhat obliquely separated, distinguished by a slight indentation on either side: the androphorous cell small, hyaline, and clearly distinguished, obliquely separated from the subbasal cell, and occupying half its inner margin; bearing a pair of large, brown, stout, nearly straight antheridia; the rest of the axis comprising two to three cells, which produce branches above and below; the latter pale brown, stout, or even vesicular, curved upward, their basal lower margins nearly opaque; the upper branches much more slender, tapering, and hyaline or but slightly suffused at the base, the lowest one branched above the basal cell. Stalk-cell of the perithecium relatively small and irregular, its upper half intruded between two of the cells above it; which are thick walled, somewhat vertically elongated, subsimilar; the base of the secondary stalk-cell prominent externally; the margin of the outer basal cell convex; the whole region hyaline, somewhat irregular, and slightly broader than the base of the venter: body of the perithecium pale brown, somewhat darker on the inner side, the regions not distinguished, the wall-cells with a more or less distinct twist of perhaps one quarter of a turn; its venter in general very slightly inflated, but conspicuously so on the right side; its distal half but slightly narrower, the inner margin curving abruptly to the nearly horizontal hyaline lip-region, which is partly hidden by the base of a well developed trigger-appendage, the broad black angular base of which, subtended by a more or less perceptible elevation, contrasts strongly with the paler cells about it, extending upward with but slight divergence, thence bending slightly inward for a short distance, and thence abruptly outward almost at right angles; this much longer portion of the appendage paler brown above, darker below, subhorizontal in position, its outline often irregular or wavy, its tip often slightly bent upward. Spores 28 × 2.5 $\mu$. Perithecia 36–40 × 14 $\mu$, basal cell region 12–13.5 × 12 $\mu$, its appendage variable, the longer 85 $\mu$, including both vertical (18 $\mu$) and horizontal (67 $\mu$) portions. Axis of the appendage about 20 $\mu$, the longest branches 28 $\mu$; the antheridia 12 × 3.5 $\mu$. Protrusion from subbasal cell of the receptacle 18 × 7 $\mu$. Total length to tip of perithecium 58–68 $\mu$.

On the upper posterior surface of the left wing of *Ilythea* sp., near the base. No. 2514, Bocas del Toro, Panama.

Although this species is closely allied to *I. manubriolatus*, it is at once distinguished by the outgrowth from the subbasal cell of the receptacle which is similar to that which arises from the secondary stalk-cell in the last mentioned species. The form of the perithecium and its appendage is also very different. Fifteen adult individuals have been examined which were obtained from a host kindly collected for me by Mrs. J. B. Rorer.

### **Ilytheomyces miniscutus** nov. sp.

Erect, straight above the subbasal cell, relatively slender; subbasal cell hyaline, longer than broad, prominent externally. Axis of the appendage blackened, the basal and subbasal cells indistinguishable, the androphorous cell minute, obliquely separated, the paired antheridia relatively pale, somewhat appressed, the remaining cells indistinguishable, bearing two or three inferior blackened distally recurved and few subhyaline superior branches, which are relatively slender and tapering. Stalk-cell of the perithecium about as large as the subbasal cell below it, deeply suffused with dark reddish brown, the hyaline region above broader, tapering slightly downward, all the cells subsimilar, longer than broad and externally convex; the perithecium dark reddish brown, straight, nearly symmetrical, the inner margin more convex, tapering slightly, its termination bluntly rounded, the lips merely indicated by a slight irregularity, the left and the posterior lip-cells combined to form a free, very slightly upcurved, bluntly tipped prolongation, diverging at an angle of about 45°, the left half pale brown, the posterior hyaline; the lip-cells misplaced by a quarter turn of the whole series of wall-cells, as a result of which the left and the posterior lip-cells both become anterior; while the right and anterior occupy the posterior (inner) side. Spores about 22 × 2 $\mu$. Perithecia 48-50 × 15 $\mu$; the stalk-cell 9 × 7.5 $\mu$; the hyaline region above it 18 × 14 $\mu$; its terminal projection 15 × 5 $\mu$. Axis of the appendage 30 $\mu$, its longest branches 35 $\mu$, the antheridia 12 $\mu$. Total length to tip of perithecium 70–90 $\mu$.

Near the base of the right wing of *Ilythea* sp., No. 2043, Clarkstown, Jamaica, W. I.

A species clearly distinguished from the related *I. calycinus* and other allied forms by the divergent projection from the apex of its perithecium, formed by a combination of two misplaced lip-cells. Sixteen specimens have been examined which show no essential variations.

### **Ilytheomyces lingulatus** nov. sp.

More or less strongly curved throughout, the basal cell of the receptacle very minute, only the hyaline edge, below the insertion of the appendage, distinguished from the foot; subbasal cell strongly convex outward. Axis of the appendage black, strongly curved outward, its basal cell somewhat narrower than the subbasal; the latter hardly larger than the somewhat obliquely separated androphorous cell which bears a pair of slightly divergent, brown, relatively large, stout antheridia: the rest of the axis comprising not more than two cells, the lower bearing no branch on the outer side and a stout several times divided faintly brownish branch on the inner side, which constitutes the bulk of the appendage, its stout branchlets curved outward; the terminal cell ending in a curved prolongation and bearing a pale simple branchlet from its upper side. Stalk-cell of the perithecium relatively small, hyaline, becoming brownish below, partly overlapped by the considerably larger, hyaline, secondary stalk-cell, which bulges outward above it, its outer margin convex, as is that of the somewhat smaller outer basal cell which is tinged with brown; the inner basal cell vertically elongated, narrow, hyaline; body of the perithecium brown or blackish brown; the venter convex externally, the inner margin nearly straight; the region of the tip and apex broad, slightly flaring, clearly defined below by a slight depression, the margins broadly suffused with deeper color; the outer convex, and ending in a slight rounded elevation, from which the distal margin turns abruptly inward at right angles; the inner lip-cell opaque, and prolonged to form a conspicuous, slightly divergent, tongue-like appendage, somewhat recurved at its tip, which is edged above and within by a perfectly hyaline prolongation of the left lip-cell. Spores 35 × 3 $\mu$. Perithecia 35 × 14–17 $\mu$; its tongue-like prolongation 16–18 × 5 $\mu$, the free portion 9–10 $\mu$. Black axis of the appendage 25–30 $\mu$, its longest branches 35 × 3.5 $\mu$; antheridia 18 $\mu$. Total length to tip of perithecium 55–65 $\mu$.

Near the base of the right wing of *Ilythea* sp., on the upper side. No. 2064 (Type), St. George, Grenada; No. 1723, Mandeville, Jamaica, and Port of Spain, Trinidad, B. W. I. No. 2514, Bocas del Toro, Panama, (Rorer).

### **Ilytheomyces major** nov. sp.

Relatively long and slender; general habit straight, with local curvature of the receptacle and in the region of the perithecial stalk-cell. Basal cell of the receptacle combined with the foot, but its hyaline distal wall and lumen clearly visible below the insertion of the appendage; the subbasal cell considerably enlarged, curved, its outer margin strongly convex, bulging below. Axis of the appendage diverging at an angle of from 45° to 50°; black, the outer margin even below, the inner margins of the basal and subbasal cells hyaline and separated by a deep indentation; the androphorous cell hyaline, much smaller than the subbasal cell; which bulges somewhat below it, its base almost horizontal, bearing a pair of slightly divergent, stout, brown antheridia; the rest of the axis comprising perhaps two or three more or less indistinguishable cells, from which stout faintly brownish branches are developed above and below; the base of the lowest outer branch deeply blackened and somewhat recurved; the upper (inner) branches stout, curved outward and downward. Stalk-cell of the perithecium hyaline, becoming brown below, deeper externally, usually bent sidewise at its contact with the hyaline cells above; the secondary stalk-cell longer than broad, extending down to form a rounded protrusion beside the stalk-cell and forming, together with the long inner basal cell, a relatively narrow region above the stalk-cell, which is abruptly broadened by the hump-like protrusion of the outer margin of the hyaline outer basal cell: the straight erect, brown venter, long narrow and slightly, almost symmetrically, inflated, tapering slightly distally; the brown tip- and apex-region, which is hardly distinguished, straight, stout and of nearly even diameter below, but distally passing to a thick tongue-like incurved termination, its convex margin deeply colored, and ending in a very slight rounded hemispherical elevation; while its upper convex, thick, perfectly hyaline margin is continuous with the outer margin of the perithecium, the transition from the suffused to the hyaline area marked by a hardly perceptible depression. Spores 45 × 3 $\mu$. Perithecia 58–68 × 18 $\mu$; the tongue-like termination 10 × 8 $\mu$; the basal cell region 25 × 10 $\mu$ below, × 14 $\mu$ above. Appendage; axis about

35 μ; longest branches 55 × 5–6 μ; antheridia about 12 × 4.5 μ. Total length to tip of perithecium 90–110 μ.

On the upper surface of the right wing of *Ilythea* sp. near the base. No. 2513, Bocas del Toro, Panama.

This species was obtained from a host for which I am indebted to the kindness of Mrs. J. B. Rorer. It is closely allied to *I. lingulatus*, but differs in its much greater size, straight habit and the conformation of its perithecium, as well as in minor points. A comparison of the fifteen adult specimens which have been examined with the abundant material available of *I. lingulatus*, shows that the two do not tend to vary toward intermediate forms.

### **Ilytheomyces calycinus** nov. sp.

General axis irregularly curved inward, somewhat zigzag below the venter. Subbasal cell relatively large, curved, or strongly bulging outward. Appendage variably divergent, its axis blackened, the basal and subbasal cells hyaline along the inner margin, distinguished by a slight indentation; the small subhyaline androphorous cell bearing a pair of rather strongly divergent, wholly brown, relatively slender, evenly tapering, distally outcurved antheridia: the axis above comprising two or three indistinguishable cells, the branchlets relatively short and scanty; the two lower blackened externally below, paler distally above; the upper branches short, simple, not well developed, somewhat brownish. Stalk-cell of the perithecium narrow, slightly longer than broad, smoky brown below; the cells above hyaline, relatively small and subequal, with convex margins; the perithecium dark blackish brown, the lower wall-cells slightly prominent, the inner margin nearly straight, the outer convex; the apex rather abruptly distinguished, slightly spreading, its four lip-cells dark brown with hyaline terminations, the brown portions externally convex, the convexities separated from one another by paler furrows, the whole suggesting the slightly spreading lobes of a calyx; the right lateral lip forming distally a hyaline flat-topped projection, which bulges outward and inward, the outer lobe broader, the whole almost as broad as the entire distal margin of the apex which it surmounts; the left lip-cell also forming distally a narrow, slightly tapering, bluntly tipped prolongation directed obliquely inward, and projecting but slightly beyond the inner lobe of the right lip-cell. Spores 25 × 2.5 μ. Perithecia 60–68 × 17–20 μ; the apex 15–17 × 15–17 μ; stalk- and basal cell region 18 × 12 μ. Subbasal cell of the receptacle 12 μ. Axis of the appendage 25–35 μ, the longest branch 20 μ; the antheridia 14 μ. Total length to tip of perithecium 90–110 μ.

On the upper surface of the left wing near the base of *Ilythea* sp. No. 1929 (Type) Battersea, and No. 2043d, Clarkstown, Jamaica, W. I.

This minute species is clearly distinguished by the peculiar development of its apex; the suffused portion of which, when viewed sidewise, has the appearance of a slightly opened four-lobed calyx. The four lower wall-cells, forming the venter proper, are relatively short and distally more distinctly prominent than in other related species. It is most nearly allied to *I. obtusus,* which also lacks anything in the nature of a trigger-organ.

### **Ilytheomyces obtusus** nov. sp.

General axis erect, with four slight successive curvatures. Subbasal cell relatively small, hardly broader than long, its outer margin but slightly convex. Axis of the appendage blackish, except along its inner margin; the basal cell somewhat longer and narrower than the subbasal; the androphorous cell relatively small, hyaline, its base but slightly oblique; bearing a pair of relatively short, brown, slightly out-curved and divergent antheridia; the third cell commonly producing only the usual inferior branch, the other two both inferior and superior branches, the upper bearing one or more slightly tapering branchlets. Stalk-cell of the perithecium about as large as the subbasal cell below it, hardly longer than broad, slightly intruded above, somewhat larger than the small subsimilar hyaline cells above it, the secondary stalk-cell producing a tooth-like, straight, slightly curved, or even recurved, blunt, somewhat tapering outgrowth, which arising externally, or usually on the left side, projects obliquely forward: the perithecium wholly dark brown, straight, the axis bent outward, the margins asymmetrically convex, the apex wholly brown with a lateral vertically elongated lighter area, bent inward, clearly distinguished, broad, short, distally asymmetrically rounded; the lips forming slight irregularities in the outline. Spores 30 × 2.5 μ. Perithecia 48 × 18–20 μ. Axis of appendage 30 μ, longest branch 35 μ. Subbasal cell 6–7 μ. Total length to tip of perithecium 80 μ.

Near base of left wing of *Ilythea* sp., No. 2043e, Clarkstown, Jamaica, W. I.

This species is well distinguished by the brown, rounded, unmodified apex of its perithecium, and by the peculiar outgrowth from its secondary stalk-cell, which resembles that which occurs in the same position in *I. manubriolatus*, to which it is probably most nearly related; although its apex is very differently shaped, and lacks the well developed trigger-organ of the latter species. It appears to be rare, only a dozen individuals having been examined, from a single host.

### **Ilytheomyces anomalus** nov. sp.

Straight, erect. Basal cell of the receptacle distally hyaline and clearly visible, subbasal cell long-triangular, pointed below, hyaline. Axis of the appendage stout, opaque below, dark brown and but slightly bent distally, the basal and subbasal cells large, opaque, distinguished by slight indentations on both sides, the androphorous cell minute, squarish, pale brownish, bearing terminally a single larger straight stout antheridium, the venter mostly hyaline, the neck purplish, except the hyaline tip; the remainder of the axis consisting of about five obliquely separated cells, the two lower much larger; the second bearing externally a blunt, spur-like projection resembling an abortive antheridium; the rest producing, from the upper side only, short, single, stout, more or less abortive simple branches. Stalk-cell of the perithecium hyaline, very large, distally prominent below the almost opaque small inner basal cell, and separated obliquely from the secondary stalk-cell, which is also obliquely separated from the small, strongly protruding, more or less suffused outer basal cell: the perithecium becoming blackish brown, the upper limits of the lower wall-cells indicated by variably conspicuous indentations, stout and nearly symmetrical below the somewhat tapering tip- and apex-region; which may become rather clearly distinguished, and is more or less distinctly curved outward to the rather broad rounded or somewhat flattened termination; the subhyaline lip-edges indistinguishable, or but slightly prominent. Spores 28 × 2.8 $\mu$. Perithecia 50 × 16–62 × 24 $\mu$: stalk-cell 20 – 50 $\mu$; the stalk- and basal cell region 32–60 × 13–17 $\mu$. Receptacle including foot 28 $\mu$. Axis of appendage 50 $\mu$; antheridia 16 $\mu$. Total length to tip of perithecium 100–140 $\mu$.

On the posterior right leg of *Ilythea* sp. No. 2043b, Clarkstown, Jamaica, W. I.

This species departs rather strikingly from the normal type of the genus and is distinguished by its straight receptacle, greatly developed perithecial stalk-cell, stout appendage in which a spur from the fourth cell replaces all the lower branches; while the upper are more or less abortive; in its solitary antheridium and entirely normal perithecium. A small number of very mature individuals which have been examined, are much larger than the others, darker; the protrusions of the cells below the perithecium, as well as the differentiation of its wall-cells, being much more conspicuous. One individual examined is associated with a simple functional male individual consisting of two cells, terminated by an antheridium. Other individuals, however, removed and still adherent, show that both spores of a pair may develop normally.

### **Laboulbenia Sapromyzae** nov. sp.

Straight and rather slender, the perithecium and outer appendage divergent; cells III and IV replaced by a single cell. Basal and subbasal cell hyaline or slightly soiled, the basal usually slightly longer, the subbasal abruptly somewhat broader; cells III–IV soiled with dirty olivaceous brown, inconspicuously striate-punctate, abruptly prominent below the insertion-cell; cell V relatively long and narrow, its inner margin distally free, so that the thick olive-black insertion-cell is quite free; the basal cells of the outer and inner appendages free, divergent; the latter obliquely terminal, small, pale, usually bearing a pair of short, olive-brown antheridial branchlets right and left; the antheridia single, large, olivaceous: the outer appendage terminal, consisting of usually three cells deeply suffused with olive brown, usually terminated by a pair of branchlets; the two lower cells each producing distally on the inner side a branch, the lower sometimes twice branched, but usually two celled with a pair of terminal branchlets; while the upper bears a similar pair, or only one, directly from its basal cell; the branchlets, of which there may thus be seven or less, rather stout, nearly uniform, blunt distally hyaline or paler, lying in a radial plane, some of them usually characteristically curled or curved outward, distally. Perithecium dark translucent olive brown, finely granular, but slightly asymmetrical, narrow, very slightly inflated, tapering evenly from below the middle to the broad blunt apex, the upper margin of which usually presents a small median rounded elevation; the wall-cells describing a quarter of a turn from right to left, so that the anterior or posterior side is normally presented, the lateral view, which is not often seen, being of the more normal type; the broad termination of the almost opaque posterior lip-cell contrasting with, and extending above and over the pale, externally slightly convex anterior lip-cell: the basal cell region concolorous, small, externally somewhat prominent. Spores 50 × 4 μ. Perithecia 85–105 × 18–22 μ. Receptacle 70–105 × 18–20 μ. Appendage, longer branches, 90–100 μ. Total length to tip of perithecium 175–225 μ.

On the wings of *Sapromyza triseriata* Coq. and on *Sapromyza* sp., Nos. 1630 and 1631, Los Amates, Guatemala, (Kellerman).

This species belongs to the section formerly separated as *Ceraiomyces*, with which it agrees except for the presence of cell V; the receptacle thus corresponding to Spegazzini's '*Laboulbeniella*'. Abundant material has been examined which, apart from slight differences in the branches of the appendages, shows no important variation. The tip, however, varies in appearance very greatly, when, as sometimes happens, it is viewed laterally in the position normal to the genus.

### **Laboulbenia muscariae** nov. sp.

Habit slender, the perithecium and outer appendage approximated; usually straight, except that the perithecium is evenly and characteristically curved inward from its stalk-cell to its apex. Receptacle becoming faintly suffused, the basal and subbasal cells of nearly the same diameter throughout, or slightly broader at the septa, becoming faintly punctate: cell III–IV more distinctly punctate-striate, conspicuously rounded outward below the insertion; cell V about half as long, narrow, clearly defined. Insertion-cell free, opaque, continuous with the outcurved concolorous axis of the outer appendage, which arises from it terminally; the basal cell of the inner appendage arising laterally from its inner side, very small, hyaline, producing a usually two-celled brownish antheridial branchlet on either side, terminated by a pair of brownish antheridia in close contact, and lying obliquely across the venter of the perithecium on either side: the axis of the outer appendage consisting of usually three cells ending in a pair of branchlets; the two lower cells each bearing a branch distally on the inner side, the lowest blackened externally, and bearing branchlets, sometimes three side by side, of the second or even third order; the branchlets distally pale, or hyaline, stout, blunt, extending somewhat above the apex of the perithecium. Perithecium dark olive, the lower wall-cells at first distinctly paler; the region of the subbasal wall-cells sometimes abruptly darker and very slightly inflated; the perithecium tapering thence to the symmetrically rounded, long, narrow, finger-like termination of the apex; which is abruptly distinguished by a large nearly black, externally slightly convex area, which subtends it on the inner side, and by an umbonate black elevation, which subtends it externally and contrasts strongly with a pale area immediately below it. Spores 52 × 4 μ. Perithecia 90–110 × 16–20 μ. Receptacle 88–15 × 18 μ. Appendage to tips of longest branches 105 μ. Total length to tip of perithecium 200–230 μ.

Near the base of the left wings of *Sapromyza muscaria* Lev. No. 1629, Los Amates, Guatemala (Kellerman).

Although this species is nearly allied to *L. Sapromyzae,* it is at once distinguished by the finger-like apex of its characteristically curved perithecium. Like the last mentioned species, it belongs to the section of the genus formerly distinguished as *Ceraiomyces,* its receptacle also corresponding to the type of Spegazzini's '*Laboubeniella*'. The infested host was found among a small lot of flies collected for me by the late Professor Kellerman.

### **Laboulbenia crispata** nov. sp.

Slender, straight, or somewhat curved. Basal cell of the receptacle nearly hyaline, slender, subclavate, obliquely and asymmetrically adjusted to the much shorter olivaceous subbasal cell, which is distally very obliquely related to the stalk cell of the perithecium (cell VI) and less so to cell III–IV, which is similar or slightly larger, externally convex, bulging but slightly below the broad insertion, and more than twice as long as the narrow, but clearly defined cell V, both cells pale olivaceous. Insertion-cell broad, flat, almost opaque: basal cell of the inner appendage dark brown, small, bearing branches right and left; their basal cells brown, the rest hyaline with dark septa: axis of the outer appendage opaque, bearing usually three branches externally and one terminally, each of the latter bearing a pair of branchlets terminally from their basal cells; the branchlets lying in a radial plane, deep brown, nearly uniform throughout, very long, slender, subparallel, curved inward across or over the perithecium, extending free far beyond it, their extremities, especially those of the outer, strongly recurved inward. Perithecium somewhat darker olivaceous, usually strongly curved across the base of the appendage, somewhat, inflated, the limits of the wall-cells, which describe somewhat more than a quarter of a turn, indicated by fine, more or less distinct dark lines of separation; the body somewhat inflated, tapering distally to the broad, slightly angular, or oblique termination; the rather coarse lip-edges hyaline, or olivaceous, irregularly prominent; the anterior lip-cell developing a slightly curved, blunt-tipped, erect free projection of nearly uniform diameter; which, owing to the twisted wall-cells, appears to be lateral (left). Spores approximately 25 × 3 μ (measured in perithecium). Perithecium 85 × 22 μ; its terminal projection 10–12 × 3.5–4 μ. Receptacle to insertion-cell, 120 μ; basal and subbasal cells 62 × 12 μ; cell III–IV 35 × 16 μ. Appendage about 225 μ. Total length to tip of apical projection about 175 μ.

On the abdomen of *Hippelates* sp., No. 2516, Bocas del Toro, Panama.

I am indebted for the host on which this very distinct and graceful species was found, to the kindness of Mrs. J. B. Rorer. Only four specimens have been examined, in which antheridia are no longer present, or are not visible on the partly concealed appendage. The insertion-cell is similar in all respects to that of normal types; but, as in the two preceding species, the receptacle is of the '*Laboulbeniella* type', cells III and IV being replaced by a single cell.

### **Laboulbenia anguifera** nov. sp.

Long and slender, straight, or more or less curved; the subhyaline basal and subbasal cells contrasting abruptly with the uniformly dark olive brown portions above. Basal cell nearly uniform throughout, one third to one fourth as long as the subbasal cell; which is broader and nearly uniform, except for an abrupt, bulb-like, subsymmetrical enlargement at its base, which may be twice the diameter of the basal cell; somewhat obliquely separated from cell VI; which is relatively small, as well as from the relatively small cell which corresponds to cells III–V, and is externally thick-walled and slightly convex. Insertion-cell broad thick and opaque, the minute hyaline basal cell of the inner appendage obliquely separated from it, and bearing a branch on either side, distinguished by a blackened septum, and producing single, straight antheridia with hardly differentiated necks; as well as two to four erect, slender, tapering, faintly brownish simple branchlets, which may reach as far as the apex of the perithecium: basal cell of the outer appendage blackened externally and distally, short and broad; bearing distally in a radial series four to seven, or even eight, closely set, erect branches with blackened basal septa; their comparatively short basal cells bearing distally elongate branchlets, usually radially associated in pairs, one member of which may occasionally be replaced by an antheridium; the branchlets long, hyaline to brownish, with here and there a secondary branchlet, somewhat uniform, diverging in a more or less compact fascicle, often irregularly nodular distally, or swollen at the remote septa; some of them ending in characteristic, rather close spirals. Perithecium nearly straight, concolorous with the small basal and stalk-cell region, from which it is distinguished below by a slight elevation, only, which marks the base of the lower tier of wall-cells; which also form a rather prominent ridge distally, the body of the perithecium in this region being slightly inflated, tapering continuously from this elevation to the narrow apex; the wall-cells describing more than a quarter of a turn: the right and posterior lip-cells misplaced, and combined to form a narrow, dark, blunt, hyaline, pointed prolongation, which is terminal and external; the two remaining lip-cells forming, below it, a hyaline, slightly prominent area. Spores 50 × 3.5 μ. Perithecium, including basal cell region (7–10 μ), 100–130 × 24–30 μ. Receptacle to insertion-cell 250–270 μ; basal and subbasal cells 218–245 × 14–18 μ, the bulbous enlargement × 35 μ or less. Longest appendages 210 μ.

On various parts of *Hippelates* sp., No. 2811B, St. George, Grenada, B. W. I.

This species has been obtained in abundance on material collected for me by Mr. Phillip, to whom I am also indebted for other interesting hosts. It is of special interest, since it has the typical receptacle of '*Ceraiomyces*,' cells III–V being replaced by a single cell, while its insertion-cell and appendages are of the normal *Laboulbenia*-type. The form of pointed perithecium is peculiar and distinctive, while the spiral terminations of many of the branches of the appendage resemble those of many trichogynes, and are quite unlike the more or less indefinite spirals seen in *L. spiralis*.

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CONTRIBUTIONS FROM THE BERMUDA BIOLOGICAL STATION
FOR RESEARCH, NO. 62.

# *ON THE PIGMENTATION OF A POLYCLAD.*

By W. J. Crozier.

With One Colored Plate.

CONTRIBUTIONS FROM THE BERMUDA BIOLOGICAL STATION FOR RESEARCH, NO. 62.

# ON THE PIGMENTATION OF A POLYCLAD.

W. J. Crozier.

Received Feb. 6, 1917. Presented by E. L. Mark.

Among early naturalists it was quite commonly held that in many cases the coloration of marine invertebrates, such as turbellaria, was directly determined by that of the pigmented substratum upon which they fed. In this way it was attempted to account for certain conspicuous resemblances between the color of species living in company with sponges, ascidians, and the like, and that of the organisms over which they crawled. Furthermore, these color agreements have frequently been considered valuable to the species concerned, according to the scheme advocated by the general theory of protective coloration (cf. Potts, 1915). Increasing knowledge of animal pigments has, however, served to develop a well-founded distrust of so simple an explanation for correspondencies in the pigmentation of associated forms. Several instances have recently come to my notice which seem of interest in connection with the idea of the origin of color in certain invertebrates from the pigmentation of their food. In the one notable case of color agreements which has been adequately studied (the prawn Hippolyte), this view has been decisively rejected (Gamble and Keeble, 1900; Gamble, 1910).

The present observations concern a polyclad turbellarian, Pseudoceros sp., found in association with various tunicates, upon the surface of which it has been seen to feed. This flatworm, of which the general characteristics may be gathered from the figures, apparently belongs to the genus Pseudoceros; but it does not agree with the diagnoses of any of the four species of this genus which Verrill (1900, 1901) has described from Bermuda.[1] I must refrain from attaching a name to it until its anatomy shall have been studied.

This form seems not to be common at any time of year. Although a somewhat careful search for it has been made on the basis of its habits, as revealed by such individuals as have been found, only a small number have been seen. It has therefore been impossible to collect a quantity of the animals for purposes of experimentation, and these notes are, in fact, based upon observations made with only six specimens.

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[1] Several of Verrill's species were founded, apparently, on color differences of single specimens.

The first individual of this type obtained was found in the branchial sac of a colorless, transparent ascidian, Ascidia curvata Traustedt, which was, as is usually the case with A. curvata, attached to the under side of a flat stone. The polyclad was of fair size, 17 mm. in length when creeping undisturbed. It was observed frequently to come out of the ascidian, and to creep about the aquarium in which the animals were kept. It was marked as shown in Figure 7, but its body, aside from the black cross-bars, was entirely devoid of coloring matter, its substance being of a velvety opaque white appearance.

Three other specimens were taken among separate colonies of the orange colored Ectinascidia turbinata Herd.[2] growing on the reefs. These colonies of the tunicate were affixed to the dead upper portions of gorgonian "whips." In one instance the colony was closely united with a mass of Rhodozona picta (Verr.), the test of which is gelatinous, transparent, and colorless.[3] The polyclads obtained from the Ectinascidia colonies are well depicted in Figure 2. It will be noted that there is a very fair agreement between the general hue of such an individual and that of the test of its host (Fig. 1).

The remaining two examples were obtained from the dark purple-black Ascidia atra Les. The one illustrated in Figure 7 came from the interior of a dead test of this species, the other, of a much darker cast, being found in the branchial sac of a healthy individual.

There is an obvious, striking parallel between the coloration of these animals and that of the tunicates with which they were individually associated. That there is any protective (concealing) value behind this, can, I think, be confidently denied. For, as a matter of fact, the flatworms when in exposed positions are perfectly conspicuous in spite of the agreeing element in their coloration. This was notably the case with the specimen found in the branchial chamber of the transparent A. curvata.

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[2] There is some reason to believe that this species is different from that recorded under this title by Van Name (1902).

[3] It is a curious fact that in the instances of compound colonies, which are not infrequent, the Ectinascidia is practically always the upper, the Rhodozona the lower, member of the group.

It is of more value to discover the source of the pigmentation than to speculate concerning its "function." A simple experiment showed that, as was to be expected from what is known of some other turbellarians,[4] a good fraction of the color seen in such specimens as those drawn in Figures 2 and 7 was due directly to food in the alimentary spaces.

The individual shown in Figures 2 and 3 was isolated from its Ectinascidia colony (cf. Fig. 1) on July 10, 1916. Five hours later it had become considerably paler (Fig. 4); and two days later it was very conspicuously so, as indicated in Figures 5 and 6. It was then returned to a small dish containing several Ectinascidia zoöids, one of which the flatworm very promptly found and began to feed upon, creeping over its surface. After six hours it had assumed a brilliant orange color, like that of Figure 2. The major portion of the orange pigment comes, then, directly from the excreted pigment-bearing cells at the surface of the tunicate test. A similar experiment was made, with a corresponding result, in the case of one of the purplish-black flatworms found with A. atra.[5]

There appears, however, to be a minimum below which the pigmentation cannot be reduced by moderate starvation. After four days in seawater, removed from their host, two of the orange specimens were still in the condition depicted by Figure 5. So far as could be made out from microscopic examination under gentle compression, no orange pigment was retained in the digestive system, although some was present in the integument.

The readiness with which these polyclads return to their particular kind of tunicate is surprising. With all of the individuals obtained, tests were made by giving them the opportunity of feeding upon E. tubinata or A. atra, placed together in the aquarium, and in two cases specimens found on E. turbinata were also offered individuals of A. curvata. When kept in aquaria the polyclads frequently left their host and wandered about the dish, but in all these instances, as well as in numerous trials in which they were artificially removed, they always returned to that type of ascidian upon which they were originally found. Nor could they be induced to feed upon a different species. I tried in this way to alter the coloration of the specimen found in A. curvata, and of two of those taken on Ectinascidia, namely by placing them upon A. atra in small aquaria containing no other tunicates. But they would not remain on this ascidian, they would not feed, and indeed died in the course of two days.

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[4] It should be stated, though, that practically no observations have been recorded respecting the food of the brilliantly colored polyclads.

[5] For a note on this pigment, see Crozier (1916).

There consequently seems to be a close correlation between the habits of the three color varieties — for I believe the evidence warrants the view that they are "physiological varieties" of one species — and their ability, on the one hand, to feed on the particular ascidian which harbors them and, on the other, to "make use" of the pigments involved. This correlation is also suggested by the size of the three varieties, which, so far as I have been able to observe, increases directly with that of their tunicate hosts (E. turbinata, A. curvata, A. atra), the average lengths of the respectively associated polyclads being 6, 17, and 20 mm.

AGAR'S ISLAND, BERMUDA.

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## EXPLANATION OF PLATE.

FIGURE 1. A group of zoöids from a colony of Ectinascidia turbinata (× 1).

FIGURES 2 & 3. Pseudoceros found on E. turbinata, freshly removed (× 5).

FIGURE 4. The same individual as that shown in figures 2 and 3, five hours after removal from its host (× 5).

FIGURES 5 & 6. The same individual, two days after removal from its host; Fig. 6 is a ventral view (× 5).

FIGURE 7. Pseudoceros found inside the tunic of a dead Ascidia atra (× 2).

The figures are by Miss H. E. Fernald.

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# THE JOULE-THOMPSON EFFECT IN SUPERHEATED STEAM: I. EXPERIMENTAL STUDY OF HEAT-LEAKAGE.

By H. M. Trueblood.

Investigations on Light and Heat made and published with aid from the Rumford Fund.

# THE JOULE-THOMSON EFFECT IN SUPERHEATED STEAM: I. EXPERIMENTAL STUDY OF HEAT-LEAKAGE.

By H. M. Trueblood.

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## I. INTRODUCTION.

Some time ago, the writer undertook the experimental determination of the Joule-Thomson effect and of the product of this by the specific heat at constant pressure, in superheated steam, as functions of pressure and temperature. The theoretical and practical importance of such measurements as these has been fully set forth in a recent paper by Davis.[1] In the experimental work, it became evident with the first results obtained that the principal problem was the reduction of heat leak to such a point that its effect might be eliminated without an excessive number of observations under varied conditions of pressure drop and flow at a single point of the $(p, T)$ plane. The experimental study of this problem developed chiefly in the direction of making measurements of $\mu$, the Joule-Thomson coefficient, with plugs of different types, under a considerable variety of circumstances in respect of the arrangement of the containing case, lagging, etc., the determination of the product $\mu C_p$ being temporarily abandoned. These experiments have involved altogether nearly one hundred separate measurements of $\mu$ with three distinct types of radial flow plug case, one of which was especially designed for the purpose of studying the effect of variations in lagging, and with one type of case for what may be called, for the sake of convenient distinction, an *axial* flow plug. Of these measurements of $\mu$, all but fourteen were made at practically the same pressure and temperature.

The chief intent of the present paper is the presentation and discussion of the results of these measurements of $\mu$ at a single point of the plane, as a study of the question of heat leak in throttling calorimetry, and particularly as a study of the behavior of the radial flow plug as applied to throttling calorimetry; for while a considerable proportion of the runs were made with axial flow plugs, the general type of plug and containing case used in all of these was of an early design, in which the heat leak was unnecessarily large, so that the interest these axial flow runs have for the purposes of this paper lies chiefly in the fact that they illustrate rather strikingly the heat-leak possibilities in throttling apparatus not designed primarily to avoid heat-leak, rather than in any basis they afford for a fair comparison of the relative merits of correctly designed axial flow and radial flow plugs.

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[1] Davis, Phys. Rev. (2) **5**, 359 (1915).

While the results obtained with the latest type of radial flow apparatus used justify the conclusion that the heat leak problem has been satisfactorily solved, there is little of value, in the results here presented, to those who are primarily interested in the thermodynamics of superheated steam, since the work covers the determination of $\mu$ at only one point. The experimental work at other points is now in progress, and it is expected that its results will be set forth in a future paper. This further prosecution of the work has been made possible by the generous loan of most of the necessary apparatus to the writer by the Director of the Jefferson Physical Laboratory of Harvard University. All of the results presented and discussed in this paper were obtained in the Jefferson Laboratory.

## II. THE THROTTLING APPARATUS.

In all the work described and discussed in this paper, the throttling apparatus, whether of the radial or axial flow type, was immersed in an oil bath, the temperature of which was automatically maintained constant. This oil bath also contained a device, hereafter called the secondary superheater, consisting of a coil of pipe or of pipes, through which the steam was passed before reaching the high side of the plug. With this arrangement, very common in throttling calorimetry, the temperature of the fluid on the high side of the plug is presumably that of the bath, at least if the velocity of flow of the fluid when in the secondary superheater is the same as it is at the high side thermometer bulb. For fluids with positive Joule-Thomson effects, the heat leakage is therefore from the bath inward to the fluid, if the fluid is merely throttled without other addition of heat.

### 1. Axial Flow Apparatus.

The general type of axial flow porous plug (a term here used in contradistinction to the term 'radial flow plug,' which is the other fundamental type involved in these experiments) is illustrated in Fig. 1. The steam enters at one of the orifices near the top of the cap *C*, passes the high side thermometer (not shown, but screwing into the bushing *M*), thence through the numerous fine holes shown in section in the soapstone block *S*, through the cross channel in the soapstone block $S'$, through another set of holes in block $S$ similar to those shown, past the low side thermometer bulb, and out through an orifice (not shown) in cap $C'$. The head of the low side thermometer may be seen projecting from the cap $C'$. The two sets of holes in the block $S$ are threaded with a continuous piece of wire, used as a heating coil when it is desired to conduct the experiment isothermally for the measurement of $\mu C_p$. This wire in certain plugs was of solid manganin, in size approximately No. 18 B. and S. gauge; in others, it was of stranded invar, ten strands, each about 0.01 inch in diameter, being used. Connections for the differential pressure gauge are

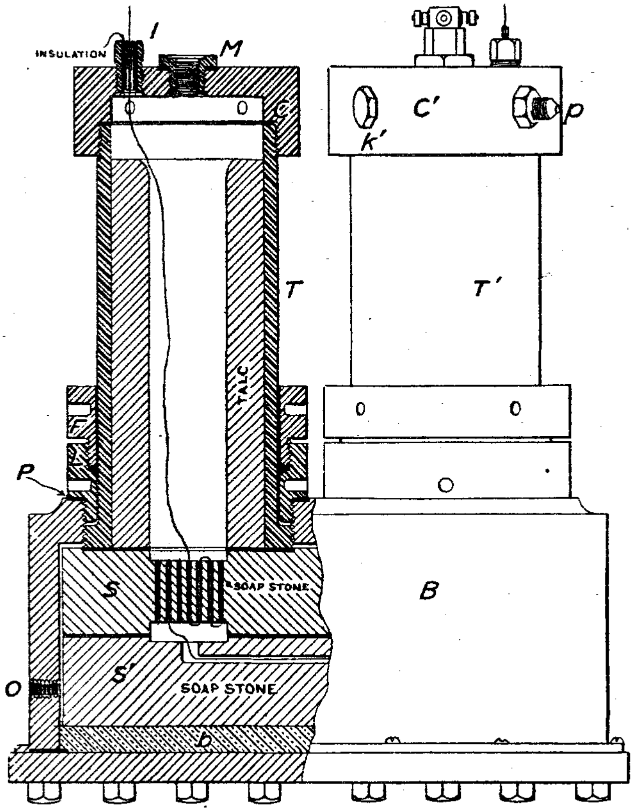

FIGURE 1. Axial flow plug and plug case.

taken from the bushing $p$ and from a similar bushing on the high side.

All axial flow plugs were of this general form and were contained in the plug case shown. The plugs used differed in the size of the holes drilled in $S$, in the dimensions and position of the cross channel in $S'$, in the character of the lagging in the vertical tubes $T$ and $T'$ (which was of talc in some plugs and of 'poplox' in others) and in the dimensions of the chamber between $S$ and $S'$. In certain plugs this chamber, somewhat enlarged for the purpose, was filled with copper baffle-plates, the object being to secure a more intimate mixing of the steam in its passage through the plug. In other plugs, these baffle plates were replaced by a number of layers of fine copper gauze. In certain plugs, additional baffles consisting of brass plates with intervening layers of copper gauze were located in the vertical tubes $T$ and $T'$ just above the holes in the block $S$. Various combinations of these baffling devices were tried, chiefly to see whether any noticeable effect in the observed values of $\mu C_p$ with varying intimacy of steam-mixing could be detected. The results of these experiments were in general negative, no consistent variations in the results on $\mu C_p$ being traceable to changes in the baffling.

In the plugs in which 'poplox' was used for lagging in the vertical tubes, the poplox, which is a very light cellular material, was contained between two thin concentric brass tubes (corresponding to the inner and outer surfaces of the annular cylinder of talc shown in the tube $T$ of Fig. 1). The inner of these tubes was used to afford an electrical connection between the heating coil and the insulating plug $I$, the wire lead shown in the figure being dispensed with. This was done because it was suspected that some part of certain rather large discrepancies in the measured values of $\mu C_p$ might be due to an accidental proximity of the lead wire, heated by the passage of the electric current, to the bulb of the thermometer on one side or the other of the plug.

The plug-case was swung in a cradle which also supported the secondary superheater, consisting in this case of a helical coil of about 40 feet of $\frac{3}{8}$ inch copper pipe (actual inside diameter = 0.49 inch) of about the height of the plug case and arranged so that it enclosed the plug case. This coil of pipe was connected at one end to the high side of the plug, at the other to a primary superheater (see Figs. 5 & 6) from which the supply of steam was drawn. The cradle, with the plug case, coil of pipe and two motor-driven stirrers, was suspended by means of a block-and-falls in the oil bath which has already been mentioned.

The most obvious defect in the type of plug shown in Fig. 1 is, of course, the opportunity for heat leakage afforded by the wide separation of the two thermometer bulbs. Nearly half of all the throttling has taken place before the steam enters the cross-channel, in which the temperature of the steam is consequently considerably less than that of the oil. This defect was not unnoticed in designing the apparatus, but its importance was much underrated. It was expected that the fairly good thermal insulation of the steam while in the cross-channel would prevent heat leakage of inconvenient magnitude (the soapstone blocks of Fig. 1 are about two inches thick and over four inches wide), and the first practicable apparatus used was built in this form, chiefly because it could be easily handled and because the design permits an easy and rapid interchange of the (resistance) thermometers for the purpose of eliminating their normal difference of resistance. Soapstone, however, is not a particularly good heat insulator, and it was found impossible to obtain reliable values of the Joule-Thomson coefficient with plugs of this type without an excessive amount of variation in the conditions of flow and pressure drop at each pressure and temperature of measurement.

## 2. Radial Flow Apparatus.

### *a. General remarks on the radial flow plug.*

The theory of the radial flow porous plug has been discussed in a paper by Burnett and Roebuck.[2] One of these authors has since applied it to the experimental determination of the mechanical equivalent of heat.[3] A radial flow plug was also used by Regnault in some porous plug experiments on air. He regarded his results as very unsatisfactory and the time spent in securing them as wasted. So far as is known to the writer, no other experimental results obtained with the radial flow plug have been published. Briefly, the radial flow plug consists of a thin cup of porous material of low thermal conductivity — porcelain, for example — and of a length some 6 or 8 times its diameter (see Fig. 2). The fluid under experiment is made to flow through the cup from the outside to the inside. The low-side thermometer bulb is located near the bottom of the cup on the inside.

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[2] E. S. Burnett and J. R. Roebuck, Phys. Rev. (1) **30**, 529 (1910).
[3] J. R. Roebuck, Phys. Rev. (2), **2**, 79 (1913).

The high side thermometer bulb may be placed at some convenient point in the path of the fluid ahead of the cup, preferably as close to the cup as possible. The whole is immersed in a bath, the temperature of which is that of the fluid at the high side thermometer. If the Joule-Thomson effect in the fluid is positive, the conductive flow of leaking heat (aside from conduction down the thin, long and poorly-conducting walls of the cup — a relatively small item) is in the direction of the flow of the fluid, and ultimately consists in the transfer of heat from a thin layer of the fluid immediately adjacent to the plug wall on the high pressure side to a similar layer on the low pressure side. This transfer of heat obviously has no effect on the temperature drop between thermometers after the steady state has been established, provided the thin layer of fluid on the outside of the plug is prevented from receiving heat through the metallic walls of the plug case from the bath. Aside from radiation, which is doubtless slight, heat may be gained, from the bath, by the fluid at any instant adjacent to the high side of the plug, by conduction, provided the temperature depression on the high side due to conduction through the plug gives rise to an appreciable temperature gradient at the walls of the plug case; that is, provided the "thin layer" of fluid mentioned above is sufficiently thick. If this is the case, any mass of fluid will, of course, also acquire heat before reaching the high side of the plug, while it is moving through regions in which the temperature is not that of the bath. Theoretically, at least in the ideal case of an infinitely thin plug of material having a finite thermal conductivity, the thickness of the "thin layer" is zero*, in the steady state, and while it is probably small, at least in the vicinity of the closed end in such

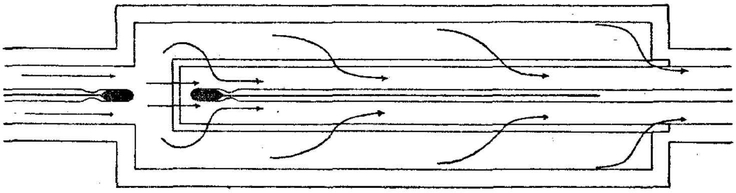

FIGURE 2. Diagram of radial flow plug.

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* The analysis on which this statement is based ignores changes in volume and specific heat due to the throttling, and also changes in temperature caused by the increased average velocity of the fluid while passing through the plug.

plugs as have actually been used, the experiments to be discussed later show that it is essential to shield the fluid from the effects of this conductive action by interposing lagging between the plug and the walls of the containing case (see Fig. 4).

An important effect which is analogous in its origin to the effect due to heat conduction through the body of the plug arises, if certain precautions are not taken, from the rather peculiar geometry of the radial flow plug. At its open end, the plug is necessarily brought, for purposes of mechanical support, into good thermal contact with the metal walls of the case. As a result, fluid which has passed the low side thermometer is provided with a thermally short path of communication with fluid which has not yet passed through the plug. This results in a depression of the temperature of the latter, which, joining the low side fluid at a temperature therefore lower than then exists there, contributes to a still further depression of the temperature of the fluid which has not yet passed. This cumulative action, similar in principle to that taking place in certain types of liquid air machines, is limited by the ability of the bath to supply heat to the low side fluid. The reason that a similar action does not take place by conduction through the plug walls is that, at least near the closed end of the plug, the direction of flow is parallel to the temperature gradient in the plug for a considerable distance on both sides of the plug; undoubtedly the effect does exist in some degree near the open end of the plug, where the stream lines even a very short distance from the plug walls are approximately axial.

The effect of the action just described on the observed temperature drop is indirect, since the fluid directly affected does not pass the low side thermometer. But the temperature depression produced in the neighborhood of the open end produces indirectly, by conduction through the walls of the plug, the walls of the case, the inside lagging and, doubtless, to some extent through the fluid itself, a depression of the temperature of fluid which does pass the low-side thermometer; that is, there is an outward leak of heat from fluid between thermometers to fluid which has passed the low side thermometer, with the result that the observed temperature drop is too high.

This effect, of the existence of which experimental evidence is given later (see IV, 2, *d*) will be called the 'regeneration effect' hereafter. The remedy for it is to provide sufficient lagging between the metallic support of the plug and the high side fluid. This lagging will be called 'internal end-lagging' or simply 'end-lagging' whenever it is necessary to refer to it in what follows.

### *b. The V-type of radial flow plug case.*

The apparatus illustrated in Fig. 3 will be referred to as the V-type of plug case, for the sake of brevity. The design, which is due to H. N. Davis, under whose direction the apparatus was also constructed, was adopted with the two-fold purpose of retaining the convenience of approximately vertical thermometers and of doing away with the lengthy cross channel of Fig. 1. It was also possible to employ the oil tank, stirrers, cradle, secondary superheater, etc., which had been used with the apparatus of Fig. 1.

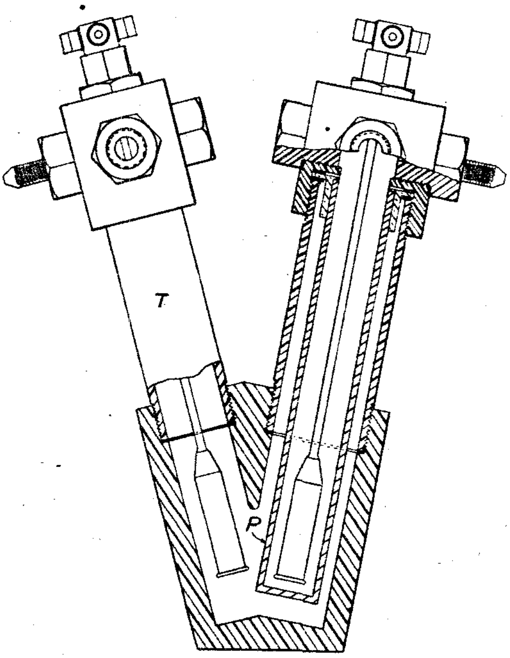


FIGURE 3. Radial flow plug and case, V-type.

The plug *P*, in Fig. 3, is of alundum, and was obtained from the Norton Alundum Co., of Worcester, Massachusetts. It was secured, as shown, to a brass collar by the device of plating from a solution of copper sulphate onto the inside of the collar, through the alundum, from a copper electrode placed just inside the open end of the cup. The copper fills up the pores of the alundum and a mechanically rigid union is thus obtained. This method of attachment is also due to Prof. Davis and is very satisfactory. It was deemed advisable to make the joint steam-tight by the use of bakelite varnish, bakelized under pressure according to standard methods of treating this material. Bakelite varnish has also been depended upon for mechanical union in some plugs, the copper plating being dispensed with; but the results are not so satisfactory as when the plating method is used.

No internal lagging whatever was used with this type of plug case; but the case was provided with a galvanized sheet iron container (not shown) resembling a small coal scuttle. The plug case being placed inside the container, the space remaining could be filled with poplox, giving an average thickness of insulation of perhaps two inches between the plug case and the oil. The container was equipped with an oil tight top, so that it, with the plug and poplox inside, could be completely immersed in the oil bath. Experiments were conducted, (i) without the container; (ii) with the container, but no poplox, the cover being omitted and the container immersed until its upper edge was flush with the surface of the oil; (iii) with the container filled with poplox, cover omitted, immersion as in (ii); (iv) with the container filled with poplox, covered and completely immersed.

### c. *The U-type of radial flow plug case.*

This name is used merely for convenience. The plug case was simply the axial flow case of Fig. 1, with certain minor changes necessary for adaptation to radial flow work. The plug was located in the low side tube, $T'$ of Fig. 1, and was supported by means of a special flange which made possible complete internal lagging, of a thickness roughly $\frac{1}{2}$ inch at the open end of the plug and rather less than $\frac{3}{4}$ inch elsewhere. The end-lagging was asbestos wood (a commercial product consisting of asbestos fibres with a binding of portland cement); the other lagging, separating the plug from the steel wall of the tube $T'$, was of talc, as in Fig. 1. It may be mentioned that asbestos wood is a much more efficient lagging material than either talc or soapstone.

The block $S$ of Fig. 1 was replaced by a block having a single large hole at the place at which the heating coil is shown in Fig. 1 and a similar hole at the corresponding place on the low side. The block corresponding to $S'$ had a half-inch channel, with baffling and mixing chambers at each end.

Only a few runs were made with this type of plug case, which was used because it was the simplest form of apparatus immediately available at the time by which internal lagging could be provided. It was found that although the regeneration effect noticed with the V-type of case was absent, or practically so, a considerable error due to heat leak in the cross-channel, and through the talc lining of the tube $T'$, persisted. This seems to have been due mainly to the depression of temperature experienced by the steam in passing through the baffling chambers. The most important bearing which the runs made with this apparatus have is an incidental one concerned with the question of the dryness of the steam; this is discussed later (see IV, 2, *g*).

### *d. Straight-away type of plug case.*

The final form of plug case adopted, and the one by means of which the various heat-leak difficulties are believed to have been solved, is illustrated in Fig. 4. It will be seen that the annular lagging (that concentric with the plug) is quite heavy; it is nearly two inches thick. The slight axial taper given to its inner surface is intended to prevent variation in the velocity of axial flow along the outside of the plug. The apparatus is designed throughout to preserve a nearly constant velocity of flow, equal to that in the secondary superheater. The end-lagging (that at the open end of the plug) is also thicker than in previous apparatus, being somewhat over one inch in thickness. All lagging is of asbestos wood.

The two thermometer bulbs are placed almost as close together as it is possible to get them. Their distance of separation may be varied, if desired, by using longer bushings at *B*, *B*.

The plug case is designed so that it may also be used for an axial flow plug, if desired. The change is effected by replacing the radial flow plug and its lagging with a suitably lagged axial flow plug. The low side cap is removed and replaced with a coupling, into which is screwed a cylindrical chamber of about the same dimensions as that which, in Fig. 4, is shown containing the radial flow plug and its lagging. The low side cap is then screwed onto the other end of this new chamber, which contains the heavily-lagged low-side thermometer. No runs have yet been made with this axial flow set-up.

## III. OTHER APPARATUS.

A general view of the essential parts of the whole apparatus is given in Fig. 5 (plan, with end elevation showing manometer connections) and Fig. 6 (front elevation). The oil bath and its contents (secondary superheater, stirrers, etc.), shown in these figures are as used with the straight-away type of plug described under II, 2, *d* above. A complete reconstruction of all of this section of the apparatus was necessitated by the adoption of this type of plug. The remainder of the apparatus, aside from a few minor changes, was unaltered by the change from the vertical to the horizontal type of plug.

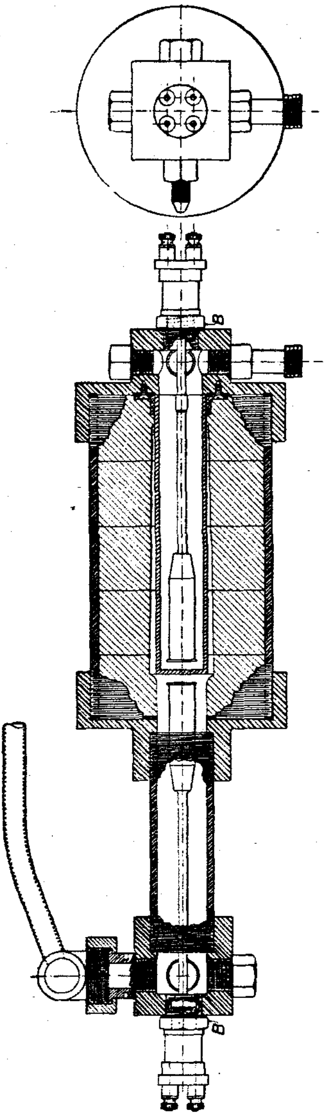


FIGURE 4. Straight-away Type of Radial Flow Plug Case.

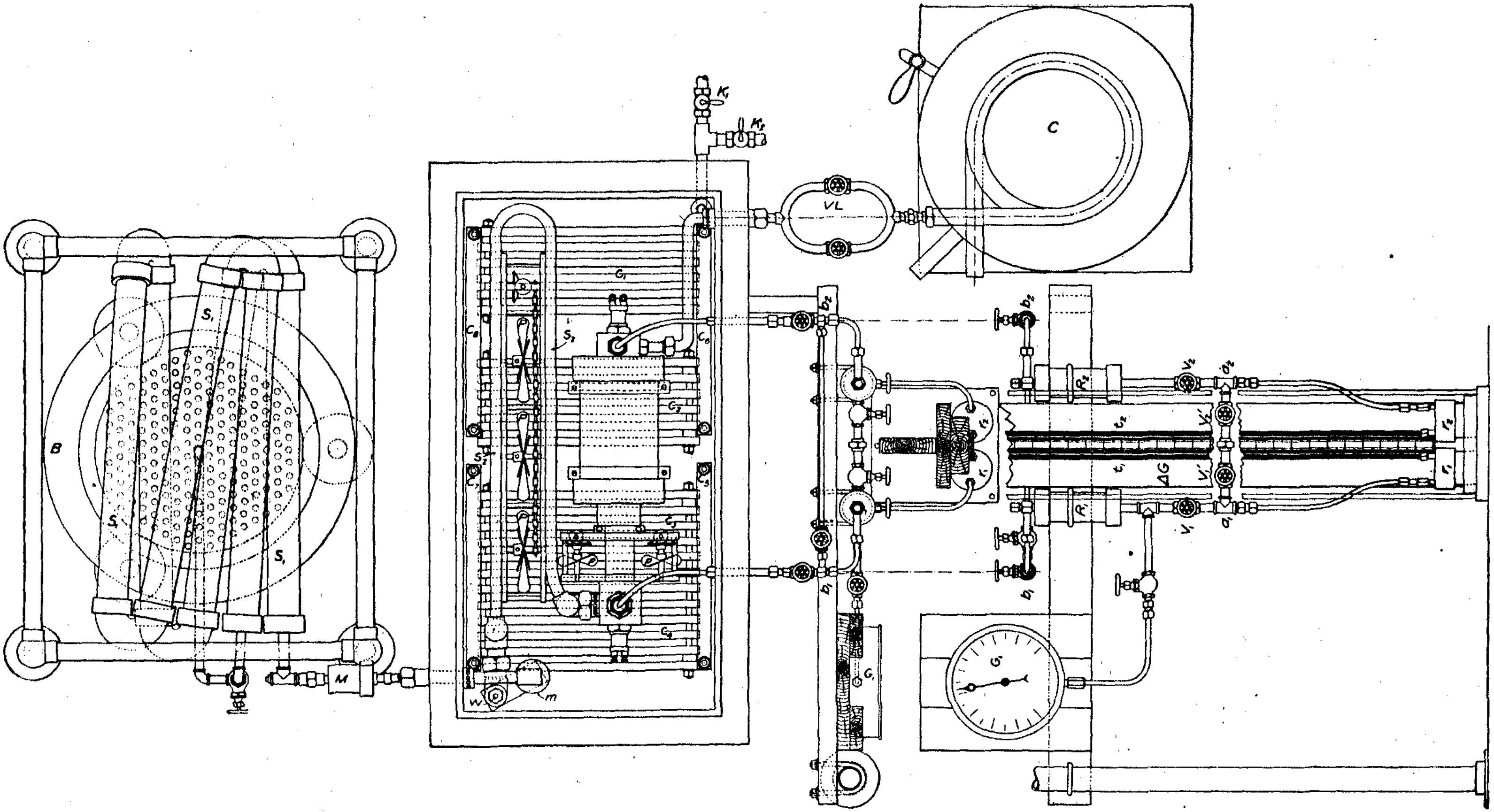
## 1. OIL BATH AND SECONDARY SUPERHEATER.

Only the bath and superheater used with the straight-away plug will be described in detail.

The oil is contained in a rectangular tank (Figs. 5 and 6), 41 × 18 × 13 inches. The last dimension is the depth. The tank is of sheet iron, galvanized after construction. It is heavily lagged, on the four vertical sides and the bottom, with asbestos, and when running, the top is completely covered with an asbestos cover about 1½ in. thick. The two long vertical sides and the bottom are lined on the inside with a series of 8 German-silver heating coils ($C_1$, $C_2 \ldots C_8$, Fig. 5); each coil takes about 300 watts at 110 volts. At each end of the tank is a small circular aperture (not shown) ordinarily closed with a flange. Removing the flanges, after draining the oil from the tank, makes it possible to interchange the thermometers without disturbing the set up. The oil, of which some 35 gallons are used, may be drained into a lagged storage tank (not shown) by opening the cock $K_1$ (Fig. 5). After the thermometers are interchanged, the oil, still hot, is pumped back into the bath-tank by means of an oil-pump (not shown). The oil that has been used is a heavy tempering oil, quite viscous at room temperature, and having a very high flash-point.

In all of the work here reported, the control of temperature has been by means of a mercury-in-glass thermostat of the familiar type, operating by means of a platinum contact, through two relays, to open or close the circuit supplying the heating coils. Owing to the rather large current (25–30 amperes) carried by this circuit, it was necessary to use alternating current, to avoid troubles with arcing on the make and break. The type of thermostat mentioned has been used with entire satisfaction at temperatures up to 220° C., and, in some earlier work, with fair satisfaction at 240° C. At higher temperatures, the operation is likely to be uncertain because of evaporation of the mercury.

The secondary superheater ($S_2$, Fig. 5) in which the final adjustment of temperature is made, consists of eight $\frac{3}{8}$ in. copper pipes bent back upon themselves as shown, and silver-soldered at both ends into manifolds. The pipes are arranged in two groups, each group consisting of four pipes in parallel; these two groups may, by suitable arrangements, be placed either in series or in parallel, thus making it possible to double the velocity of steam flow. In the earlier form of the apparatus, with the vertical or U-type, and also with the V-type, of plug case, the secondary superheater consisted of a single coil of $\frac{3}{8}$ in. copper pipe, the total volume of which (0.052 ft.$^3$) was not very different from that of the copper pipes of $S_2$. The steam velocity in this earlier superheater was necessarily much greater than the velocity of flow in the talc-lined tube $T$, Fig. 1, or the tube $T$ of Fig. 3, where the diameter of the circular cross-section was about 1 in.; as a result of this, the temperature of the steam at the high side thermometer was higher than that of the oil, by an amount depending on the flow, because of the decrease in the kinetic energy of the steam as it passed from the small cross-section of the superheater to the larger cross-section of the tube $T$. It was chiefly in order to avoid this defect, with its obvious possibilities, direct and indirect, that the parallel arrangement of pipes was adopted in the later form of superheater. The velocity of flow here is very nearly the same as in the tube $T$ of the plug case (Fig. 3), or as in the conical annular space surrounding the plug, (Fig. 4), when the two groups of superheater


B
S₁
S₁
S₁
M
W
m
C₈
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C₆
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C₅
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K₁
K₂
VL
C
b₂
b₁
r₂
r₁
R₂
R₁
t₂
t₁
V₂
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V₂'
V₁'
a₂
a₁
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FIGURE 5. General view of main parts of apparatus. Plan and end elevation.

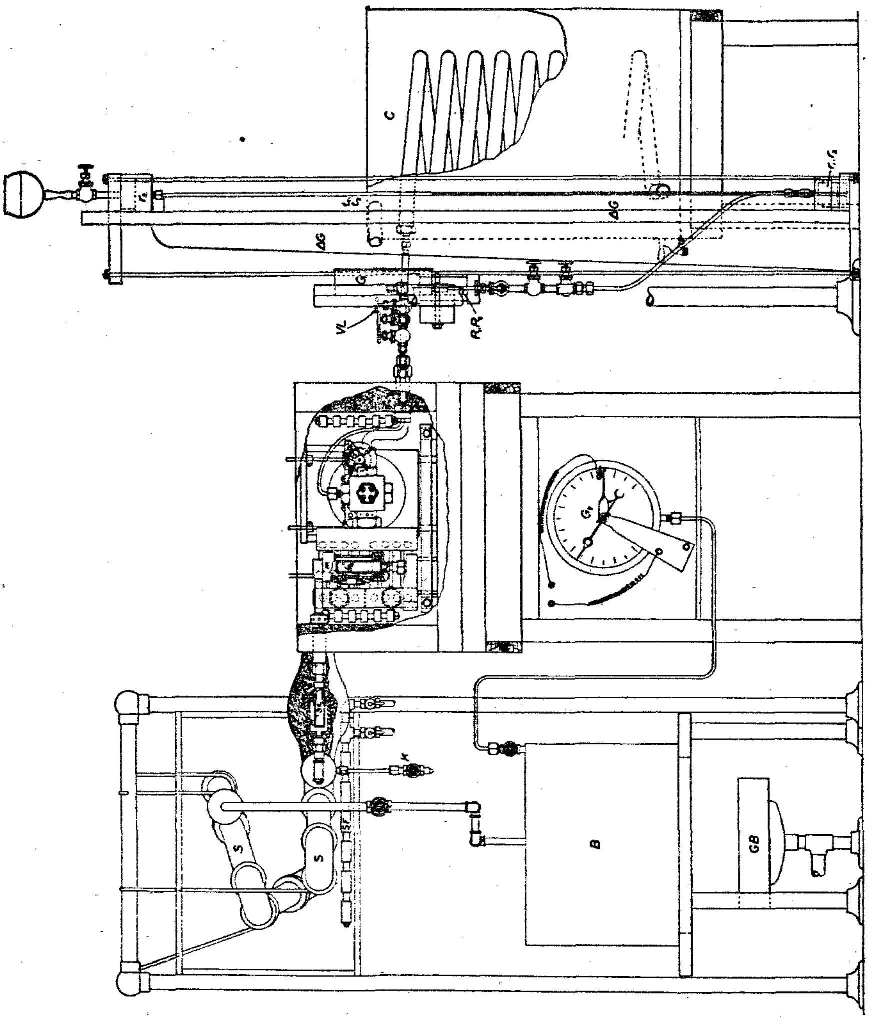
C
ΔG
ΔG
G
VL
R.R
G₁
S
S
SF
K
B
GB

FIGURE 6. General view of main parts of apparatus.— Front elevation.

pipes are in parallel; in fact, for a considerable area of the ($p$, $T$) plane, either velocity is so small, for the highest flow rates used, that the steam could be considered at rest so far as any temperature effect of its kinetic energy is concerned. By altering the connections of the superheater so that the groups of pipes are placed in series instead of in parallel, tests of the effect of different velocities in the superheater and in the plug case may be made.

In the operation of the apparatus, it has been customary to bring the steam to a temperature exceeding the high-side working temperature by from five to fifteen degrees C., or even more, before it enters the secondary superheater. This superheater thus really acts as a cooler. The chief purpose of this method of operation is to avoid the possibility of the introduction of moisture into the steam by condensation in the short length of heavily lagged pipe connecting the primary superheater ($S_1$, Fig. 5; $S$, Fig. 6) with the secondary superheater. (The lagging on this pipe is shown in Fig. 6, but is omitted in Fig. 5). A well ($W$, Fig. 5) for a mercury thermometer is provided at the entrance of the secondary superheater, to facilitate this regulation of temperature.

As a further precaution against moisture, the steam passes through a strainer (contained in the chamber $M$, Figs. 5 and 6) just before entering the secondary superheater, and through another strainer (chamber $m$, Figs. 5 and 6) just after entering it. These strainers are both of alundum and are, in fact, nothing but small editions of the plug in which the throttling takes place. By omitting either or both, tests of the need for them and of their efficacy may be made. Some remarks and conclusions, from experimental results, on the utility of such strainers will be found in section IV, 2 $g$, which deals with the data derived from the type U plug case.

The stirring devices are indicated at $P$, $P$, $P$, $p$, $p$, Fig. 5. The three larger four-bladed propellers, $P$, $P$, $P$ provide the main circulation, which is across the lower four pipes of the superheater, under and around the plug case, and back across the upper four pipes of the superheater. Axial circulation is provided by the two smaller four-bladed propellers $p$, $p$.

## 2. Boiler, Primary Superheater and Condenser.

The steam is generated in an 18 inch Stanley automobile boiler ($B$, Figs. 5 and 6), designed to operate at from 500 to 600 pounds per square inch and tested to 1800 pounds per square inch by hydraulic pressure. Heat is supplied to the boiler by a gas burner beneath it (*GB*, Fig. 6). Special water level indicators are necessary at the higher pressures, gauge glasses being impracticable. Two different types have been used, one being that regularly supplied by the makers of the boiler. This device, which is very reliable, indicates whether the water level is below or above a not very sharply defined position, but requires a rather large variation in level for its operation. It was found desirable to have a sharper regulation of water-level than this appliance affords, because of the danger of introducing moisture into the steam if the level is raised too high. To secure this, a float enclosed in a vertical steel tube has been used. The float is of glass, copper-plated and then silver-plated. It completes an electrical circuit through a relay and thereby turns off an otherwise lighted incandescent lamp, when the surface of the water is raised to a certain definite level. The silver-plating is necessary to secure reliability of contact; it was found necessary also to silver-plate the inside of the containing tube. The only difficulty with this appliance arose from the gradual penetration of the water to the interior of the float through small flaws in the metal-plating — a rather striking example of the slow solvent action of hot water upon glass.

Pressure regulation is obtained by means of the hack-gauge $G_2$, Fig. 6. The pointer of this gauge makes an electrical contact between platinum terminals when the boiler pressure rises to the desired value. The completion of this circuit opens, through a relay, an electro-magnetically operated valve which admits air under pressure to one side of a U-tube containing mercury. The ensuing rise of the mercury level in the other side of the U-tube shuts off a portion (or, if desired, all) of the supply of gas to the burner. The electro-magnetic valve closes again when the pressure has fallen sufficiently to permit the platinum contacts mentioned to separate.

As is indicated in Figs. 5 and 6, the steam passes from the boiler to the primary superheater ($S_1$ in Fig. 5; $S$ in Fig. 6). This superheater consists of six 30-inch lengths of two-inch double strength wrought iron pipe, joined in series by cast-steel return bends, the whole capped at each end with cast-steel caps, tapped to fit the smaller entrance and exit pipes. The pipe connecting the boiler and superheater is heavily lagged, and is heated, throughout its length, by the hot gases from either the boiler furnace or the superheater furnace *SF* (Fig. 6). The superheater and its furnace are enclosed in an asbestos box, which is shown in part in Fig. 6, but not in Fig. 5. The total volume of the primary superheater is about 0.33 ft.

Each length of pipe in this superheater is filled from end to end with a system of copper baffle-plates arranged as shown in Fig. 7. It will be seen that this device imparts to the steam a rotary motion, the direction of which is reversed at each baffle. This insures thorough mixing and the projection of particles of moisture against the hot walls of the pipe. The baffle plates fit snugly into the pipe and make good thermal contact with the walls. To provide for the elimination of such moisture as may be separated from the steam by the action of this series of baffle plates, the drip-cock *K* (Fig. 6) is placed at the end of the superheater nearest the throttling apparatus. This cock is normally slightly opened during operation.

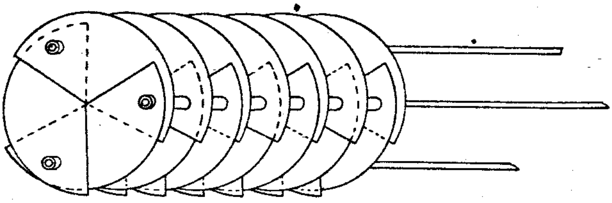

FIGURE 7. Arrangement of baffling plates in primary superheater.

The furnace for the superheater consists of six lengths of $\frac{3}{8}$ in. pipe, screwing into manifolds at each end. It is fed with gas and air at each of two opposite corners. A uniform fire is secured by independent control of fuel and air for each corner. The furnace is shown at *SF*, Fig. 6; it is omitted from Fig. 5, to avoid confusion.

After leaving the plug, the steam passes through two valves in parallel (*VL*, Figs. 5 and 6) to the condenser (*C*, Figs. 5 and 6). The use of two valves permits a nicer adjustment of pressure drop than is possible with a single valve, and, by halving the valve throttling, makes it possible to operate at a lower mean pressure without the use of an air pump on the condenser. The condenser consists of a helical coil of 1 in. wrought iron pipe in a galvanized sheet iron containing tank, through which passes a continuous supply of cooling water. The condensed steam is received in glass jars and weighed.

## 3. PRESSURE MEASURING APPARATUS.

The high side pressure is measured by means of a precision Bourdon gauge ($G_1$, Figs. 5 and 6), made by the Crosby company, and of the type known as 'standard test gauge.' The gauge was calibrated at the company's factory.

The drop of pressure through the plug is measured directly on the mercury manometer $\Delta G$, Figs. 5 and 6. As is indicated in Fig. 6, this manometer is so arranged that all joints are packed against water rather than against mercury. The two vertical glass tubes $t_1$, $t_2$ (barometer tubing) extend to the bottom of the separate reservoirs $r_1$, $r_2$. These reservoirs are each filled to about one-fourth of their heights with 500 gms. of mercury. The remainder of the volume of each contains water; the top reservoir $r_3$, Fig. 6, together with the space in the glass tubes above the mercury columns, also contains water. As the figure shows, the reservoirs $r_1$ and $r_2$ are connected respectively to the high and low sides of the plug by means of flexible copper tubing, through the reservoirs $R_1$ and $R_2$. The purpose of these latter reservoirs is partly to provide a considerable mass of comparatively cool water, all of which must pass through a constricted passage before any steam can escape, in the event of the bursting of a glass manometer tube. The time required for blowing out this reserve of water is sufficient to enable the experimenter to close the necessary valves, thus preventing further damage to the manometer or other apparatus near it by reason of the presence of hot steam or hot water. The 'constricted passages' referred to are at the bottom of each of the reservoirs $R_1$ and $R_2$, where the diameter of the passage is reduced to about 0.04 inch for a length of about 0.25 inch. These constrictions also serve to damp oscillations of the mercury in the manometer tubes.

A further purpose of the reservoirs $R_1$, $R_2$ is to prevent variations in the amount of water in the connections from the gauge to the plug from affecting the registered pressure difference. Such variations may be due to gradual and undetected condensation or to a sudden change in the pressure difference, which would force the water down in one connection and up in the other, or to other causes. Whatever the origin of these variations, the portion of either connecting passage first affected is that between the oil bath and the reservoirs $R_1$, $R_2$; as this part is nearly level, the effect on the registered pressure difference of removing even a large part, or all, of the water in it, is slight. If the change in the amount of water is great enough to extend beyond this portion of the connection, the large area of cross-section of the reservoirs $R_1$, $R_2$ prevents even a large change in the volume of the contained water from greatly affecting its level. It may be stated that the connections between the plug and the differential gauge shown in Figs. 5 and 6 were adopted only after a considerable amount of experimenting, and after other forms (in particular, one in which a part of the tube between either plug connection and the corresponding reservoir was vertical) had proven themselves unsatisfactory. With these earlier forms, it was not found possible, even with the most careful attention to filling the entire connections with water before starting a run, to prevent the formation of troublesome bubbles of air or steam in the parts of the tubes lying between the reservoirs and the plug. The presence of these bubbles was indicated by a failure of the mercury columns to equalize in height on shutting the low side flow valve *VL;* they were localized in the parts of the connections mentioned by opening the test valves $V_1'$, $V_2'$ in the cross-connection $a_1$ $a_2$ (Fig. 5) and closing one of the valves $V_1$, $V_2$, thus applying the pressure existing on one side of the plug to both sides of the gauge through the parts of the connections lying between the cross-connection $a_1$ $a_2$ and the gauge. No differences of level of any significant amount were ever observed when this was done.

The upper cross-connection $b_1$ $b_2$ (Fig. 5) did not exist in the earlier types of the connections between the plug and the manometer, but was added when the type shown in Fig. 5 was adopted. It was added because it was found that a part of the difference in level of the mercury columns which persisted after closing the valve *VL* could not be ascribed to the presence of air or water vapor in the upper connections, but was due to a lag of the equalization of pressure on the two sides of the plug behind the more or less regular pulses of the high side pressure caused by the action of the piezostat: the throttling in the plug, particularly in the casé of radial flow plugs, prevented immediate equalization of the pressure following a pulse, and the upper cross-connection, by short-circuiting the plug, removes the cause of this lag. All these refinements in what would seem to be a comparatively insignificant part of the apparatus were necessary because the difference of pressure which persisted after stopping the flow of steam had to be applied as a correction to the pressure difference observed during a run. With the earlier forms of connections, this correction was relatively large, being frequently in the neighborhood of 0.5 cm. of mercury and occasionally as much as 1.0 cm. or even more. It was, moreover, uncertain, being frequently different at the end of a run or a series of runs from what it had been at the beginning. With the type of connection shown in Fig. 5, the correction was reduced to so small a figure that it could almost be ignored, for it now ranged from zero to 1 or 2 mm. of mercury at the most (averaging about 0.8 mm.) —

an error of little consequence in total pressure drops of from 40 to 130 cm. of mercury when the errors due to other causes are considered.

The orifices in the bushings which screw into the plug case, and through which connection is established with the manometer, were covered with several circular discs of copper gauze, capped and held in place by a similar disc of sheet copper fastened to the bushing with screws. The gauze discs and the plate were of a diameter about three times that of the orifice, and they were located concentrically with the orifice. The purpose of this arrangement was to avoid Pitot tube effects. It has been used before by other experimenters.

The possibility of moisture entering the apparatus through the pressure connections is obviated by the shape of the connecting tubes inside the oil bath. (See Fig. 6.)

The possibility of an error in the measurement of the difference of pressure due to the fact that the pressure at the bulb of the thermometer may not be that at the point at which the connection is taken off from the plug case to the manometer, was carefully investigated. It was found that, if such an error exists at all, it is of inappreciable magnitude. The method employed in settling this question consisted in measuring the difference in pressure between these two points directly on an auxiliary manometer while the pressure difference on the two sides of the plug was maintained at various values such as were commonly used in regular runs, and further, in making rapid and large changes in the rate of flow while noting the resulting changes in the pressure difference registered by the auxiliary manometer. The actual conditions of a regular run were very closely imitated by using a dummy thermometer as the means of impressing the pressure existing at a thermometer bulb in a regular run on one side of the auxiliary manometer. The observations were made on both sides of an axial flow plug such as is shown in Fig. 1. The readings of the auxiliary manometer were small, averaging 0.09 cm. Hg. on the high side and 0.07 on the low side. On the high side, all readings happened to be zero or of the same sign, which was not that to be expected from the physical circumstances of the case. On the low side, differences of both signs as well as zero were observed. When the main pressure difference was varied rapidly from 20.8 cm. to 86.5 cm., no change was noted in the reading of the auxiliary manometer, although its reading changed by 0.20 cm. when the main pressure difference was brought back to 20.8 cm. There is no doubt that the actual differences registered by the auxiliary gauge were accidental and due to variations in the water level in the connections thereto.

## 4. THERMOMETERS AND THE MEASUREMENT OF TEMPERATURE.

### *a. Description of thermometers.*

All temperature measurements in this work have been made by means of platinum resistance thermometers. A large amount of experimenting, the details of which are not of much interest, was done before a satisfactory type of thermometer was evolved. The requirements of a thermometer suited to exact calorimetry at fairly high temperatures and pressures are rather severe. The chief ones may be said to be: (*a*) mechanical durability, (*b*) permanence of calibration, (*c*) ability to follow fluctuations of the temperature of the surroundings with reasonable rapidity, (*d*) similarity in all respects of the two members of a pair of thermometers. The last-named is particularly important where thermometers are to be used differentially, because the advantage of the differential method obviously decreases as the difference in the resistance of the two thermometers at the same temperature increases.

The type of thermometer which was finally adopted and used is illustrated in Fig. 8. It is essentially an adaptation of the calorimetric thermometer of Dickinson and Mueller,[4] but is naturally much more substantial mechanically, and much less sensitive. It is of the Callendar type, the leads being of No. 26 copper wire, silver soldered at one end to the platinum, at the other end to copper binding posts. In the instrument shown, the binding posts are silver-plated. It was found desirable to do this and also to silver-plate the lugs at the

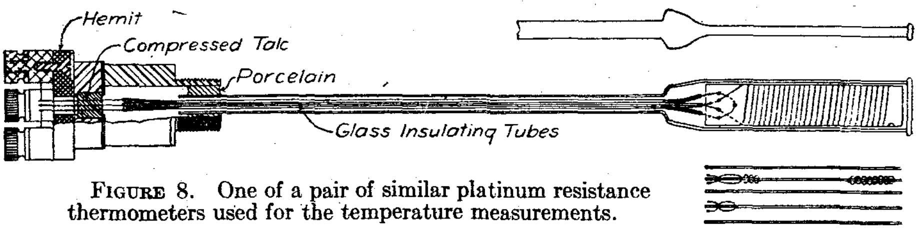

FIGURE 8. One of a pair of similar platinum resistance thermometers used for the temperature measurements.

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[4] Bulletin of the Bureau of Standards **3**, 641 (1907).

thermometer ends of the leads which connect the thermometers to the bridge, in order to reduce contact resistance to as low a figure as possible. One thermometer, with flat leads similar to those of the Dickinson-Mueller thermometer, enclosed in a flat tube, was constructed, but difficulty in securing adequate insulation resistance was encountered, and the type was abandoned. The bulb and stem, which in certain thermometers were separated by a layer of porcelain, are of steel, the stem being $\frac{1}{4}$ inch in diameter and having a wall thickness of about 0.03 inch. The stem is secured to the head by means of a porcelain joint. The bulb is flat, about $\frac{5}{8}$ inch wide, and with a space of about $\frac{3}{32}$ inch between the two flat walls inside. The length of a thermometer from the bottom of the bulb to the upper porcelain joint is about $9\frac{1}{2}$ inches.

Certain experiments with these thermometers, having for their object the investigation of the magnitude of various errors in the measured difference of temperature on the two sides of the plug, will now be briefly described.

### *b. Effect of conduction of heat down the stem of the thermometer.*

The porcelain joints which have been mentioned were employed for the purpose of minimizing any error which might arise from this source. As nothing was actually known regarding the possible magnitude of this error, it was considered desirable to make some experimental tests on it. To this end, a thermometer having two porcelain joints, one between the bulb and the stem and one between the stem and the head, was used. Two sets of clamps, made of copper and soft iron, were constructed: one set of clamps could be applied to the upper joint so that it short-circuited the heat-insulation afforded by the porcelain; the other set could similarly be used to short-circuit thermally the lower joint. Both sets were carefully made to fit the joint concerned closely and to give good thermal contact with the metal on each side of the joint. Comparative readings of the thermometer when immersed in the vapor of boiling naphthalin, exactly as for calibration at this point, were made, with the following different arrangements: (i) neither joint short-circuited; (ii) both joints short-circuited; (iii) upper joint free, lower short-circuited; (iv) lower joint free, upper short-circuited. Taking (i) as standard, case (ii) lowered the apparent boiling point by 0°.01 C; case (iii), by

0°.02 C; case (iv), by 0°.005 C. While the relative magnitudes of these results are hardly such as would be expected, the fact that a depression of the apparent temperature was obtained in each case indicates the existence of a real error from the cause under investigation and that the porcelain joints are of some value in diminishing its magnitude. The experiments were not refined, as will be inferred from the method used. They were carried no further than was necessary to secure the information desired. They can hardly be regarded as conclusive regarding the size of the error to be expected if no thermal insulation is used, for it is difficult to calculate even approximately the relative insulating effects, as regards the bulb, of a porcelain joint and of the long, thin stem. Without joints, or even with them, the temperature gradient in the stem must be concentrated near its upper end, and the insertion of an insulating joint at this point may be expected to have a much larger effect on the effective thermal conductance of the combination than would be calculated on the assumption of a uniform temperature gradient longitudinally in the stem and radially in the joint. However, it is at least reasonable to infer from the above experiments that the error in the *measured temperature drop through the plug* due to the cause we are considering would be of insignificant magnitude, whether insulating joints are used or not. For such an error would be that occasioned by conduction down the stem of the low side thermometer, with a temperature difference between the head and the bulb not exceeding 5°, while in the above experiments this temperature difference was at least 140°, since it was observed that naphthalin would freeze on the head of the thermometer. On the basis of the above data, the omission of insulating joints could then affect the measured difference of temperature by not more than $0.02 \times \frac{5}{140} = 0°.0007$, and if we assume that the effective thermal resistance of the stem is even as much as 10 times that of the insulating joints, conduction down the stem of a thermometer without joints would produce an error in the temperature difference of only 0°.007, which is not greater than arises accidentally from uncontrollable causes.

The thermometers used in all the work on the Joule-Thomson effect described in this paper have the upper porcelain joint, but not the lower one. It is difficult to get an absolutely steam-tight joint, and even a small leak at the bulb would obviously be disastrous, while such a leak at the upper joint would be of much less moment. Besides, if porcelain joints are to be used at all, the upper one is probably of more value than the lower.

### *c.* *Radiation effects.*

In a further study of the question of heat-conduction down the walls of the thermometer tube, undertaken about a year after the experiments just described, some unexpected effects due to radiation were noticed. These experiments were made with a specially-constructed thermometer, the coil of which is illustrated in Fig. 9. As is indicated, the coil is wound on a mica cross supported by means of four quartz tubes which also serve to insulate the leads. The containing tube, of uniform bore throughout and large enough to permit the coil to slide in or out easily, may be of either steel or glass. No insulating joints were used. This thermometer is called No. 8 in what follows.

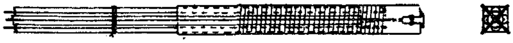

FIGURE 9. Coil of the experimental platinum resistance thermometer No. 8.

The experiment consisted in inserting this thermometer, together with a comparison thermometer (the one used in the experiments just described, hereafter called No. 4) in the vapor of naphthalin boiling in the hypsometer (see below, page 762) regularly used for comparing thermometers. Several measurements, extending over twelve days, were made with both the steel tube and the glass tube. The apparent temperature by No. 8 was invariably *higher* when the steel tube was used. Omitting one or two measurements, doubtful because some naphthalin penetrated to the interior of the glass tube before a proper method of packing it was discovered, the results were as follows, in chronological order:

TABLE I.

| Experiment No. | Kind of tube | $R_8 - R_4$ Ohms | °C* |
|---|---|---|---|
| 1 | glass | 0.06833 | 3.578 |
| 2 | glass | 0.06820 | 3.571 |
| 3 | steel | 0.06913 | 3.620 |
| 4 | steel | 0.06887 | 3.606 |
| 5 | glass | 0.06824 | 3.573 |
| 6 | steel | 0.06891 | 3.608 |

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* This is the temperature equivalent of $R_8$–$R_4$ on the scale of 4. It would be slightly different on the scale of 8, but the hundredths of degrees would be the same on the two scales. It is of course immaterial which scale the difference is referred to, as far as the comparison is concerned.

The mean apparent *rise* in the temperature when the glass tube is replaced by the steel tube is 0°.037 C; as will be seen, the difference is quite uniform. Being in the direction opposite to that of the difference which could be caused by conduction along the tube, it was attributed to radiation. To test this, a further measurement was made with the inside of the glass tube blackened to resemble the inside of the steel tube. The result was $R_8$–$R_4$ = 0.06895 ohm = 3°.610 C., which is practically the value obtained with the steel tube. In addition to verifying the suspicion that the difference was due to radiation, this result substantiates the conclusions reached in the experiments described under *b* regarding the effect of heat conduction along the thermometer tube. The steel tube of No. 8 was of twice the diameter of the stem of No. 4 and also somewhat thicker, having an area of cross-section perhaps three times as large as that of No. 4.

The surprising magnitude of this radiation error was sufficient effectually to discourage the use of thermometers of the type of No. 8 in the work of the main research. If it is thought that it ought also to arouse suspicion as to the existence of a comparable error in the case of the thermometers actually used, it may be replied that the coil of 8, wound in contact with quartz tubes of high reflecting power and separated by a comparatively thick layer of air from the walls of the containing tube, is necessarily far more susceptible to radiation effects than the coil of a thermometer of the type of No. 4, wound on a strip of mica and fitting snugly into its flat tube, from either side of which it is separated by only a thin layer of mica. Moreover, such radiation error as actually exists must be practically the same for both high side and low side thermometers, and hence without effect on the *difference* of temperature.

### *d. Effect of heat conduction along the thermometer leads, and of unequal temperature distribution in these leads.*

To test this, a wing of thin sheet copper was attached to one of the binding posts of the coil circuit of thermometer No. 8 during an ice point calibration. The apparent elevation of the freezing point observed when this wing was heated to redness to within one inch of the binding post was 0.00102 ohm = 0°.53 C. The heat was sufficient to oxidize the copper binding post very appreciably. While the magnitude of this difference seems disturbingly large at first glance, a simple calculation shows that the actual effect in the case of a measured difference in temperature in a Joule-Thomson experiment would be insignificant. In such an experiment, the temperature difference existing between the bulb of the thermometer and the binding-posts at the head is not, as a rule, in excess of 5° C., whereas, in the above test, this difference must have been at least 200° C. This would signify nominally an error of 0°.013 in the measured temperature difference; but in the above-described test, only one of the four binding-posts was heated, so that the ordinary function of the compensating circuit was wholly inoperative. This is of course not the case with the Joule-Thomson experiment, where we may reasonably expect the nominal difference of 0°.013 to be eliminated to within 10 per cent. of itself at most.

To test the effect of inequalities in the temperature of the leads connecting the thermometers with the bridge, large differences of temperature in the coil-circuit and compensating-circuit leads were artificially created. Several tests of this general nature showed that no error from this cause need be feared with the leads as arranged under normal operating conditions. The leads used were of No. 12 solid copper wire, about 22 feet long; resistance of a pair, about 0.037 ohm.

### *e. Contact resistances.*

With unplated copper binding posts on the thermometers, it was found necessary to keep them and the lugs attached to them carefully polished, and to set up tightly on the binding post nuts, in order to be sure that the net contact resistance might not exceed $0.0_{4}44$ ohm (0°.0022 C), the limit of sensitivity of the bridge. With silver-plated binding posts, much less care was necessary, and the comparatively small amount of work needed to put on a thin plating of silver is amply repaid by the abolition of the continual necessity of fussing over contacts and by the additional security felt regarding experimental results.

### *f. Variations in lead-resistance.*

It is necessary in work of this sort to keep track constantly of the resistances of the leads from the apparatus to the bridge, particularly if all or portions of these are of stranded wire, as is sometimes necessary for the sake of flexibility. Mere bending of a stranded lead will sometimes produce a noticeable change in its resistance. An unnoticed parting of a strand will frequently cause a very appreciable change in the lead resistance. Solid leads are much to be preferred, wherever it is possible to use them.

### *g. Calibration of differential thermometers.*

The measurement of a change of temperature in continuous flow work involves the use of two thermometers, which must be as nearly alike as it is possible to get them, if the benefits of a direct measurement of the difference of their resistances (as against the separate measurement of the resistance of each) are to be completely retained. Since it is not possible to make two thermometers exactly alike, there will always exist a difference in their resistances when they are at the same temperature. It will be convenient to speak of the combination of two thermometers used in this way as a *differential thermometer*, and to call the difference in their resistances at a common temperature the *normal resistance* of the differential thermometer. A particular differential thermometer will be referred to by means of a number made up of the numbers of the component thermometers, the number of the component thermometer having the higher resistance coming first: e. g., 67 is the differential thermometer composed of the individual thermometers 6 and 7. The normal resistance of this thermometer will be denoted by $R_{67}$, etc.

In order to get the difference of temperature corresponding to a measured difference of resistance, the normal resistance of the differential thermometer must be eliminated from the latter. This may be done by calibrating the differential thermometer, determining its normal resistance as a function of the temperature, and subtracting from a measured difference of resistance the normal resistance of the thermometer as calculated for the temperature in question; or, the experiment in which the measured difference of resistance is obtained may be repeated with the component thermometers interchanged. In most of the work described in this paper, both methods have been used; but, except where for some reason the interchange method was not used, the actual elimination has been effected by this method, the calibration serving merely as a check. There is little evidence to indicate that the value of the normal resistance as given by one of these methods is different from that given by the other; thus, the mean of six values of ($R_{67}$ by calibration — $R_{67}$ by interchange) at 165° C. is 0.00003 ohm or about $0^{\circ}.001_5$ C. with an average deviation of 0.00025 ohm or about 0°.012 C. The time interval involved in this work is about a year and a half. Of the six calibrations of the differential thermometer involved, two were made necessary by reason of accidents necessitating repairs to the thermometers which involved changes in their constants. Each of the values of $R_{67}$ (a single one excepted) by interchange is the mean of from three to five determinations with the same set-up of apparatus, but with varying flow rates. The greatest average deviation involved in any of these five sets of interchange values of $R_{67}$ was 0.00018 ohm, ($0^\circ.009$ C.); the least, 0.00007 ohm ($0^\circ.003_5$ C.); the mean, 0.00010 ohm ($0^\circ.005$ C.). Thus the interchange method gives results which are more consistent among themselves than they are with the results obtained by calibration. It cannot be supposed, however, that this signifies a real difference of the indicated magnitude in the values of the normal resistance by the two methods, for the reason that uncontrollable fluctuations in this resistance between calibrations are at least as great as the discrepancy between its values as obtained by interchange and by calibration. The size of these fluctuations is shown in Fig. 10, which exhibits the behavior of $R_{67}$ at 0° C. over a period of eighteen months. It is partly because it is not always convenient to calibrate the thermometer immediately before or after (or both before and after) a set of runs that the interchange method has been used to eliminate the normal resistance; partly also because, by this method, the thing eliminated is the normal resistance under actual conditions of use. This last reason was of more weight before experience had proven that the use of the differential thermometer under pressure did not affect its normal resistance, as conceivably might have been the case: in calibration, of course, the normal resistance is determined at atmospheric pressure.

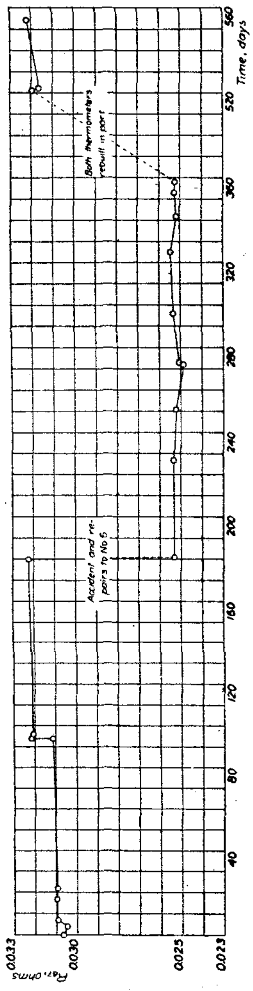

FIGURE 10. Resistance at 0° C. of the differential thermometer 67.

In spite of the fact that the calibration is not, in general, used directly in calculating the temperature difference from the resistance measurements, it is distinctly worth while, in the writer's opinion, for the following reasons:

First, it gives a valuable check on elimination by interchange; second, it affords a very useful indication of the precision of the calorimetric work; for the value of the normal resistance as determined by interchanging thermometers in a Joule-Thomson experiment includes the accidental errors of the experiment, so that the divergence of this value from the calibrated value is an index of the magnitude of these errors; third, if through some accident or oversight in setting up the apparatus one of a pair of runs has to be rejected, the other may still be used; for example, of a certain set of nine runs, the first four (with thermometer No. 7 on the high-pressure side of the plug) had to be thrown out because of a fault in one of the leads from the plug to the bridge which was not discovered until after these runs had been completed. The value of the remaining runs, with No. 6 on the high side, was not affected, because a calibration made immediately after afforded the means of eliminating the normal resistance.

In calibrating the differential thermometer, the chief difficulty lies in getting the two thermometers simultaneously at the same temperature when this temperature exceeds that of the room by say 150° or more. At least two such points are necessary. It is probable that the most satisfactory method of obtaining them would be to employ a double oil bath, provided a sufficiently sensitive thermostat for the higher temperatures could be devised. The method employed by the writer made use of a large, specially-constructed hypsometer of the well-known Regnault type, in which the two thermometers could be exposed to the vapors of substances which boil at convenient temperatures. The hypsometer was heavily lagged with asbestos except for the reservoir containing the liquid and for an air condenser by means of which most of the vapor was liquefied and returned to this reservoir. The substances used in this apparatus were water, cumol (about 165° C.), naphthalin (218° C.), diphenylamin (310° C.). Water, of course, gives no trouble whatever. Cumol, in spite of its comparatively low boiling point, has always been unsatisfactory, chiefly because its boiling point rises continuously as vaporization progresses, even if the liquid has been previously distilled. Naphthalin is very satisfactory. Diphenylamin has been little used, as its boiling point lies beyond the temperature range required in the work discussed in this paper. It has the same disadvantage as cumol as regards rise of the boiling point during vaporization, though in a lesser degree.

It is doubtful, however, whether the hypsometer method of calibration would be satisfactory at temperatures as high as that of boiling diphenylamin, even if the boiling point were perfectly steady. It was found early in the work that even when naphthalin was used, the temperature in the region occupied by the two thermometer bulbs was not uniform, but that the apparent value of the normal resistance of the differential thermometer could be changed by an amount of the order of one or two hundredths of a degree on the scale of either thermometer by merely interchanging the positions of the thermometers. This effect was not accidental, but could be obtained repeatedly with great uniformity and definiteness; because of this, it could with confidence be regarded as eliminated by taking the mean of a number of measurements equally distributed between direct and reversed positions of the thermometers. To facilitate this method of observation, which was always used, the top of the hypsometer, from which the thermometers were suspended, was made capable of rotation about a vertical axis, with respect to which the two thermometers were symmetrically arranged; thus the thermometers could be made to exchange positions without withdrawing either from the hypsometer or disconnecting it from the bridge.

This non-uniformity in temperature of the vapor column in which the two thermometers were immersed was first noticed in a much simpler comparator than that which has been described. This earlier comparator consisted merely of a large glass tube (3 inches in diameter) of uniform bore, closed at one end and lagged on the sides with asbestos. With this apparatus, the apparent normal resistance of the differential thermometer could be changed by an amount equivalent to 0°.16 C. by merely interchanging the positions of the thermometers,—which signifies a difference of 0°.08 C. in the temperature of the vapor (naphthalin) at two points at the same level and about 1.5 inches apart. This difference was not accidental; in an experiment lasting over three hours, the apparent differential resistance in either position of the thermometers remained constant to within less than 0°.01 C. The only evident asymmetry of arrangement which might account for this rather remarkable effect consisted in the fact that the thermometer bulbs were always on a line perpendicular to a brick wall near which the apparatus was set up; but the phenomenon was not in the least altered by the use or omission of radiation shields around the thermometer bulbs, and has been noticed, in the Regnault comparator, even when the above-mentioned asymmetry was avoided.

The results of the calibration of a differential thermometer are expressible in the form of a relation quadratic in the temperature or in the resistance of one of the component thermometers. As far as convenience in computation of temperature differences is concerned, there is not much to choose between these two methods of representation. As a check on the accuracy of the latter, three differential thermometers, made up of thermometers 4, 6 and 7, were calibrated at the ice, steam and napthalin points, and the differential thermometer 67 was also calibrated at the boiling point of cumol. The relation $R_{64} = R_{67} + R_{74}$ should hold for all temperatures. The observed values of the three resistances satisfied this relation to within 0.00013 ohm (0°.0065 C.) at the steam and naphthalin points, and to within 0.00006 ohm (0°.003 C.) at the ice point. These results may be regarded as satisfactory, since the closing errors are to be divided among the three thermometers. After forcing the closure at each point, the following equations were obtained:

$$\begin{aligned} R_{67} &= 0.03071 + 0.000532\,(R_7-5.3003)-0.0003554\,(R_7-5.3003)^2 \\ R_{74} &= 0.01823 + 0.027436\,(R_7-5.3003)-0.0000354\,(R_7-5.3003)^2 \\ R_{64} &= 0.04894 + 0.028009\,(R_6-5.3310)-0.000396\,\;(R_6-5.3310)^2 \end{aligned}$$

From these one finds at the boiling point of cumol (165°.7, $R_6 = 8.748$, $R_7 = 8.720$):

$$R_{67} = 0.02837,\quad R_{74} = 0.11163,\quad R_{64} = 0.13992$$

so that the closing error is 0.00008 ohm (0°.004). The observed value of $R_{67}$ at cumol was 0.02842 ohm. Similarly one finds, for example, at 130° C. ($R_6 = 7.998$, $R_7 = 7.969$)

$$R_{67} = 0.02960,\quad R_{74} = 0.09120,\quad R_{64} = 0.12082$$

from the above equations: here the closing error is only 0.00002 ohm.

It should, however, be stated that the satisfactory results set forth in the preceding paragraph must be regarded as partly accidental — or, at least, that more care was taken in the experimental work from which they were derived than limitations of time ordinarily permit. Thus three later calibrations of 67 give 0.00033, 0.00021 and 0.00013 ohm respectively for the difference between the observed normal resistance in the vapor of boiling cumol and the value of this resistance as calculated from an equation of the above form in which the constants are determined from observations at the ice point and at the boiling points of water and naphthalin. These discrepancies are undoubtedly due in large measure to the difficulties of the calibration at the cumol point, where it has always been hard to secure conditions sufficiently steady to warrant confidence in the results. In fact, cumol was finally abandoned altogether because of this trouble.

The difference in magnitude between the coefficients in the equation of calibration of the differential thermometer 67 and the corresponding coefficients in the other two equations is due to circumstances which are worth a brief remark. The two thermometers 6 and 7 were made of consecutive pieces of platinum wire taken from a spool marked 'Heraeus' purest,' and no pains were spared to make these thermometers exactly alike. The two pieces of platinum were annealed at a red heat, and their resistances were adjusted to equality within less than 0.0005 ohm at room temperature. After winding, the platinum was once more annealed and the net resistances of the two thermometers were again equalized by making slight adjustments in the compensating circuits. The resistance of each was also now very nearly that of No. 4. The two thermometers were then inserted in their respective tubes and the caps which close the bottoms of these tubes were brazed on. This operation nullified all the efforts at equality of resistance which had been made. The resistances at zero differed by nearly 100 times as much as before brazing, and each exceeded that of 4. Moreover, as the equation of calibration shows, the normal resistance of 67 *decreases* slightly with rising temperature, although it is so nearly constant over a considerable range of working temperature that the actual temperature need not be known accurately so far as elimination of the normal resistance is concerned; this is a convenience rather than otherwise.

Thermometer No. 4 was made of Baker platinum, and it is therefore not to be expected that it would combine with either 6 or 7 to form a differential thermometer as satisfactory as 67 itself; in fact, both $R_{74}$ and $R_{64}$ vary much more rapidly with the temperature than $R_{67}$, as is indicated by the equations of calibration. The dissimilarity of 4 to 6 and 7 is further shown by a comparison of the Callendar constants for the three thermometers. The constant $\delta$ has the values 1.54, 1.65, 1.58, and the constant $K \left(= \frac{R_{100} - R_0}{100\, R_0}\right)$ the values 0.003795, 0.003859, 0.003882, for 4, 6 and 7 respectively. The value of $\delta$ is ordinarily taken as a criterion of the purity of the platinum: if $\delta$ exceeds about 1.50, the platinum is regarded as impure. On the other hand, values of $K$ as large as those given for 6 and 7 indicate a high degree of purity in the platinum. There thus seems to be some doubt whether a large value of $\delta$ is, by itself, a reliable indication of impure platinum. The writer has learned in conversation of a thermometer made of the best platinum obtainable, the $\delta$ of which was 1.68; and in a paper on the mechanical equivalent of heat,[5] Roebuck describes two thermometers made from the same sample of platinum, for which the $\delta$'s were respectively 1.748 and 1.567, with fundamental coefficients of nearly equal magnitudes (0.0035127, 0.0035387).

### *h. Calibration of individual thermometers.*

The temperature scale in all the work here described depends on calibrations in ice, steam and naphthalin vapor. The naphthalin used was Kahlbaum's Reagent, its boiling point being taken as

$$t_n = 218.0 + 0.058\,(H - 760)$$

in which $H$ is the pressure in millimeters of mercury under standard conditions.[6] Ordinary drug-store naphthalin boils at almost the same temperature. The apparatus used for the boiling point of naphthalin was that which has become standard for the sulphur point.[7] Drip-cones of asbestos paper and aluminum radiation shields were regularly used, but little effect was noted when either or both were omitted.

Some care was found necessary at the ice point. A thermos bottle was used in the earlier work, but the apparatus finally found most reliable consisted of a 3-inch glass cylinder, lagged with asbestos, and provided with a funnel-shaped bottom, to which a rubber tube with stop-cock could be attached. The cylinder was long enough to permit immersion to the head of the thermometer. It was found expedient to cool the thermometer bulb and stem in a separate ice-bath before immersion in this ice-point apparatus. Experiments conducted to determine whether there was any difference in the freezing points as given by commercial (natural) ice and by ice made from distilled water showed no difference to the order of the accuracy of the resistance measurements, and the commercial ice was thereafter regularly used, because of the lack of facilities for making the other.

Distilled water was always used at the steam point.

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[5] Phys. Rev. (2), **2**, 79 (1913).

[6] See Waidner and Burgess, Bull. Bur. St., **7**, 1–9 (1911).

[7] See Waidner and Burgess, Bull. Bur. St., **6**, 149–230 (1909–10).

### *i. Resistance Measurements.*

All resistance measurements have been made by means of a Carey-Foster bridge and depend ultimately on Wolff standards certified by the Physikalisch-Technische Reichsanstalt. All contacts on the bridge except those in the galvanometer and battery branches are mercury contacts. The bridge has two slide wires, both of manganin, the finer (0.054 ohm/cm.) being used for measuring the resistance of the high side thermometer, balanced against one or more secondary standard coils. The other wire (0.00445 ohm/cm.) was used in measuring the difference of resistance of the two thermometers, balanced against each other. The galvanometer was sufficiently sensitive to indicate a motion of the slider of 0.1 mm. from the point of balance on the coarse wire, so that the order of the precision of the differential measurements is $0.0_445$ ohm, or about 0°.0022 C. The precision of the measurement of high-side temperature, in so far as it depends on the resistance measurement, is about 0°.03 C.

## IV. EXPERIMENTAL RESULTS ON THE JOULE-THOMSON EFFECT.

### 1. Preliminary Discussion. Methods of Eliminating the Effect of Heat-leak.

Before beginning the consideration of the results obtained with the various types of throttling apparatus which have been described, it seems desirable to outline briefly a few points concerned with the general question of heat leakage and the methods which have been or may be used to eliminate its effect from the directly measured ratio of temperature drop to pressure drop. If we call the last mentioned quantity the 'apparent Joule-Thomson effect' and denote it by $\mu'$, we shall have

$$\Delta T = \mu \cdot \Delta p - \frac{\delta Q}{f C_p} \tag{1}$$

and hence

$$\mu' = \mu - \frac{\delta Q}{f \Delta p C_p} \tag{2}$$

in which $\mu$ is the true Joule-Thomson effect, $f$ is the flow (mass of fluid passing in unit time), $\delta Q$ is the heat leakage in unit time (counted positive if heat is received by the fluid), $C_p$ is the specific heat at constant pressure and $\Delta T$ and $\Delta p$ respectively the temperature and pressure drops, assumed to be so small that $\mu$ and $C_p$ are sensibly constant over the temperature-pressure interval covered in the experiment.

To eliminate the effect of heat leakage, experiments at approximately the same pressure and temperature, but with different rates of flow, must be conducted. The flow may be varied without varying the temperature drop by using different plugs, and if it may be assumed that the various circumstances which affect the total heat-leakage remain the same for all members of a set of experiments made in this manner, it will be possible to eliminate the effect of heat-leak from the results. A second method of varying the relative magnitude of that portion of the observed effect which is due to heat-leak is to vary the pressure drop and the flow together, using the same plug throughout the set of experiments.

The first of these methods involves more labor and greater experimental difficulty than the other, and although, in the work under discussion, a number of plugs giving widely different flows at the same pressure drop have been used, the second method is the one which has been employed in arriving at a result that is believed to be free from leakage errors.

The application of this method to the elimination of the heat leak effect depends, 1°, upon an experimental fact, and 2°, upon an assumption.

1°. The experimental fact is that when a set of $\mu'$'s obtained with a single set-up of the apparatus are plotted against the reciprocals of the corresponding flows, the curve so determined is a straight line, within the limits of experimental error. (One or two possible exceptions to this statement will be noted below.) It follows from this that, within the range of experiment, the leakage term in equation (2) must be of the form

$$\frac{\delta Q}{f \Delta p C_p} = A + \frac{B}{f} \tag{2a}$$

in which $A$ and $B$ are constants. $B$ is evidently the negative of the slope of the experimentally determined straight line just mentioned.

Plots of $\mu'$ vs. $\frac{1}{f}$ are shown in Figs. 12 and 13. Table III contains the data on which these plots are based, together with certain other significant data.

2°. The assumption is that $A = 0$. The confidence one feels in the correctness of this assumption is based on the fact that to deny it is to assert that $\delta Q$, the heat leak per unit time, contains a term which is proportional to $f\Delta p$. As will appear later (see equation (4) and Table II) this is the same thing as saying that $\delta Q$ contains a term approximately proportional to $(\Delta p)^{3/2}$ (or to $f^3$) in the case of axial flow plugs, and to $(\Delta p)^{7/4}$ (or to $f^{7/3}$) in the case of radial flow plugs. Now the probability of the existence of an appreciable heat leak of this sort is, on the face of things, extremely slight, because $\Delta T$ is always at least roughly proportional to $\Delta p$ (see Figs. 15 and 16) and it is hardly credible that there can be any part of $\delta Q$ which increases more rapidly — 50 to 75 per cent. more rapidly — than $\Delta T$. In putting $A = 0$ we are therefore not merely making an assumption which seems the most plausible of several alternative ones; we are, on the contrary, unable reasonably to make any *other* assumption.

If $A$ is zero, $\mu$ may be calculated from *any* point on the line mentioned above by adding the amount $B/f$ to the ordinate at this point. Obviously the easiest method of doing this is to extrapolate the line to the axis $1/f = 0$. This 'extrapolation to infinite flow' presupposes nothing whatever as to the practicability, or even the theoretical possibility, of obtaining an 'infinite flow' of steam, or any other flow lying outside the experimental range. It is merely a graphical method of arriving at the value of $\mu$ which is necessarily involved in the straightness of the line and the absence of any part of $\delta Q$ proportional to $f\Delta p$.

It is consequently necessary to establish only the fact that the graph is truly rectilinear over any range of observation, however short, in order to avail one's self of this method of elimination. For example, if the entire range of observation included a portion known to be rectilinear, with a piece of curved graph at one or both ends, the latter could be entirely ignored. But as a practical matter, it is possible to be reasonably certain that the graph is really straight over any range only by observing that it is straight, within experimental error, over a reasonably extended range of observation. It is easy to imagine that terms of higher orders in $\frac{1}{f}$ might be present in the right hand side of equation (2a), and that these terms, although having so comparatively insignificant an effect within the range of observation as to escape detection if the range is small, might nevertheless seriously affect the value of $\mu$ obtained by extrapolation. If one actually obtained an experimental curve like that imagined above, with a portion rectilinear within experimental errors and another, curved portion nearer the $1/f$ axis, for example, it would not be possible to convince one's self that the true $\mu$ could be obtained by extrapolating the straight part. On the contrary, the extrapolation of the curved line would certainly be expected to lead to the true value of $\mu$, and, if it were possible to perform it, the process would differ in no essential respect from the rectilinear extrapolation — that is, it would be merely a mathematical process, with no implied assumption regarding the possibility of physically realizing the inferred portion of the curve. The true value of $\mu$ is reached by the extrapolation because the terms multiplying the first and higher powers of $1/f$ vanish for $f = \infty$; and the true value of $\mu$ could equally well be obtained by direct calculation for any $1/f$, since the coefficients of these powers must theoretically be known to perform the extrapolation.

The results obtained by the rectilinear extrapolation of the data obtained with several plugs are certainly considerably too small. This is true of the axial flow plugs A1, A2 and A3 and of the radial flow plugs U1 and U2. (Cf. Figs. 12 and 13.) In all of these cases, the origin of the error must be ascribed to the presence of curvature which either was ignored (because it was impossible to take account of it in the extrapolation), as in plugs U1 and U2, or was not detectable, owing to insufficient range of observation, as in plugs A1, A2 and A3.

It is hardly to be supposed that any line would prove to be absolutely straight, if it were possible to obtain results entirely free from accidental error. Hence, although it is perfectly true that the rectilinear extrapolation is merely a convenient method of obtaining a result which could just as well be obtained by calculations confined to the region of observation if the plotted line were known to be strictly rectilinear, there will exist errors which may be properly called extrapolation errors, in the sense that their magnitude will depend upon the probability that the rectilinearity of the graph is a correct inference from the data, and in the sense that this inference and the value of $\mu$ deduced from it are the less certainly correct the greater the accidental errors and the greater the ratio of the range of extrapolation to the range of observation. It is also evident that, of two extrapolations which are alike in the two respects just named, that one for which the variation of the leak effect ($\delta Q/f\,C_p\Delta p$) with the flow is the greater — that is to say, that one for which the slope of the line is numerically the greater — will lead to a less trustworthy value of $\mu$ than the other.

Mr. S. A. Moss has pointed out to the writer that if one may assume that the total heat-leak per second, $\delta Q$, is proportional to the drop of temperature, $\Delta T$, — certainly an assumption which appears more plausible than that $\delta Q$ is proportional to the drop of pressure — then a plot of $\frac{1}{\mu'}$ vs. $\frac{1}{f}$ should be straight, and the extrapolation of this plot should lead to the correct value of $\frac{1}{\mu}$. This appears on dividing equation (1) by $\mu \cdot \Delta T$; this gives

$$\frac{1}{\mu} = \frac{1}{\mu'} - \frac{\delta Q}{\mu C_p \Delta T} \cdot \frac{1}{f} \tag{3}$$

which represents a straight line in the variables $\frac{1}{\mu'}$, and $\frac{1}{f}$, provided $\delta Q / \Delta T$ is constant.

A few plots of $\frac{1}{\mu'}$ vs. $\frac{1}{f}$ are shown in Fig. 14.

A moment's consideration of the geometrical relations of the two plots will show that if either is straight, with a slope not zero, the other cannot be straight; that if either is straight and approximately horizontal, the other will be nearly straight, and will also be approximately horizontal; and that if 'best representative straight lines' are drawn through corresponding experimental sets of points on the two plots, these lines will have intercepts on the axis of $\frac{1}{f}$ which will be, approximately, the reciprocals of each other, if the two lines are nearly straight and nearly horizontal. The consequence of this is that where the heat-leak effect varies so little within the range of experiment that the extrapolation seems worthy of confidence, it makes little difference which plot is used. At the same time, the results of extrapolating both lines, even in such a case as this, may be seriously in error; that is, *mere* absence of large slopes in both plots does not signify necessarily that their intercepts are true values of $\mu$ or of $\frac{1}{\mu}$; see, for example, the curves U1 in Figs. 12, 13 and 14.

Still another method of exhibiting graphically the results of Joule-Thomson experiments is illustrated in Figs. 15 and 16, in which the observed temperature drop is plotted as ordinate against the pressure drop as abscissa. In the absence of heat-leak, all plotted points should lie on a straight line passing through the origin, and that portion of any ordinate lying between the locus of observed points and a line through the origin having a slope equal to the true value of $\mu$, represents, for the pressure drop in question, the heat-leak per unit mass of fluid, occurring during transit through the plug, divided by the specific heat at constant pressure.

It will be seen from Figs. 15 and 16 that a simple linear equation can express the relation between $\Delta T$ and $\Delta p$ with fair accuracy in most cases. With the further aid of a relation connecting flow and pressure drop, both of these quantities may be eliminated from equation (2), leaving an equation expressing $\delta Q$ as a function of $\Delta T$ only, provided $\mu$ is known. Such relations are of some value in a comparative study of heat leakage in different types of apparatus.

The relation between flow and pressure drop which has just been mentioned is expressed quite accurately by

$$f = K\,(\Delta p)^r \tag{4}$$

for the present experiments. Here $r$ is approximately $\frac{1}{2}$ for axial flow plugs and approximately $\frac{3}{4}$ for radial flow plugs. In Fig. 11, logarithmic plots of flow against pressure drop for several plugs are given. Table II gives the constants of the equation (4).

TABLE II.

Constants of equation (4). Flows in kgm./hr. Pressure drops in kgm./cm².

| Plug | $K$ | $r$ | Plug | $K$ | $r$ |
|---|---|---|---|---|---|
| A 1 | 18.6 | 0.462 | V 1* | 35.2 | 0.689 |
| A 2 | 29.8 | 0.444 | V 2* | 24.1 | 0.799 |
| A 3 | 29.4 | 0.446 | U 1 | 28.0 | 0.749 |
| A 4 | 39.2 | 0.498 | U 2 | 25.6 | 0.765 |
| | | | S | 16.4 | 0.751 |

## 2. Discussion of the Experimental Results.

### *a. General.*

In the plots of Figs. 11 to 16 inclusive, the designations attached to the several curves have the following significances:

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* The relation (4) is only roughly true for these plugs, owing to choking of the plug while in service. This is very noticeable with radial flow plugs, especially where the steam has not been previously strained through an alundum strainer. The same porous cup was used in all radial flow plugs, and the variations in the constants are due to the gradual 'aging' of the cup. The plugs are listed in chronological order.

1°. A1, A2, A3, A4 refer to four axial-flow set-ups, all similar in general respects to that shown in Fig. 1 and described in Section II, 1. In all of these plugs, there are 37 holes 1½ inches long arranged in a hexagonal pattern at each end of the soapstone block S; in plugs A2, A3, A4, the diameter of each of these holes is 0.086 inch. In plug A1 the diameter of each hole is 0.076 inch. In plugs A1, A2 and A3, a single piece of solid manganin wire having a diameter of 0.044 inch is threaded through all 74 holes; in plug A4, this is replaced with a stranded invar wire having ten strands, each of about 0.01 inch diameter. The block S′ is the same in plugs A1 and A2, in which the cross-channel is about ¼ inch in diameter and is located on the axis of the block. The thickness of each of the blocks $S$ and $S'$ is about two inches and the length of the cross-channel is about 4½ inches.

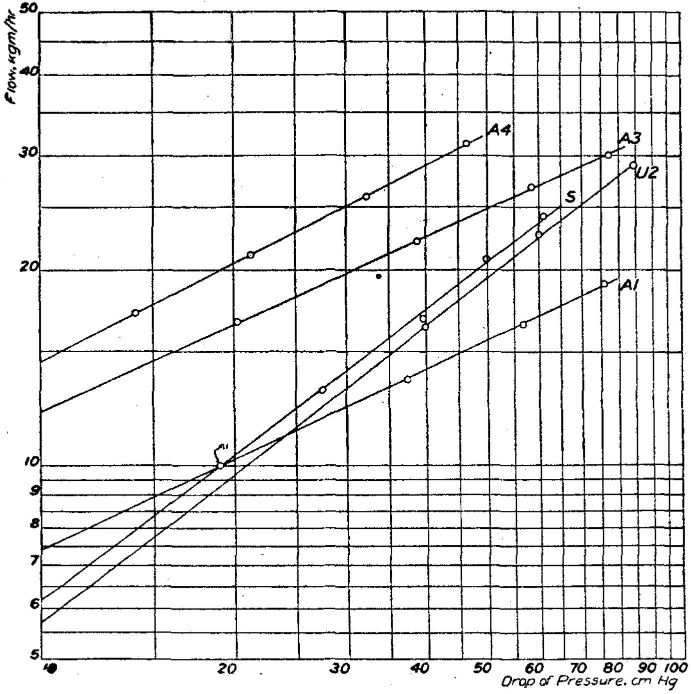

FIGURE 11. Logarithmic plots of flow against pressure-drop. Abscissae, pressure drops in cm. Hg. except for plug S, for which abscissae are one-half of pressure drops in cm. Hg. Ordinates, flow in kgm. per hour.

In the block $S'$ of plug A3, the cross-channel is located with its axis on a line about $1\frac{3}{8}$ inches from the lower face of the block. In other respects, the block $S'$ of A3 is similar to that of A1 and A2. The purpose of locating the cross-channel of A3 further from the lower face of the block than in A1 and A2 was to reduce the heat-leakage taking place in the cross-channel. In the block $S'$ of plug A4, the diameter of the cross-channel is $\frac{1}{2}$ inch, and it is located with its axis a little above the axis of the block, which is otherwise similar to the blocks used with the other plugs. By thus enlarging the area of the cross-channel, the velocity of the steam while passing through it was made practically the same as in the secondary superheater in the oil-bath. In plugs A1, A2 and A3, the lining in the tubes $T$ and $T'$ is of talc having a radial thickness of $\frac{3}{4}$ inch and an inside diameter of $1\frac{1}{2}$ inches. In plug A4, this lining is of poplox of about the same thickness, and a system of brass baffle plates separated by layers of copper gauze is located in each passage just above the soapstone block $S$. Copper baffles are also located in the chambers between blocks $S$ and $S'$. These chambers, in plugs A2 and A3, contain a few layers of copper gauze, and in plug A1 are empty. No baffling devices are located in the talc-lined passages of tubes $T$ and $T'$ in any of the plugs A1, A2, A3.

2°. V1 and V2 refer to two radial-flow set-ups in the V-type of plug-case, illustrated in Fig. 3 and described in Section II, 2, *b*. V1 represents results with the plug case externally lagged with poplox and wholly submerged in the oil bath, V2 results with the plug-case directly immersed in the bath with no intervening lagging. The two other arrangements used with this type of plug case ((ii) and (iii), page 742) resulted in large heat-leaks, as might have been anticipated, and are of no interest.

3°. U1 and U2 refer to two radial flow set-ups in the U type of plug-case, described in Section II, 2, *c*. These set-ups are similar in all respects except that, with U1, a copper gauze strainer was used in the chamber $M$, Figs. 5 and 6, while with U2, an alundum strainer was used. (Only one strainer was used with plugs A1, A2, A3, A4, V1, V2, U1 and U2, there being no strainer corresponding to that shown in chamber $m$, Figs. 5 and 6).

4°. $S$ refers to a radial-flow set-up of the straight-away type, described in Section II, 2, *d* and illustrated in Figs. 4, 5 and 6.

All the $\mu'$ vs. $\frac{1}{f}$ plots (Figs. 12 and 13) represent least-square adjustments of the observations. All the straight lines drawn on the $\frac{1}{\mu'}$ vs.

$\frac{1}{f}$ plots (Fig. 14) also represent least-square adjustments. The dotted curves on this diagram represent the corresponding least-square $\mu'$ vs. $\frac{1}{f}$ lines: where no dotted curve is drawn, the two loci are so close together as to be barely distinguishable. The numbers adjoined to each plotted point represent the number of runs involved, and, as a rule, also the weight attached to the point in making the least square adjustment.

The extrapolated value of $\mu$ (3°.182 C. cm.$^2$/kgm.) as given by the set-up $S$ is believed to be the true value of $\mu$ within $\frac{1}{2}\%$, for reasons which will appear in the following discussion. This value of $\mu$ is indicated on Figs. 15 and 16 by the slope of the straight dotted line passing through the origin. Throughout the following discussion, to avoid circumlocution, the value of $\mu$ with which these lines are associated will be spoken of as the true value of $\mu$.

The actual mean temperatures of the various experiments range from 162°.6 C. to 168°.9. In reducing the observed $\mu$'s to the standard temperature of 165° C., the value $-0.031$ cm.$^2$/kgm., taken from Fig. 7 of Davis' paper, 'On the Applicability of the Law of Corresponding States to the Joule-Thomson Effect in Water and Carbon Dioxide,'[8] was used for $d\mu/dT$. This value is undoubtedly sufficiently accurate for the purpose. A few measurements of $\mu$ for steam at 220° C. made by the writer verify the trend of Davis' curve. Moreover, the deviations of the temperatures at which the individual $\mu'$'s belonging to a given curve were experimentally determined, from the mean of these temperatures for this set of $\mu'$'s, are what are chiefly significant as regards error due to an erroneous value of $d\mu/dT$. This is because any error in the reduced values of these $\mu'$'s will be magnified in the extrapolation. In the case of every curve, these deviations were much smaller than the 2.4 to 3.9 degrees C. by which the temperatures of the extreme $\mu'$'s depart from 165° C.

In general, it is the high-side pressure rather than the mean pressure which was held approximately constant in the experimental work, so that, strictly, a reduction for pressure should also have been made before plotting the observations. No such reduction has been made. Its effect would certainly be almost negligible and would probably be to increase slightly the slopes of all the plotted straight lines of Figs. 12, 13 and 14, and thus also the $\mu$'s obtained by extrapolation.

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[8] Davis, Proc. Am. Acad., **45**, 243–264 (1910).

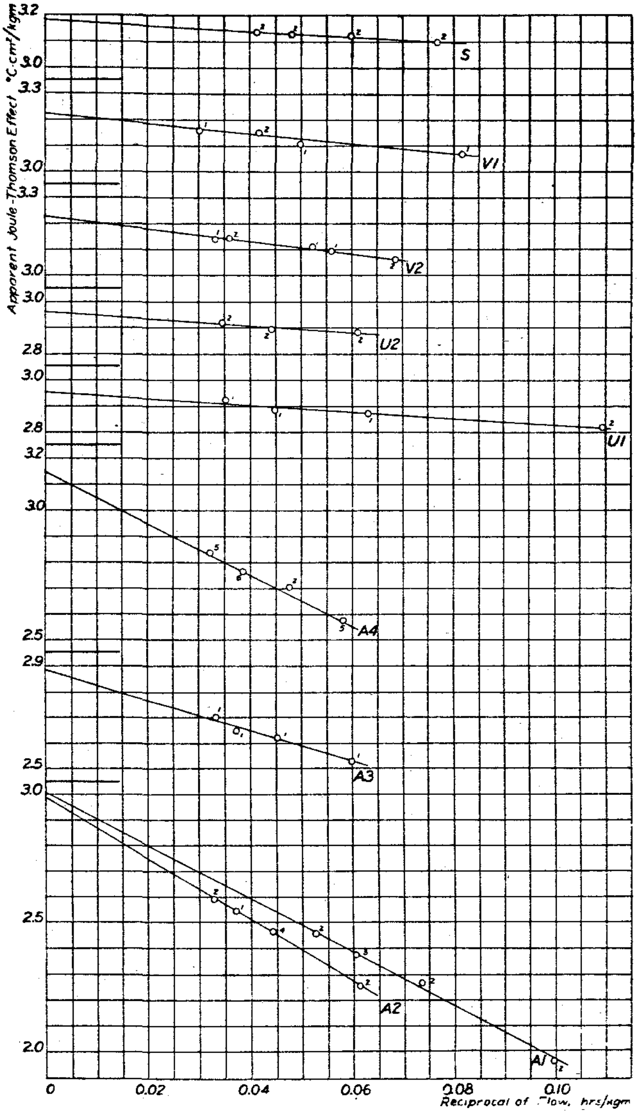

FIGURE 12. Apparent Joule-Thomson effect ($\mu' = \Delta T/\Delta p$) vs. reciprocal of flow. All $\mu'$'s are reduced to a mean temperature of 165° C. The average mean pressure, all values of $\mu'$, is 4.02 kgm./ cm.$^2$ abs. The average mean pressure for the runs with plug S is 3.86 kgm./cm.$^2$ abs. A1, A2, A3 and A4 refer to axial flow plugs, U1, U2, V1, V2, and S to radial flow plugs. The intercept of the line S on the axis of $\mu'$ is believed to be the true value of $\mu$, within $\frac{1}{2}$%. The numbers adjoining the plotted points indicate the number of runs involved.

It was deemed best to omit it because no reliable data for calculating it are available. The question of the pressure coefficient of $\mu$ is briefly considered in a later section of this paper (Section IV, 2, *h*). The pressure given in the title to Fig. 12 (4.02 kgm./cm.$^2$) is the average mean pressure for the nine types of plug. The average mean pressure for the set-up S is also separately given.

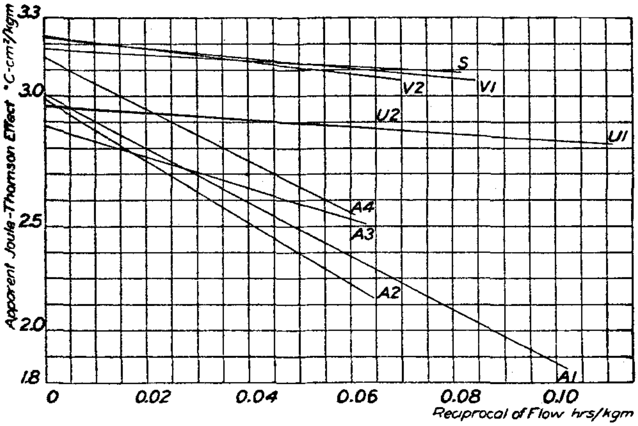

FIGURE 13. Curves of Fig. 12 shown with a common origin for comparison purposes.

### b. *Axial-flow Plugs A1, A2 and A3.*

It will be noticed immediately (Fig. 14) that the plots of $\frac{1}{\mu'}$ vs. $\frac{1}{f}$ for plug A2 shows very well marked curvature. The plots for plugs A1 and A3 are not shown, but are similarly curved. The plots for the other plugs (excepting possibly U1) do not present certain evidence of curvature which can be detected by the eye; but, with one exception (curve U1), in all cases in which the constants of the plot have been determined by the method of least squares, the residuals indicate that the $\mu'$ line represents the observed values of $\mu'$ with considerably more accuracy than the $\frac{1}{\mu'}$ line represents the observed values of $\frac{1}{\mu'}$. One concludes, therefore, that the heat-leak in unit time is not even approximately proportional to the temperature drop for the axial flow plugs, in which it is large, and that in general it

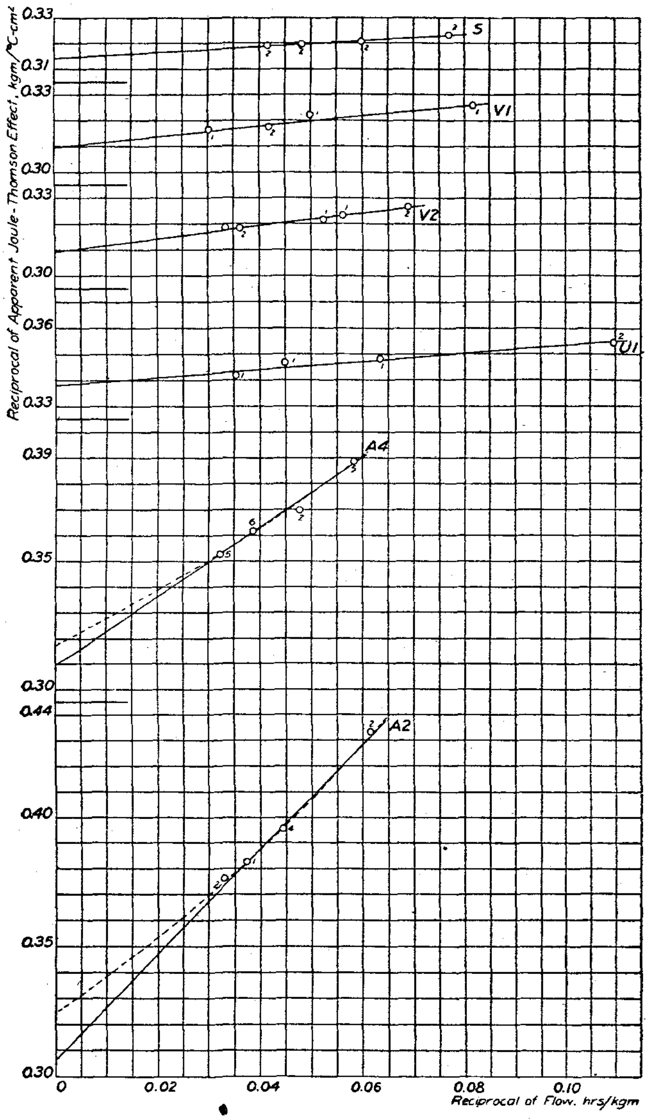

FIGURE 14. Reciprocal of apparent Joule-Thomson effect ($1/\mu' = \Delta p/\Delta T$) vs. reciprocal of flow. All $1/\mu'$'s reduced to a mean temperature of 165° C. The average mean pressure, all values of $1/\mu'$, is 4.02 kgm./cm.$^2$ abs. A2 and A4 refer to axial flow plugs, U1, V1, V2 and S to radial flow plugs. The numbers adjoining the plotted points indicate the number of runs involved.

varies more nearly as the pressure drop — is, in fact, in practically every case, proportional to pressure drop, within the limits of experimental error. Since the temperature drop increases more rapidly than the pressure drop in all cases, this means that the heat leak per unit time increases less rapidly than the temperature drop.

The very large leakage in plugs A1, A2 and A3 is partly due to the unusually happy combination for the promotion of heat-leak afforded by the long and rather constricted cross-channel used in these plugs. The comparatively small area of cross-section of the cross-channel, by forcing a high steam velocity and thus increasing the kinetic energy of the steam at the expense of its internal energy, depresses the temperature of the steam while it is in the cross-channel by an amount which, for high velocities, varies nearly as the inverse fourth power of the diameter of the cross-channel, for a given flow, and as the square of the flow for a fixed cross-section of the cross-channel. Since for these plugs the flow is approximately proportional to the square root of the pressure-drop, the temperature depression due to kinetic energy effect alone in the cross-channel will vary directly as the pressure drop. To this must be added the depression due to the throttling which has taken place in the high-side half of the plug, which is also proportional to pressure drop. Of course this total cross-channel depression of temperature is, in the steady state, reduced to some extent by the heat-leakage itself. It might be expected that the kinetic energy effect would predominate. This is not borne out by the facts, however; for plug A2, which permits more than 50 per cent. greater flow for a given pressure drop than plug A1 (see Table II) would produce a kinetic energy temperature depression more than twice as great as plug A1 for a fixed pressure drop, while the throttling depression would be the same for both. If then the kinetic energy effect were the dominant one, the heat leak per gram of steam at a fixed pressure drop might be expected to be greater for plug A2. The contrary is the case, as is shown by the $\Delta T$ vs. $\Delta p$ plot. The heat-leak per gram of steam per unit pressure drop for a given flow is, however, greater for plug A2, as is indicated by the $\mu'$ vs. $\frac{1}{f}$ plot. This is because the pressure drop for a given flow is smaller for A2 than for A1.

The $\Delta T$ vs. $\Delta p$ plots for these three plugs are all concave upward, though only slightly so; this means that the heat leak per gram of steam, which increases fairly rapidly with the pressure drop throughout the range of the experiments for all three plugs, increases a little

## TABLE III.

Summary of plotted values of apparent Joule-Thomson effect and of its reciprocal, with other pertinent data.

All $\mu'$'s reduced to 165° C mean temperature. $p_m$ signifies mean of high- and low-side pressures, $f$ signifies steam flow. The last column indicates the number of runs involved for each plotted point.

| Plug | $p_m$ kgm/cm² | $f$ kgm/hr. | $\frac{1}{f}$ hr./kgm | $\mu'$ $\frac{°\text{C cm}^2}{\text{kgm}}$ | $\frac{1}{\mu'}$ $\frac{\text{kgm}}{°\text{C cm}^2}$ | No. of runs |
|---|---|---|---|---|---|---|
| A1 | 4.30 | 10.00 | 0.10000 | 1.964 | 0.5091 | 2 |
| | 4.06 | 13.56 | 0.07373 | 2.266 | 0.4413 | 2 |
| | 4.00 | 16.49 | 0.06065 | 2.376 | 0.4208 | 3 |
| | 3.91 | 19.00 | 0.05262 | 2.414 | 0.4070 | 2 |
| Mean $p_m$, | 4.07 | | | | | |
| A2 | 4.19 | 16.28 | 0.06144 | 2.256 | 0.4432 | 2 |
| | 4.06 | 22.54 | 0.04437 | 2.465 | 0.4057 | 4 |
| | 3.92 | 26.73 | 0.03742 | 2.545 | 0.3929 | 1 |
| | 3.83 | 30.31 | 0.03299 | 2.590 | 0.3861 | 2 |
| Mean $p_m$, | 4.00 | | | | | |
| A3 | 4.27 | 16.68 | 0.05998 | 2.528 | 0.3956 | 1 |
| | 4.20 | 22.08 | 0.04531 | 2.620 | 0.3817 | 1 |
| | 3.88 | 26.82 | 0.03728 | 2.645 | 0.3781 | 1 |
| | 3.75 | 30.00 | 0.03333 | 2.699 | 0.3705 | 1 |
| Mean $p_m$, | 4.02 | | | | | |
| A4 | 4.23 | 17.18 | 0.05821 | 2.575 | 0.3883 | 5 |
| | 4.20 | 21.02 | 0.04756 | 2.704 | 0.3699 | 2 |
| | 4.11 | 25.96 | 0.03852 | 2.765 | 0.3616 | 6 |
| | 4.06 | 31.18 | 0.03207 | 2.836 | 0.3527 | 5 |
| Mean $p_m$, | 4.15 | | | | | |
| V1 | 4.20 | 12.24 | 0.08172 | 3.068 | 0.3259 | 1 |
| | 4.16 | 20.07 | 0.04983 | 3.106 | 0.3220 | 1 |
| | 4.07 | 23.92 | 0.04179 | 3.150 | 0.3174 | 2 |
| | 3.95 | 33.24 | 0.03008 | 3.159 | 0.3166 | 1 |
| Mean $p_m$, | 4.09 | | | | | |
| V2 | 4.10 | 14.53 | 0.06878 | 3.062 | 0.3266 | 2 |
| | 3.97 | 18.46 | 0.05418 | 3.103 | 0.3223 | 2 |
| | 3.83 | 27.70 | 0.03610 | 3.144 | 0.3181 | 2 |
| | 3.78 | 30.04 | 0.03329 | 3.140 | 0.3185 | 1 |
| Mean $p_m$, | 3.92 | | | | | |

| Plug | $p_m$ kgm/cm² | $f$ kgm/hr. | $\frac{1}{f}$ hr./kgm | $\mu'$ $\frac{°\text{C cm}^2}{\text{kgm}}$ | $\frac{1}{\mu'}$ $\frac{\text{kgm}}{°\text{C cm}^2}$ | No. of runs |
|---|---|---|---|---|---|---|
| U1 | 4.24 | 9.12 | 0.10958 | 2.820 | 0.3546 | 2 |
| | 4.13 | 15.82 | 0.06320 | 2.873 | 0.3480 | 1 |
| | 4.06 | 22.29 | 0.04486 | 2.885 | 0.3466 | 1 |
| | 3.94 | 28.43 | 0.03517 | 2.926 | 0.3418 | 1 |
| Mean $p_m$, | 4.09 | | | | | |
| U2 | 4.15 | 16.33 | 0.06122 | 2.882 | 0.3470 | 2 |
| | 4.03 | 22.66 | 0.04414 | 2.896 | 0.3453 | 2 |
| | 3.88 | 28.96 | 0.03454 | 2.920 | 0.3425 | 2 |
| Mean $p_m$, | 4.02 | | | | | |
| S | 4.13 | 13.01 | 0.07692 | 3.096 | 0.3230 | 2 |
| | 3.92 | 16.78 | 0.05982 | 3.120 | 0.3205 | 2 |
| | 3.76 | 20.78 | 0.04818 | 3.128 | 0.3197 | 2 |
| | 3.64 | 24.14 | 0.04148 | 3.136 | 0.3189 | 2 |
| Mean $p_m$, | 3.86 | | | | | |

Mean $p_m$, all plugs, 4.02 kgm./cm².

## TABLE IV.

### Characteristics of $\mu'$ vs. $\frac{1}{f}$ plots.

$\mu'_0$ is the extrapolated value of the Joule-Thomson coefficient; $\mu'_2$, the apparent value at the highest flow; $\mu'_1$, the apparent value at the lowest flow. The quantities in the fifth column may be taken as representing the percentage part of the extrapolated coefficient which the extrapolation is depended upon to supply.

| Plug | $\mu'_0$ | $\mu'_2$ | $\mu'_1$ | $\frac{100}{\mu'_0}(\mu'_0-\mu'_2)$ | $\frac{100}{\mu'_0}(\mu'_0-\mu'_1)$ | Geometrical Slope | $\frac{\text{Range of extrapolation}}{\text{Range of observation}}$ |
|---|---|---|---|---|---|---|---|
| A1 | 3.007 | 2.465 | 1.974 | 18.0 | 34.4 | 0.517 | 1.11 |
| A2 | 2.988 | 2.599 | 2.260 | 13.0 | 24.4 | 0.592 | 1.16 |
| A3 | 2.885 | 2.688 | 2.529 | 6.8 | 12.3 | 0.297 | 1.25 |
| A4 | 3.149 | 2.832 | 2.572 | 10.1 | 18.3 | 0.493 | 1.23 |
| V1 | 3.224 | 3.167 | 3.065 | 1.8 | 4.9 | 0.096 | 0.58 |
| V2 | 3.230 | 3.150 | 3.061 | 2.5 | 5.2 | 0.121 | 0.94 |
| U1 | 2.956 | 2.912 | 2.877 | 1.5 | 2.7 | 0.064 | 0.47 |
| U2 | 2.961 | 2.917 | 2.878 | 1.5 | 2.8 | 0.073 | 1.30 |
| S | 3.182 | 3.139 | 3.098 | 1.3 | 2.6 | 0.055 | 1.17 |

less rapidly as the pressure-drop is raised. The general trend of these plots as compared with that of the line representing the true value of $\mu$

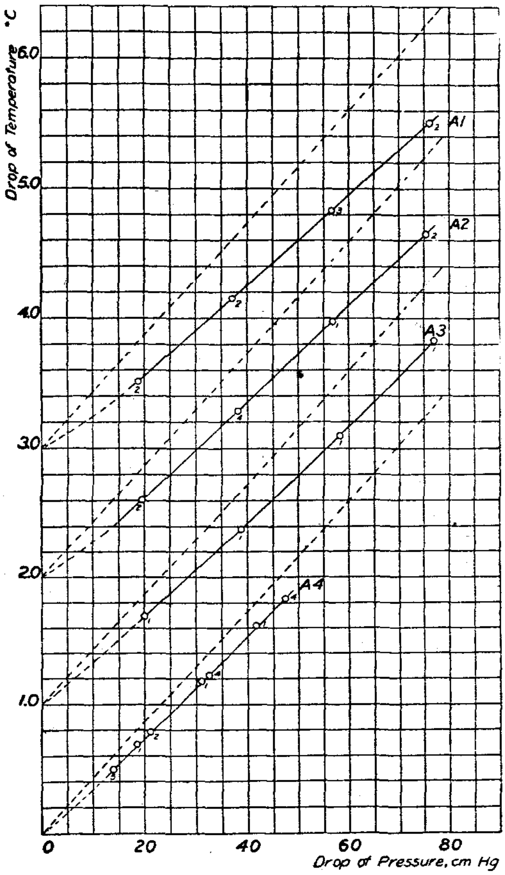

FIGURE 15. Temperature drop vs. pressure drop, axial flow plugs. The numbers adjoining the plotted points indicate the number of runs involved. The numbers along the axis of ordinates are for plug A4, the curves being spaced at intervals of 1° to avoid confusion. The straight dotted lines have slopes equal to the extrapolated $\mu$ for plug S.

illustrates, probably better than anything else, the general hopelessness of attempting to obtain reliable values of $\mu$ with such apparatus as this. The $\Delta T$ vs. $\Delta p$ plot for plug A3 shows smaller heat-leakage

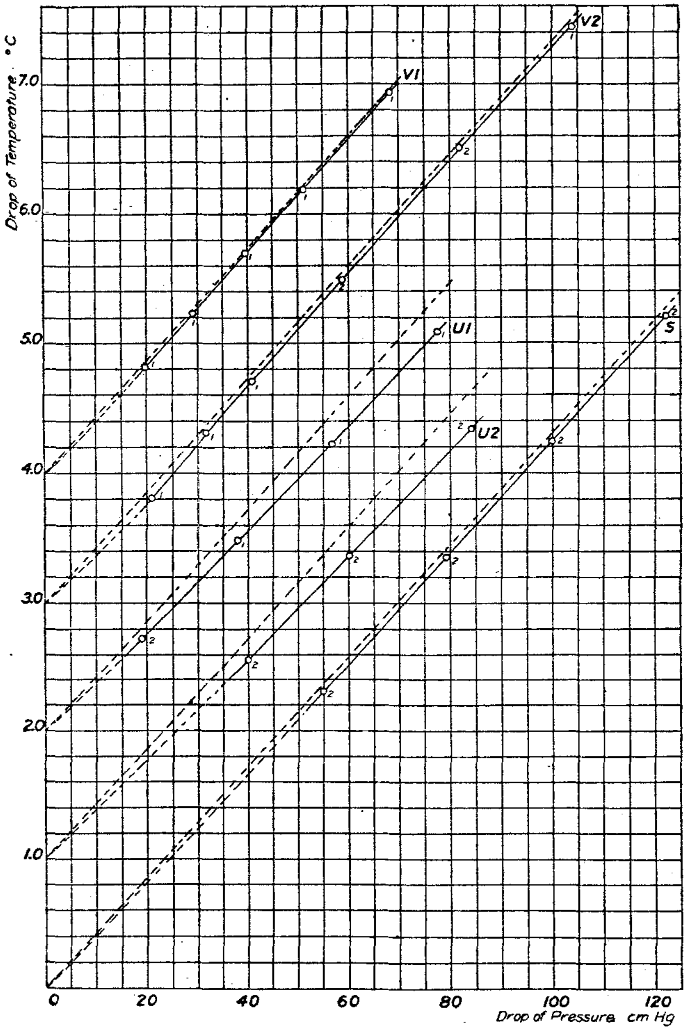

FIGURE 16. Temperature drop *vs.* pressure drop, radial flow plugs. The numbers adjoining the plotted points indicate the number of runs involved. The numbers along the axis of ordinates are for plug S, the curves being spaced at intervals of 1° to avoid confusion. The straight dotted lines have slopes equal to the extrapolated $\mu$ for plug S.

per gram of fluid than the others; this is due to the better insulation of the cross-channel in this plug.

The $\mu'$ vs. $\frac{1}{f}$ plots for these three plugs indicate clearly that the range of observation would have to be much greater than that actually covered by the experiments to justify any expectation of reliable results, and that, if the range were extended, these loci would necessarily be curved and hence that their extrapolation would be difficult or impossible. The extrapolated $\mu$'s from these plots are from 6 to 9 per cent. too small. It is probable that the small value of the extrapolated $\mu$ for A3 is due in part to an uncertainty regarding the pressure-drops for some of the runs involved in this plot, due to a difficulty with the zero correction to the manometer which has already been mentioned (page 752).

### c. *Axial flow plug A4.*

The results obtained with the set-up A4, and represented by the lines so marked, are the best that have been secured with an axial flow plug and are probably as good as can be obtained with apparatus of the particular type used. The number of runs involved is excessive, amounting in all to seventeen, for a single value of $\mu$ which is about 1% too low. The very marked improvement, as regards heat-leak, over the axial flow plugs A1, A2, and A3, is due chiefly to the larger cross-channel, the kinetic energy depression of temperature being done away with. It is probable also that the improvement in thermal insulation effected by the use of poplox lagging in the vertical tubes $T$, $T'$ of Fig. 1 is in part responsible for the reduction of heat-leak errors.

The $\Delta T$ vs. $\Delta p$ plot probably shows to the best advantage the superiority of this plug to the other axial flow plugs. The heat leak per gram of steam is, generally speaking, less than half as great for a given pressure-drop as it is for plugs A2 and A3, and increases much less rapidly with increasing pressure drop than in these plugs. This is partly due to the larger flow per unit $\Delta p$ in plug A4. The slope of the $\Delta T$–$\Delta p$ plot at its upper extremity approximates the true value of $\mu$ much more closely for plug A4 than for any of the other axial flow plugs.

All of the $\mu'$ vs. $\frac{1}{f}$ plots for the axial flow plugs have large slopes, and it is therefore to be expected that the $\frac{1}{\mu'}$ vs. $\frac{1}{f}$ plots will exhibit curvature. They all do, but the curvature is least distinct for plug A4, and, owing to the way in which the experimental errors distribute themselves, the curvature of the A4 plot might almost be considered accidental. It is interesting to note, in this connection, that if either of the $\mu'$ vs. $\frac{1}{f}$ or the $\frac{1}{\mu'}$ vs. $\frac{1}{f}$ plots is straight, the other will be concave upward; and that a rectilinear extrapolation of a $\mu'$ vs. $\frac{1}{f}$ plot which is slightly concave upward will lead to too small a value of $\mu$, while an extrapolation of a similar $\frac{1}{\mu'}$ vs. $\frac{1}{f}$ plot will lead to too small a value of $\frac{1}{\mu'}$ i. e., to too large a value of $\mu$. Hence if, in any given case, both plots seem sensibly straight, it might be expected that the true $\mu$ would lie somewhere between the two values obtained by these extrapolations. In the case of plug A4 the mean of these two extrapolated values is 3°.190 C. cm$^2$/kgm, which is within 0°.008 C. cm.$^2$/kgm. or $\frac{1}{4}$ of 1% of what has been taken as the true $\mu$ in this discussion.

### *d. Radial flow plugs V1 and V2. The regeneration effect.*

The plotted points on the $\mu'$ vs. $\frac{1}{f}$ diagram for plug V2 (no lagging) and, possibly, also for plug V1 (external poplox lagging) are so situated as to admit the possibility of some real curvature in these plots, the curvature being negative (concavity downward). This curvature, reversed, is more evident in the $\frac{1}{\mu'}$ vs. $\frac{1}{f}$ plots for these two plugs. It is probable, therefore, that rectilinear extrapolation of either plot for either plug is of rather doubtful reliability, in spite of the comparatively small slope of the lines concerned. However, the deviations of the plotted points from the least square straight lines are not too large to be ascribable to accidental errors, at least in the case of the $\mu'$ vs. $\frac{1}{f}$ plots, and even if they were, and if some better representation could be devised, the plots in their present form are useful for comparison purposes. They are therefore shown.

Both of the $\mu'$ vs. $\frac{1}{f}$ plots, on this rectilinear representation, extrapolate to values of $\mu$ which exceed the true value by about 1 per cent., the result for the unlagged plug (V2) being slightly greater than that for the other. (Both extrapolated values would of course be too large if there is genuine negative curvature in the loci.) The slope is greater for the unlagged plug than for the other.

The $\Delta T$ vs. $\Delta p$ plots exhibit definite characteristics unlike those of any other plug. Both are concave downward within the region of observation, the heat-leak per gram of fluid decreasing with increasing pressure drop, with a well-marked tendency to approach a limiting value which is nearly zero for V1, the lagged plug, and which, for V2, is small and of about the same magnitude as the corresponding leak for plug S. Both plots would show points of inflexion if observations had been taken at smaller pressure drops, since $\Delta T$ and $\Delta p$ vanish together; this is indicated by the dotted portions of the plots.

This peculiarity is believed to be due to the operation of the regeneration effect discussed under II, 2, *a*. As is there pointed out, this effect produces an outward heat-leakage from the steam between the thermometers, the effect of which is to oppose the normal inward heat-leakage due to conduction. The external lagging used with plug V1 assists the operation of this regenerative action, since it interferes with the flow of heat from the bath to those portions of the passing fluid upon which the regenerative action takes place. It is on this account that the net inward heat-leak is larger for the unlagged than for the lagged plug. It seems probable that the total outward heat leak due to regeneration would increase at about the same rate as the pressure drop, and since the flow varies approximately as the $\frac{3}{4}$ power of the pressure drop, the outward heat leak per gram of fluid due to regeneration would increase with the pressure drop; moreover, since the inward leak per gram of fluid due to conduction, etc., increases with pressure drop in the case of every plug in which this is the only leak occurring (including plug S), it is evident that the net result might be to produce a heat-leak per gram which would tend toward some limiting value such as the experimental results actually indicate. It is hardly possible to be more definite than this regarding the effect which might reasonably be anticipated from the joint operation of the several agents at work, because of the complicated nature of the phenomena concerned.

The $\mu'$ vs. $\frac{1}{f}$ plots also point, though less clearly, to the operation of regenerative action. The greater slope of the V2 (unlagged plug) line is due to the inhibition of the corrective action of the regenerative leakage by the flow of heat from the bath. While this line extrapolates to a larger value of $\mu$ than the other, the difference is slight and is of much less significance than the definite difference of slopes, in view of the comparatively large deviations of the plotted points from the representative straight lines, (or possibly, the genuine curvature of both plots).

The important conclusion from the work with these set-ups is that the effect of regeneration is not great. This inference is drawn from the observation that only relatively insignificant changes in heat leak were produced by making changes in lagging which must have had a very important effect on regeneration. This is the principal justification for the belief that regenerative action is practically negligible in plug S, in which heavy *internal* lagging is provided.

It is to be noticed that the slopes of the upper portions of both $\Delta T$–$\Delta p$ plots are practically identical with what has been called the true $\mu$.

### *e. Radial Flow Plug S.*

This represents the final form of the apparatus. The $\mu'$ vs. $\frac{1}{f}$ and $\frac{1}{\mu'}$ vs. $\frac{1}{f}$ plots both have smaller slopes than with any other plug, and neither exhibits any evidence of curvature. The deviations of the plotted points from the representative straight lines are small, and the value of $\mu$ as derived from the $\frac{1}{\mu'}$ plot exceeds its value as derived from the other by only 0°.002 cm.$^2$/kgm., or less than $\frac{1}{15}$ of 1%. All the points of the $\Delta T$ vs. $\Delta p$ plot lie on a straight line, the slope of which is very slightly less than the true $\mu$. The heat-leak per gram of steam, which is probably due chiefly to conduction down the walls of the plug, thus still increases slightly with the pressure drop, but the increase is so slight that it would disappear or become a decrease if the value of the 'true $\mu$' were changed by only a small amount. There is no evidence like that which, in the case of plugs V1 and V2, was interpreted as pointing to regenerative heat-leak.

The reasons for believing that the value of $\mu$ (3.182), given by the rectilinear extrapolation of the $\mu'$ vs. $\frac{1}{f}$ plot for this plug, is the true value within $\frac{1}{2}$ per cent., may be summarized as follows:

1°. The results show that the set-up was to a large degree successful in avoiding the sources of error it was designed to avoid. The ordinary conductive heat leak is very small and, as already noted, there is no evidence of regenerative leak. The actual percentage of the true $\mu$ which the extrapolation is depended upon to supply is only 1.3 per cent.

2°. It is very probable that the true value of $\mu$ lies somewhere between the values given by plugs V1 or V2, which are too high because of regeneration (or possibly because the $\mu'$ vs. $\frac{1}{f}$ plots for these plugs are really concave downward instead of rectilinear — and, if this is so, it is doubtless itself due to regeneration), and the value as given by the $\mu'$ extrapolation for plug A4, which is too low. (Plug A4 is the best of a series of axial flow plugs, all of the earlier of which give extrapolated $\mu$'s which are certainly much too small.) The extrapolated $\mu$'s for plugs V1 and V2 are respectively 3.224 and 3.230 while that for plug A4 is 3.149. Also, it has already been noted that the mean $\mu$ as determined from the two A4 plots differs by only 0.008 — about $\frac{1}{4}$ per cent.— from the extrapolated $\mu$ of plug S.

3°. Where the slopes of the $\Delta T$ vs. $\Delta p$ plots reach, or approximately reach, a limit at or below the highest pressure drops, these limiting slopes may be regarded as representing approximately the true value of $\mu$. If the calculations of slope are made by subtracting one observed temperature drop from another and dividing the result by the difference of the corresponding pressure drops, it is found that none of the set-ups except V1, V2 and S show evidence of a limiting value of the slope such as has just been mentioned. In the case of these three plugs, the values of the slope, calculated as described from the two highest pressure drops, are 3.20, 3.18, and 3.18 degrees C. per kgm. per cm.$^2$, respectively. These values would probably be slightly changed if the smoothed curves, instead of observed points, were used as a basis for the calculation; but in either case, the method of calculation would necessarily be somewhat inaccurate, and the results are offered only as confirmatory of the result (3°.182 C. cm.$^2$/kgm.) obtained by extrapolating the $\mu'$ vs. $\frac{1}{f}$ plot for plug S.

4°. The $\mu'$ vs. $\frac{1}{f}$ plot for plug S has the least slope of all such plots, and the points plotted to determine it show that the accidental errors were smaller for plug S than for any other plug (see Table IV and Fig. 12). There are several other plugs for which the ratio of the range of extrapolation to the range of observation is less than for plug S, but, as an inspection of Fig. 13 will show, the inferiority of plug S in this respect in a comparison with any of these plugs is outweighed by its marked superiority in one or both of the other two respects.

In the case of plug S, as has already been stated, the difference between the value of $\mu'$ at the highest flow and its value on the axis of $\frac{1}{f}$ is only 1.3 per cent. of the latter. The extrapolation must therefore be supposed to be so uncertain that the part of the extrapolated $\mu$ which depends upon it is in error by 40 per cent. of itself, if one is to believe that an error as great as 0.5 per cent. is involved from this cause alone. A rough idea regarding the probability of a percentage extrapolation error of this magnitude in the curve of plug S may be obtained by considering the extrapolation errors of other plugs.

If it is assumed that the constant $A$ in equation (2a) is zero, all of the deviation of any extrapolated $\mu$ from the true $\mu$ must be assigned to extrapolation error, in the sense in which this expression has heretofore been used. Now, whether the precision claimed for the $\mu$ given by plug S is allowed or not, we may certainly suppose that 3.18 represents the true value of $\mu$ within, say, 2 per cent. — that is, that $\mu$ certainly lies somewhere between 3.15 and 3.21. For any plug, the extrapolation error will then lie between $3.15 - \mu'_0$ and $3.21 - \mu'_0$, where $\mu'_0$ is the extrapolated $\mu$. The percentage error of extrapolation is 100 times the ratio of the difference between the true $\mu$ and $\mu'_0$ to the difference between $\mu'_0$ and $\mu'_2$, where $\mu'_2$ is the apparent value of the coefficient observed at the highest flow. Hence the percentage error of extrapolation lies between $100\ (3.15-\mu'_0)/(\mu'_0-\mu'_2)$ and $100\ (3.21-\mu'_0)/(\mu'_0-\mu'_2)$. The values of these limits may be calculated from the data given in Table IV; they are found to be, for plug A1, 20 and 52 per cent.; for plug A2, 34 and 64 per cent.; for plug A3, 128 and 192 per cent.; for plug U1, 320 and 560 per cent.; for plug U2, 400 and 700 per cent. For the other plugs, $\mu'_0$ lies too close to 3.18 for the calculations to have much significance.

It is thus seen that, for an error as great as $\frac{1}{2}$ per cent. to be present in the $\mu$ obtained by extrapolating the plot of plug S, we must suppose the percentage extrapolation error with this plug to be of the same order of magnitude as with plugs A1 and A2 and from a third to a fifth as large as with plug A3. One is justified in believing that the error is considerably smaller than this. It is true that it is necessary to suppose the percentage extrapolation error with plug S to be only $\frac{1}{8}$ to $\frac{1}{18}$ of this error with plugs U1 or U2, in order thereby to account for an error of $\frac{1}{2}$ per cent. in what we have called the true $\mu$. But it is extremely doubtful that the plots for plugs U1 and U2 can be regarded as straight within experimental error, and therefore difficult to say by how much the extrapolation error with these plugs might reasonably be expected to exceed that with plug S, for which the rectilinearity of the plot is more secure than for any other plug. Moreover, the large values of the percentage extrapolation error for plugs U1 and U2 are due to the small slopes of the plots and would be materially changed by comparatively small changes in these slopes.

5°. Nothing whatever as to the legitimacy of the assumption that the $A$ of equation (2a) is zero can be inferred either from the straightness of a given plot or the smallness of its slope, and this assumption has exactly the same validity in the case of plug S as it has in the case of any other plug for which the value of the exponent $r$ of equation (4) is the same. But certainly it is not easy to see how or why a heat leak per unit time varying as the 1.75 power of the pressure drop — practically as the 1.75 power of the temperature drop — and of any appreciable magnitude, could be present.

6°. The effect of failure to correct for the pressure coefficient of $\mu$ probably is to make the extrapolated $\mu$ too small, but not, it is believed, by an amount as great as 0.3 per cent. (See Section IV, 2, *h*.)

7°. The precision of the temperature measurements is at least as good as $\frac{1}{5}$ per cent., and that of the pressure measurements $\frac{1}{10}$ per cent., or even better at the higher flows.

### *f. Measurements of $\mu C_p$ and the calculation of $\mu$ from them.*

To measure directly the value of the product $\mu C_p$, it is necessary to supply energy to the fluid during its passage through the plug, of an amount sufficient to prevent the ordinary Joule-Thomson drop of temperature. In the case of such axial flow plugs as have been used in this work, this energy is supplied electrically by means of the heating coil shown in Fig. 1. If the supply of energy is just sufficient to maintain the low-side temperature equal to the high side temperature, it may be assumed, in the absence of such effects as varying kinetic energy in different parts of the plug, that the temperature of the steam is that of the oil-bath during the whole of its passage through the plug, and hence that there is no heat-leakage. Under these circumstances, the energy supplied per unit mass of fluid is equal to $\mu C_p \cdot \Delta p$, in which $\mu$ is evaluated at the mean of the high and low side pressures, and at the mean of the high side temperature and of what would be the low side temperature if the experiment were adiabatic, and $C_p$ is evaluated at the mean of these temperatures and at the low side pressure. The product $\mu C_p$ is in several respects of more importance than $\mu$ alone, and obviously a direct measurement of $\mu C_p$ has advantages over its determination by separate measurements of $\mu$ and of $C_p$, particularly as it is the manner of variation of $\mu C_p$ with pressure and temperature which it is useful to know.

In the present work, eleven measurements of $\mu C_p$ have been made, all at approximately the same pressure and temperature at which the $\mu$-measurements were made. Two were made with plug A2, four with plug A3, and five with plug A4. The measurements, when combined with the low values of $\mu$ obtained with the earlier apparatus, led to values of $C_p$ which were so much in excess of the reliable results obtained by Knoblauch[9] and by Knoblauch and Mollier,[10] that the inference seemed unavoidable that some large source of outward heat-leak must have been operative. In addition to this, the measurements were discordant to a degree indicating accidental errors of much greater magnitude than occur in the adiabatic experiments. The results on $\mu$ obtained with the later apparatus bring the mean of these determinations of $\mu C_p$ into much closer agreement with Knoblauch and Mollier's values of $C_p$, however. These $\mu C_p$ measurements thus afford another check on the accuracy of the value of $\mu$ obtained in the experiments already discussed. They are not offered as representing dependable values of $\mu C_p$ but merely as having a certain bearing on the legitimacy of the claim which has been made in respect to the reliability of $\mu$ as determined by the S-type of radial flow plug.

If each of the several measurements of $\mu C_p$ is divided by the proper value of $C_p$, taken for the conditions of the experiment from table 4 of the paper by Knoblauch and Mollier above referred to, there result eleven values of $\mu$. After these are reduced to 165° C., their mean is 3°.26 C. cm$^2$/kgm., with an average deviation for the entire eleven of 0°.09 C. cm.$^2$/kgm. If one divides this deviation by the square root of the number of determinations, one finds $0^\circ.02_7$ as the deviation of the mean. This is little more than nominal, for it is highly improbable that the large errors involved were all accidental. Accepting it, however, the difference between the value of $\mu$ as derived from these measurements and the value obtained from the experiments with the S-type of plug is about three times the deviation of the mean of the results of the $\mu C_p$ measurements — that is, about 2.5 per cent. of $\mu$ itself. There does not appear to be any reason for suspecting that the experimental work of Knoblauch and Mollier was such as to involve a constant error in either direction. Their experimental errors, as indicated by the values of $C_p$ plotted on Fig. 5 of their paper, were of the order of about one per cent. in the region of the plane here involved. It happens that their $C_p$ curve for 4 kgm./cm.$^2$, the position of which largely determines the values of $C_p$ used in the calculations of $\mu$, is for some reason drawn by them along the lower edge of their band of experimental points in the temperature region within which the $\mu C_p$ measurements were made. If the curve had been here so located as to pass through the mean position of their experimental points, about one-third of the discrepancy of 2.5 per cent. noted above would disappear.

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[9] Knoblauch and Jakob, Mitteilungen über Forschungsarbeiten des Vereines deutscher Ingenieure, **35**, 109 (1906).

[10] Knoblauch and Mollier, ZS. des Vereines d. Ing., 1911, p. 655.

As has been already stated, the $\mu C_p$ experiments were dropped rather early in the present research in order to settle the heat-leak question in the adiabatic work, and there is not sufficient experimental evidence to yield any conclusion of consequence regarding a constant error of either sign, or regarding the cause of the large experimental errors, in them. With plugs A2 and A3, the effect of high steam velocity in the cross-channel would be to make the measured values too small. This effect ought to be absent from the results obtained with plug A4, which are, in fact, larger than with the other plugs. Variations in the provisions for mixing the steam produced no consistent results. It was suspected at first that an outward heat-leak might have been due to the fact that, in order to get heat from the heating-coil into the steam, it had been necessary to make the temperature of the coil considerably higher than that of the oil bath. Measurements of the resistance of the manganin coil used with plugs A2 and A3 seemed to indicate this, but the indication was not reliable, owing to the small temperature coefficient of manganin. The matter was more thoroughly tested in plug A4 by employing a heating coil of stranded invar. This alloy has a high specific resistance and, for an alloy, a high temperature coefficient.* The resistance of the coil, at constant temperature, decreased slightly during the course of the experimental work, doubtless because of an aging effect, recovering a portion of the decrease during idle periods. An increase in the energy input at constant flow resulted in a decrease of the measured resistance in five cases and in an increase in three. The evidence is thus against the existence of a heat-leak of the kind suspected; it is probable that the variations of resistance actually noted were due to other causes than variations of the temperature of the coil.

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* Mean coefficient, 30° to 170° C., 0.00145 per °C.

### *g. The wet-steam question. Radial flow plugs U1 and U2.*

It is improbable that perfectly dry superheated steam is ever secured in practice, at least at temperatures within thirty or forty degrees Centigrade of saturation. Particles of liquid are always brought away, suspended in the vapor, during vaporization. These particles, spherical in shape, are able to persist in the liquid form after the application of superheat because of the excess pressure on the interior of the drop produced by the surface tension. The effect of evaporation from the surface of the drop is a reaction against a continuation of the process, because of the increase in the curvature of the surface. Further, if, in some manner, steam free from moisture were actually obtained, it would be very difficult to demonstrate the fact.

As far as calorimetric effects are concerned, the presence of a small amount of moisture in superheated steam is not objectionable, provided the quantity present does not change as a result of the experimental processes. For example, if the steam contains 0.1 per cent of water, any temperature change undergone by the fluid will differ by only about 0.1 per cent. from what it would be if no water were present at all, *provided* the water-content is not changed; but if, for each degree of temperature change, as little as 1 per cent. of the 0.1 per cent. water-content is vaporized, the effect on the temperature change will amount to 1 per cent. (These statements suppose, as is roughly the case, that the latent heat of vaporization and the specific heats of water and steam are in the proportion 1000:2:1.)

What an experimenter should aim at, therefore, is to reduce the amount of moisture to as low a figure as possible and to endeavor to have the moisture which is unavoidably present in a form such that no change in its amount is likely to occur during passage through the calorimetric apparatus. After taking all the precautions which seem necessary to secure these conditions, they should be doubled. If no noticeable changes result, it may be regarded as reasonably certain that the effect of moisture is negligible — not, however, that moisture is necessarily absent.

The presence of varying amounts of moisture in supposedly dry steam has been the occasion of so considerable an error in the results of at least one experimenter that it was deemed advisable, in the present work, to provide with some care against the entrance of error from this source. The precautions taken have already been described. They consisted in attention to the design of the primary superheater, in operating the secondary superheater as a cooler, and in providing strainers. The object of strainers is in part to get rid of moisture by the operation of throttling, on the principle embodied in the ordinary throttling calorimeter used for determining the quality of moist steam, and in part to break up particles of moisture into small drops, from which further evaporation is less likely to occur. The only actual tests on the efficacy of these precautions are given in the results obtained with plugs U1 and U2. The arrangements of the apparatus in these two cases differed only in that a copper gauze strainer was used with U1 and an alundum strainer with U2. A glance at the curves of Figs. 12 and 13 will show that this test indicates no appreciable wet steam effect. In fact, such discrepancy as appears is in the wrong direction for explanation on the supposition that wet steam was the cause of it, since the alundum strainer was far more effective than the gauze strainer.

The low values of $\mu$ obtained by extrapolation with the two U-type plugs are mainly due to heat-leakage into the annular space surrounding the radial flow plug. The steam is here spread out into a rather thin layer not effectively insulated from the bath, and as its temperature has been depressed, before it reaches this space, by throttling in the mixing chambers at the two ends of the cross channel, the opportunity for heat-leakage is excellent. The need of locating the high side thermometer in close proximity to the radial flow plug is well illustrated by these results. Although the set-up is similar to that used with the axial flow plug A4, the results do not form a fair basis of comparison between the two types of plug, since the chief advantage of the radial flow plug is sacrificed by the arrangement.

In both of the U-type plots of $\mu'$ vs. $\frac{1}{f}$, there is evidence of curvature, which, if actual, would lead to larger $\mu$'s than are given by the straight line extrapolation. Whether this curvature is real or not has no particular bearing on the evidence of the plots as to the question of wet steam, and the plots have practically no other value than this.

### *h. Existence of a pressure coefficient of $\mu$.*

In 1909, Davis [11] made a very complete study of all experimental results on the Joule-Thomson effect in steam which had at that time been published, and of some which had not been published. His purpose was to test the applicability of the law of corresponding states to the Joule-Thomson effect in steam and in $CO_2$. He states that a careful scrutiny of all the observations with which his paper concerns itself failed to reveal any systematic variation of $\mu$ with pressure, and concludes that if such a variation exists at all, it is within the limits of error of the experimental work. These limits are not narrow, but the wide range of pressure covered by the four experimenters of whose results Davis made use certainly justifies the inference that the pressure-coefficient of $\mu$ must be small to have escaped detection.

In the present research, a few experiments have been made at mean pressures of 83.0 lbs./in.$^2$ (5.84 kgm./cm.$^2$) and 32.7 lbs./in.$^2$(2.30 kgm./cm.$^2$), these pressures being such that the mean pressure of the other experiments is approximately their arithmetic mean. The experiments referred to were made with the V1 plug, without interchange of thermometers. The $\mu'$ vs. $\frac{1}{f}$ plots are shown in Fig. 17, together with the curve for plug V1 already shown in Fig. 12. Owing to unusually large accidental errors (or possibly, genuine curvature of the plot) in the case of the highest pressure, and to the fact that the flow through the plug was comparatively small for the lowest, the rectilinear extrapolation of the $\mu'$ vs. $\frac{1}{f}$ plot is unreliable in both cases. Moreover, the presence of the regeneration effect in this plug diminishes one's confidence in employing the results to arrive at a value of the pressure coefficient of $\mu$, particularly as this effect seems more prominent in the 5.84 kgm./cm.$^2$ runs than in the others. There is also the possibility of wet steam in the 5.84 kgm./cm.$^2$ runs, which were fairly close to the saturation line. A rough estimate of the magnitude of $(\partial\mu/\partial p)_t$ can, however, be obtained by considering those parts of the three curves on the $\mu'$ vs. $\frac{1}{f}$ diagram which overlap, ascribing the difference in the average apparent Joule-Thomson coefficients over identical ranges of flow to the effect of pressure. The three values of $(\partial\mu/\partial p)_t$ obtained in this way are:

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[11] Proc. Am. Acad., **45**, 243–264 (1910).

+0.036 °C. cm.$^4$/kgm.$^2$ for the overlapping of the 5.84 and the 4.09 kgm./cm.$^2$ lines
+0.028 " " " of the 4.09 and the 2.30 kgm./cm.$^2$ lines
+0.024 " " " of the 5.84 and the 2.30 kgm./cm.$^2$ lines

Taking the mean of the first two and the mean of this mean with the third, there results + 0.028 °C. cm.$^4$/kgm.$^2$ as the value of $(\partial\mu/\partial p)_t$. This is of course only a very rough approximation, and is probably an upper limit.

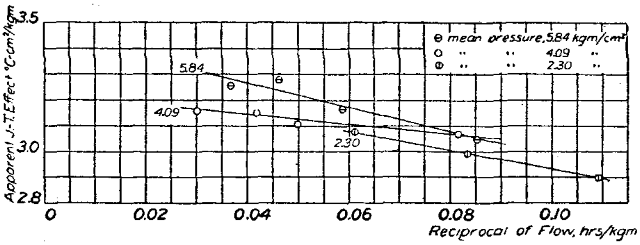

FIGURE 17. $\mu'$ vs. $\frac{1}{f}$ plots for three sets of runs at three different mean pressures, plug V1. Mean temperature, 165° C. in each case. Numbers attached to lines indicate average mean pressure, kgm./cm.$^2$

In a paper on the properties of saturated and superheated ammonia vapor, Goodenough and Mosher[12] have re-examined the data discussed by Davis, in an effort to verify an empirical equation of state for ammonia vapor by comparing reduced values of $\mu$ as calculated from it with the reduced observed values used in Davis' paper. For this purpose they recomputed Davis' reduced results, using later critical data for water than Davis employed. In this way they were able to show that their calculated $\mu$ curves for reduced pressures corresponding to ammonia pressures of 4, 5, 7 and 9 atmospheres agreed well with the observed results of which Davis made use, when these results were grouped in such a way that the mean reduced pressure for each group was approximately the same as that of the curve with which comparison was made. Inferior agreement was obtained in the case of the curves corresponding to the 2, 3 and 10 atmosphere ammonia curves, of a sort indicating that the Joule-Thomson coefficient does not vary with the pressure so rapidly as their equation demands.

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[12] Goodenough and Mosher, Bulletin No. 66, Univ. of Illinois Engineering Experiment Station, January 27, 1913.

The reduced mean coordinates of the runs with the set-up S are 0.0172 for the pressure and 0.676 for the temperature. The corresponding pressure and temperature for ammonia are respectively 29.05 lbs./in.$^2$ abs. and 495° F. abs. On differentiating equation (p) of Goodenough and Mosher's paper with respect to the pressure at constant temperature, and inserting the above values of pressure and temperature, the values of the various constants as given by the authors and their values of $C_p$ and its pressure-derivative as given by their equation (9), one finds the value $-0°.0832$ F. in.$^4$/lb.$^2$ for $(\partial\mu/\partial p)_t$ for ammonia in the state specified. Multiplication of this by $(1690)^2/(460 + 273)$, which is the ratio of the square of the critical pressure of ammonia in lbs./in.$^2$ to its critical temperature in °F. abs. gives $-11.96$ as the value of the derivative in question in reduced units. This value may be checked by reference to Figs. 9a and 9b of Goodenough and Mosher's paper. (Figs. 9a to 9e inclusive, except the lower part of Fig. 9c, of this paper, are all incorrectly labelled; 9a and 9b are for reduced pressures of 0.0174 and 0.0261 respectively, instead of 0.0870 and 0.0783. A first glance at this set of figures gives the impression that $\mu$ increases with increasing pressure at constant temperature, which is in contradiction of the equation (p) of the paper.) Finally if $-11.96$ be multiplied by $648/(225.0)^2$, we find $-0.152$ °C. cm.$^4$/kgm.$^2$ for $(\partial\mu/\partial p)_t$ for steam under the conditions of the runs of set-up S, by this somewhat round-about invocation of the law of corresponding states.

There is no doubt that this result is far too large as regards its magnitude, and possibly incorrect in sign. Its only justification is that at other pressures the throttling experiments used by Davis verify Goodenough and Mosher's ammonia equation. The actual observations in the vicinity of the pressure we are here concerned with are not in agreement with this equation. If the experiments made by Dodge, which exhibit much larger accidental errors than those of the other observers, are neglected, it may fairly be doubted whether even the negative sign of the pressure derivative of $\mu$ is verified. At all events, the accidental errors of the experimental work are so large compared with the effect sought that any quantitative estimate would be hazardous, and none will be here attempted. Of course the very good verification obtained by Goodenough and Mosher at higher pressures raises the presumption that at the pressure of the present writer's experiments, the pressure-coefficient of $\mu$ would still be negative, in spite of the contrary evidence yielded by the work which has been described. Enough weight is attached to this evidence, however, to discount to some extent the deductions from the ammonia equation, of which experimental verification at the pressure in question is lacking, and, in view of the conflict of evidence as to sign, to regard the pressure-corrections to the writer's observed $\mu$'s as negligible within the accuracy claimed for the $\mu$ of set-up S.

Goodenough [13] has proposed the empirical equation

$$v = c + \frac{BT}{p} - (1 + 3ap^{\frac{1}{2}}) \frac{m}{T^n} \tag{5}$$

for superheated steam. He finds that this equation is in excellent agreement with the specific volume determinations of Knoblauch, Linde and Klebe.[14] By means of well-known thermodynamic relations, the equations

$$C_p = \varphi(T) + \frac{Amn(n+1)}{T^{n+1}} p (1 + 2a\, p^{\frac{1}{2}}) \text{ and} \tag{6}$$

$$\mu = \frac{A}{C_p} \left[ \frac{m(n+1)\ (1 + 3ap^{\frac{1}{2}})}{T^n} - c \right] \tag{7}$$

are deduced from it. By adopting for the arbitrary function $\varphi(T)$, in (6), the form

$$\varphi(T) = \alpha + \beta T + \frac{\gamma}{T^2} \tag{8}$$

Goodenough further secures good agreement with the $C_p$ determinations of Knoblauch and Mollier [15] and other experimenters. If we calculate the value of $(\partial\mu/\partial p)_T$ from (7), using the proper values of the several constants,* and taking $p = 4.0$ kgm./cm.$^2$, $T = 165°$ C. ordinary $= 438°$ C. abs., we find $(\partial\mu/\partial p)_T = +\,0.014_7$ °C. cm.$^4$/kgm.$^2$, which is of the same sign as the roughly approximate value derived from the experiments with plug V1, though only about half as great.

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[13] Properties of Steam and Ammonia, Wiley and Sons, 1915, p. 6.

[14] Mitt. über Forschungsarbeiten des Ver. d. Ing., **21**, 33–55 (1905).

[15] Op. cit.

* If $p$ is in kgm./cm.$^2$, $v$ in m.$^3$/kgm., $T$ in °C. abs., $C_p$ in kgm. cals./kgm. °C., $\mu$ in °C. cm.$^2$/kgm., then $\log_{10} \beta = \bar{3}.67213$, $\log_{10} m = 8.59929$, $3a = 0.19339$, $n = 4$, $A = \frac{1}{0.04266}$, $\alpha = 0.320$, $\beta = 0.0002268$, $\gamma = 7371$. $c$ is not strictly constant, being the specific volume of the liquid at the pressure under consideration. It has the value 0.001083 m.$^3$/kgm. in the present calculation.

In Goodenough and Mosher's ammonia equation, the constant *a* is zero and the constant *c* is negative. This accounts for the difference as to sign between the value of $(\partial\mu/\partial p)_T$ just calculated and that derived from the ammonia equation.

If the pressure coefficient of $\mu$ is actually negative, the right hand ends of all $\mu'$ vs. $\frac{1}{f}$ plots should be tilted upward, decreasing slightly the heat-leak effect. If it is positive, as the writer believes and as Goodenough's steam equation indicates, the extrapolated $\mu$'s will all be slightly increased. In the case of the plot of $\mu'$ against $\frac{1}{f}$ for plug S, for which the effect would be greatest, a reduction of all $\mu'$'s to 3.87 kgm./cm.$^2$ on the assumption that $(\partial\mu/\partial p)_T = + 0.028$ °C. cm.$^4$/kgm$^2$ raises the four plotted points by 0.006, 0.003, -0.002 and -0.008 °C. cm.$^2$/kgm. respectively, counting from the one for the smallest value of $1/f$ outward in order, and the extrapolated value of $\mu$ is increased from 3.182 to 3.200 — about 0.56 per cent. It is doubtful whether the actual correction is as great as half of this.

### *i. Comparison with the results of other experimenters.*

Only three accounts of Joule-Thomson experiments by other investigators on superheated steam over ranges of temperature which include 165° C. have been published. These experiments were conducted by Peake,[16] Grindley [17] and Griessmann.[18] All were what engineers usually call 'throttling' or 'wire-drawing' experiments,—that is, experiments in which steam known or assumed to be dry and saturated is throttled to various lower pressures under conditions of negligible loss of energy during the throttling process. The curve which passes through a set of points representing the states of the fluid on the low side of the plug for a number of different pressure-drops from the same original high side pressure, is called a 'throttling curve,' and is a curve of constant total heat, if the throttling has been performed without gain or loss of energy. In all of the experiments of Grindley and in some of those of Peake, the steam on the high side was not devoid of moisture; hence, in these experiments, the state point of the high-side steam does not lie on the throttling curve. However, in these as well as in experiments in which the high side steam was really dry, the slope of the tangent to any throttling curve plotted in the $(p, T)$ plane is the value of $\mu$ at the point in question, provided, of course, that the throttling has been adiabatic.

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[16] Proc. Roy. Soc., **76 A,**185–205 (1905).
[17] Phil. Trans. **194A**, 1–36 (1900).
[18] ZS. des Vereines d. Ing., **47**, 1852–1857 and 1880–1884 (1903).

Since the experiments in question were primarily undertaken for the purpose of determining sets of throttling curves rather than the slopes of these curves, none of the experimenters used differential apparatus in either the drop of pressure or the drop of temperature measurement. Both drops were usually much larger than any in the work of the present writer. Thermometers were not read more closely than 0°.1 C. and this place was, as a rule, uncertain. It is therefore to be expected that large accidental errors will appear in values of $\mu$ calculated from the work of these three investigators by taking differences between successive low-side pressure and temperature readings obtained with the same high side conditions. Such calculations have been very carefully carried through by Davis in a paper already referred to [19] and the results are embodied in tables I, II and III and in the figure 6, of that paper. Davis also exhibits in figure 7 of the same paper a curve representing the Joule-Thomson effect for steam as a function of the temperature. This curve is based, not only on the work of Peake, Grindley and Griessmann, but also on steam experiments at other temperatures by Dodge and on a number of experiments on carbon dioxide, it being assumed, with regard to the last-named, that the law of corresponding states holds for the Joule-Thomson effect in carbon dioxide and water. This curve gives about 3°.11 C. cm.$^2$/kgm. as the value of $\mu$ for steam at 165° C.— about 2.2 per cent. lower than the value (3.182) obtained by the present writer with plug S.

The influence of the carbon-dioxide points is probably scarcely felt at the temperature in question, as a glance at Davis' figure 6 will show (165° C. = 0.687 reduced, using 365° C. for the critical temperature of water, as Davis did). However, to avoid any possible effect of this sort, and also to avoid an effect due to the manner in which Davis grouped the results of the throttling experiments in obtaining the coordinates of the points of his figure 6, the values of all Joule-Thomson coefficients given in Davis' tables I, II and III for temperatures lying within the interval 0.671 to 0.703 reduced (corresponding to 155° to 175° C. for steam) have been considered directly. There are thirty-five of these in all — eight of Grindley's, thirteen of Griessmann's and fourteen of Peake's. Of the thirty-five, eighteen lie above and seventeen below 3°.182 C. cm.$^2$/kgm. The mean of Grindley's values is 3.28$_5$, at 164°.5 C., or 3.27$_0$ at 165°, using $d\mu/dt = -0.031$ cm.$^2$/kgm.; the mean of Griessmann's values is 3.10$_0$ at 162°.5, or 3.02$_2$ at 165°; the mean of Peake's values is 3.19$_2$ at 165°.5, or 3.20$_8$ at 165°. The mean of all thirty-five is 3°.17$_9$ at 164°.2 or 3.15$_4$ at 165°. The actual vertical width of the band of points at 0.687 reduced temperature in Davis' figure 6 is about one-third of a reduced unit, or a little over 1° C. cm.$^2$/kgm. This is about three times the difference between the two extreme values obtained by the present writer (See Table IV).

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[19] Proc. Am. Acad., **45**, 243–264 (1910).

The precautions taken by these experimenters to avoid heat-leakage do not seem in all respects adequate, in view of the writer's experiences with this problem. Griessmann located his plug in a wooden pipe bound with iron hoops, and unlagged. The connected apparatus on both sides was well-lagged. Grindley lagged his low-side steam by means of a steam jacket supplied from the boiler used for the main steam supply. The temperature of the jacket steam could be regulated by means of throttle valves, but one infers from Grindley's paper that it was not always practicable to make it that of the low-side steam. Peake used a self-lagging arrangement on the low side of his plug. His steam passed from the plug into and through a glass tube, thence down on the outside of this tube. He also used thermally insulating joints for his low-side pressure and flow connections, but the writer's experience with similar joints is that they are of slight value. Only Peake appears to have made any systematic attempt to determine whether his results were in error from heat leakage. His plug consisted of a mica disc with a single small hole. Copper gauze was used on the low side to destroy the kinetic energy of the steam jet. Peake found that throttling curves obtained with identical high side conditions, but with orifices of different sizes, coincided in their overlapping portions, and from this inferred that heat-leakage was negligible. 'Coincidence' here must mean agreement within experimental error, which is probably not better than 0°.1 or 0°.2 C.

There is very little upon which to base a discussion of the probable effects of heat-leakage on the results of Grindley and Griessmann. Peake's affirmative tests for the absence of heat-leak effects can scarcely be regarded as of much authority in connection with the work of the other two experimenters, because of the difference between his lagging arrangements and either of theirs. It would seem quite likely that Griessmann at least must have had a not inconsiderable outward heat-leak, the effect of which would have been to make his observed low-side temperatures lower than they would otherwise have been. Nevertheless, Griessmann's result for $\mu$ at 165° C. is the least of the three. It must be remembered in this connection that, because of the subtraction method employed in calculating $\mu$ from the data of all three observers, a considerable heat-leak may easily have been present without a proportionate effect, or even, necessarily, an effect of the same sign, on the calculated $\mu$. But it is hardly conceivable that all of the discrepancy between the results of Griessmann and of Peake, or, still more, those of Griessmann and of Grindley, at 165° C., can be due to errors of observation; or, what amounts to the same thing, that the width of 1° C. cm.$^2$/kgm. of the band of points of Davis' figure 6 can be due to this cause. It is not unlikely that the high value of Grindley's result is in part a low-side kinetic energy effect. His plug consisted of a quarter-inch glass plate through which a single $\frac{1}{16}$-inch hole was drilled, and his low side thermometer (a thermo-junction) was located in the jet from the orifice, about two inches from the orifice. He made tests to determine whether altering the position of the thermo-junction would affect the apparent low-side temperature, and concluded that, on the whole, the indication of the thermo-junction was independent of its position in the low-side channel. His tests do not seem absolutely convincing, however. Peake, who also used a single small hole (in a mica plate) as his main throttling device, found on one occasion, when he accidentally omitted the several layers of copper gauze regularly placed between the orifice and his low-side thermometer, that, under the most unfavorable conditions, the temperature registered by the low side thermometer was thereby depressed by 7° C. Peake's maximum flow was nearly twice Grindley's, however.

Aside from these probabilities, one must ascribe a part of the lack of agreement among Grindley, Griessmann and Peake to heat-leakage effects about which it is impossible to be in any degree specific. It happens that the result of that one of the three observers who made tests showing that his heat leakage effects were negligible is the most nearly in agreement with the value which the writer's work leads him to believe correct. For the rest, the most that can be said is that the effects of heat leakage were probably small and may have been of both signs, and thus minimized in a mean of the results.

Goodenough's empirical equation (equation (5)) gives, for $\mu$ at $p$ = 4.0 kgm./cm.,$^2$ $T$ = 165 + 273° C. abs., the value $\mu$ = 3°.34 C.

cm.$^2$/kgm. as compared with the value 3°.182 C. cm.$^2$/kgm. obtained by the present writer.

## V. SUMMARY.

1. In arriving at an experimental method of controlling and eliminating the effect of heat leakage in Joule-Thomson experiments, one type of axial flow, and three types of radial flow throttling apparatus, were used, and are described in detail. Broadly speaking, an axial flow throttling apparatus may be defined as one in which the direction of flow of the fluid is in general perpendicular to the leakage temperature gradient, and a radial flow throttling apparatus as one in which the plug, or throttling partition, is so shaped as to cause the fluid to flow through it in a direction generally parallel to the leakage temperature gradient.

Attention is called to the possibility of a secondary leakage effect (called the regeneration effect) in radial flow apparatus. The remedy for this effect is internal lagging.

2. Other apparatus, incidental to Joule-Thomson experiments and used in this research, is described. Particular attention is paid to the fundamental measurements of pressure and temperature differences, and the results of an experimental study of several important details connected with these measurements are given.

3. A short discussion of possible methods of eliminating heat-leakage effects from the immediate data of a set of throttling experiments is given.

4. The results of a large number of adiabatic Joule-Thomson experiments on superheated steam, with four axial flow plugs and five radial flow plugs, are presented and discussed in detail, chiefly as an experimental study of heat leakage. The best of the four particular axial flow plugs used is found to be incapable of dependable results without an excessive amount of experimental work. Evidence pointing to the probable presence of a small regeneration effect is shown to exist in the results obtained with certain radial flow plugs. Affirmative evidence of the absence of error from moisture in the steam is given, with a short discussion of experimental methods for avoiding this difficulty. It is shown that the results obtained with the final form of throttling apparatus — one of the radial flow type with heavy internal lagging — may reasonably be supposed to be free from appreciable heat-leak errors. Such results may also be

obtained without a great outlay of time in experimental work. While the radial flow plug thus appears to be much superior to the axial flow plug, it is pointed out that the axial flow plugs used in this research have a serious structural defect which a better design would avoid; hence the superiority of the radial flow plug, though doubtless actual, is not so great as a cursory inspection of the results exhibited graphically would indicate. The final apparatus is designed to permit more exact tests of the relative merits of the two types of plug: these have not as yet been made.

5. The value 3.182 degrees Centigrade per kilogram per square centimeter is given for the Joule-Thomson coefficient in steam at 165° C. and 3.86 kgm./cm.$^2$ This is believed to be reliable to within 0.5 per cent.

6. The results of some early isothermal experiments giving directly the product $\mu C_p$ are briefly summarized. An agreement within about 2.5 per cent. with the above value of $\mu$ is obtained by dividing the mean $\mu C_p$ as derived from these experiments by Knoblauch and Mollier's value of $C_p$.

7. The question of the existence of a pressure coefficient of $\mu$ is discussed. This is of interest because the experiments from which the above value of $\mu$ is derived were not conducted at exactly the same mean pressure and were not corrected for the existing small differences in their mean pressures. A few experiments made by the writer indicate the existence of a small positive pressure coefficient. This is in verification of the recent steam equation of Goodenough. It is believed that the resulting correction in the above value of $\mu$, which can not yet be made accurately, is less than 0.3 per cent.

8. It is shown that the above value of $\mu$ is in good agreement (within less than 1 per cent.) with the mean of the results of Grindley, Peake and Griessmann at the same temperature.

The work which has been described in the foregoing paper was undertaken at the suggestion and under the direction of Professor H. N. Davis, to whom the writer has been indebted for constant advice and for much assistance of a practical and positive sort, only part of which has been explicitly mentioned in the body of the paper. Acknowledgment should also be made to the Rumford Committee of the American Academy of Arts and Sciences for a grant to Prof. Davis, a considerable part of which was used to further this work.

**Proceedings of the American Academy of Arts and Sciences.**

Vol. 52. No. 13. — October, 1917.

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RECORDS OF MEETINGS, 1916–17.

OFFICERS AND COMMITTEES FOR 1917–18.

LIST OF THE FELLOWS AND FOREIGN HONORARY MEMBERS.

BIOGRAPHICAL NOTICES.

STATUTES AND STANDING VOTES.

RUMFORD PREMIUM.

INDEX.

(Title Page and Table of Contents.)

# RECORDS OF MEETINGS.

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### One thousand and fifty-eighth Meeting.

OCTOBER 11, 1916.— STATED MEETING.

The Academy met at its House.

The PRESIDENT in the Chair.

There were fifty-two Fellows and two guests present:—

The following letters were presented by the Corresponding Secretary: from J. W. Baird, W. B. Clark, L. H. Gray, A. B. Hart, Ellsworth Huntington, C. K. Leith, F. T. Lewis, W. A. Setchell, P. G. Stiles, W. C. Sturgis, accepting Fellowship; from Thomas Hardy, accepting Foreign Honorary Membership.

The Chair announced the deaths of the following Fellows and Foreign Honorary Members: Emory McClintock, Class I., Section 1; Eugene Waldemar Hilgard, Class I., Section 3; Josiah Royce, Class III., Section 1; Elie Metchnikoff, Class II., Section 3; Sir Victor A. H. Horsley, Class II., Section 4; Sir Thomas Lauder Brunton, Class II., Section 4; Arthur Sampson Napier, Class III., Section 2.

The Corresponding Secretary announced the receipt of biographical notices of deceased Fellows, as follows:— Class I., Erasmus Darwin Leavitt, by G. R. Agassiz; Simon Newcomb, by E. W. Brown; Sir Henry Roscoe, by Ira Remsen; William Thomson, Lord Kelvin, by Elihu Thomson; Class II., Thomas Jonathan Burrill, by W. G. Farlow; George Edward Davenport, by F. S. Collins, Sir Michael Foster, by Alexander Forbes, Class III., Melville Weston Fuller, by Moorfield Storey.

On the recommendation of the Council, it was

Voted, To appropriate three hundred ($300) dollars from the income of the General Fund for Library expenses, and two hundred ($200) dollars for the use of the Publication Committee in printing biographical notices of deceased Members in the Proceedings.

The following communication was presented: Professor Alfred G. Mayer, "On the Races of the Pacific."

On motion of the President, the thanks of the Academy were unanimously voted to Professor Mayer for his communication.

The meeting then adjourned.

### One thousand and fifty-ninth Meeting.

### NOVEMBER 15, 1916.— STATED MEETING.

The Academy met at its House.

The PRESIDENT in the Chair.

There were eighty-three Fellows present:—

The Rumford Medals were presented to Dr. Charles Greeley Abbot for his researches in Solar Radiation.

Dr. Abbot addressed the Academy on "The Heat of the Sun." The communication was illustrated by lantern slides.

The following paper was presented by title: "Natural and Isogonal Families of Curves on a Surface," by Joseph Lipka. Presented by H. W. Tyler.

The meeting then adjourned.

### One thousand and sixtieth Meeting.

### DECEMBER 13, 1916.— STATED MEETING.

The Academy met at its House.

VICE-PRESIDENT THOMSON in the chair.

There were forty-nine Fellows present.

The following letters were presented by the Corresponding Secretary: from Wm. Trelease, president of the Illinois Academy of Science, inviting a representative of the Academy to be present at its tenth anniversary, on February 23 and 24, 1917; from the University of Illinois, an invitation to the dedication of its new Ceramic Engineering Building; from Dr. Vincent Y. Bowditch, enclosing a notice of the Academy meeting of May 25, 1825.

The following deaths were announced: Cleveland Abbe, Fellow in Class II., Section 1; Percival Lowell, Fellow in Class I., Section 1; Charles Pomeroy Parker, Fellow in Class III., Section 2;

Arthur Auwers, Foreign Honorary Member in Class I., Section 1.

The Corresponding Secretary announced the receipt by the Council of the following biographical notices of deceased Fellows: A. T. Cabot, by F. C. Shattuck, W. R. Ware, by H. L. Warren, Friedrich Kohlrausch, by A. L. Day, S. F. Emmons, by Waldemar Lindgren, L. P. di Cesnola, by Arthur Fairbanks.

On the recommendation of the Council, Professor Farlow was invited to represent the Academy at the tenth anniversary of the Illinois Academy of Science.

The presentation by Dr. B. L. Robinson, of a portrait print of the Kruell etching of Dr. Asa Gray, President of the Academy, 1863–73, was announced.

The following communications were presented:

Mr. Arthur Foote. "On the Composition of Music," with piano illustrations.

Professor Wm. M. Davis. "Sublacustrine Glacial Erosion of the Mission Range, Montana," illustrated by lantern slides.

The meeting then adjourned.

### One thousand and sixty-first Meeting.

### JANUARY 10, 1917.— STATED MEETING.

The Academy met at its House.

The PRESIDENT in the Chair.

There were forty-three Fellows and three guests present:

The Council reported that the Treasurer had caused the following advertisement of the Francis Amory Fund to be published, in accordance with the terms of the will, in the Boston Transcript and the Boston Daily Advertiser of December 30, 1916.

AMERICAN ACADEMY OF ARTS AND SCIENCES.

FRANCIS AMORY of Boston, who died on the tenth day of November, 1912, bequeathed to the American Academy of Arts and Sciences a sum of money for the purpose of establishing a Septennial Prize and a Gold Medal to encourage the invention and discovery of measures for the relief of maladies peculiar to the bladder and the various organs connected with it.

In default of such invention or discovery, the prize and medal may be awarded for any treatise of exceptional value upon the anatomy of these organs or upon the treatment of these diseases.

Mr. Amory directed that public notice should be given of the nature and design of the Fund, and that contributions should be expressly solicited in aid of it.

The attention of the public is therefore called to the existence of this Fund and contributions are hereby solicited.

Under the provisions of the will the Income of the Fund cannot be awarded until twenty-one years from the date of Mr. Amory's death.

HENRY H. EDES, *Treasurer*.

Boston, 10 November, 1916.

### AMERICAN ACADEMY OF ARTS AND SCIENCES.

FRANCIS AMORY of Boston died on the tenth day of November, 1912. In satisfaction of the requirements of his last will and testament, duly proved and allowed as such, with certain modifications, in and by the Probate Court for the County of Suffolk in the Commonwealth of Massachusetts on the thirteenth day of March, 1914, the following detailed and itemized account of the Fund bequeathed to the American Academy of Arts and Sciences, and of the administration and application thereof during the fiscal year 1915–16, is published in two of the principal newspapers printed in Boston.

### FRANCIS AMORY FUND.

| | |
|---|---|
| Received from the Estate of Francis Amory | $25,000.00 |
| Less 10% as per Agreement of Compromise | 2,500.00 |
| | $22,500.00 |
| Interest thereon to 31 July, 1915 | 517.50 |
| On Investments | 567.50 |
| | $23,585.00 |
| Investments made, as follows: | |
| $5,000 Mortgage on improved Real Estate in Boston at 5% | 5,000.00 |
| 5,000 Chicago Junction Railway and Union Stock Yards Company 5% bonds | 4,950.00 |
| 5,000 Western Electric Company 5% bonds | 5,045.00 |
| 5,000 Western Telephone and Telegraph Company 5% bonds | 4,808.75 |
| 3,000 New York Telephone Company 4½% bonds | 2,876.25 |
| Interest in adjustment | 243.80 |
| Paid Probate Office, for attested copy of Mr. Amory's will and the Agreement of Compromise | 6.25 |
| Balance of Cash on hand, uninvested | 654.95 |
| | $23,585.00 |

HENRY H. EDES, *Treasurer*.

Boston, 10 November, 1916

The following gentlemen were elected Fellows of the Academy:—

Edward Weston, of Newark, N. J., to be a Fellow in Class I., Section 2 (Physics).

James Walter Goldthwait, of Hanover, N. H., to be a Fellow in Class II., Section 1 (Geology, Mineralogy and Physics of the Globe).

Thomas Wayland Vaughan, of Washington, D. C., to be a Fellow in Class II., Section 1.

Joseph Augustine Cushman, of Sharon, to be a Fellow in Class II., Section 3 (Zoology and Physiology).

The following communications were presented:

Professor F. C. Shattuck, "On the Development of Medical Science."

Professor W. E. Story, "On Big Numbers."

The meeting then adjourned.

### One thousand and sixty-second Meeting.

### FEBRUARY 14, 1917.— STATED MEETING

The Academy met at its House.

The PRESIDENT in the Chair.

There were thirty-one Fellows present.

The following letters were presented by the Corresponding Secretary:— from Isaiah Bowman, T. W. Vaughan and J. A. Cushman, accepting Fellowship.

On recommendation of the Council, it was

Voted, To increase the subscription for the Union List of Periodicals from fifty ($50) dollars to seventy-five ($75) dollars.

The following communication was presented:

Professor W. T. Bovie, "The Effect of Rays on Protoplasm."

The meeting then adjourned.

### One thousand and sixty-third Meeting.

### MARCH 14, 1917.— STATED MEETING.

The Academy met at its House.

The PRESIDENT in the Chair.

There were fifty-three Fellows and eleven guests present.

The Corresponding Secretary announced the receipt of a biographical notice of Percival Lowell, by George R. Agassiz.

The Chair announced the death of Edward Dyer Peters, Fellow in Class I., Section 4.

The Chair appointed the following Councillors to act as Nominating Committee:

Desmond FitzGerald, of Class I.
John C. Warren, of Class II.
Mark A. DeW. Howe, of Class III.

On recommendation of the Council, the following appropriations were made for the ensuing year:—

From the income of the General Fund, $5400, to be used as follows:—

| | |
|---|---|
| for General and Meeting expenses | $ 500. |
| for Library expenses | 2000. |
| for Books, periodicals and binding | 800. |
| for House expenses | 1600. |
| for Treasurer's office | 500. |

From the income of the Publication Fund, $3000. to be used for publication.

From the income of the Rumford Fund, $2926.60 to be used as follows:—

| | |
|---|---|
| for Research | $1000. |
| for Books, periodicals and binding | 200. |
| for Publication | 600. |
| for use at the discretion of the Committee | 1126.60 |

From the Warren Fund, $1500. to be used at the discretion of the Committee.

The following communications were presented:—

Dr. Francis G. Benedict. "Human Energy and Food Requirements."

Dr. Arthur G. Webster. "Physics and War."

The following paper was presented by title:

"New Laboulbeniales, chiefly Dipterophilous American Species." By Roland Thaxter.

The meeting then adjourned.

### One thousand and sixty-fourth Meeting.

### APRIL 11, 1917.— STATED MEETING.

The Academy met at its House.

The PRESIDENT in the Chair.

There were twenty-six Fellows and two guests present:

The following letters were presented by the Corresponding Secretary:— from J. W. Goldthwait, accepting election to the Academy; from A. S. Hardy, declining Fellowship.

The Corresponding Secretary announced the recepit of three biographical notices: James Clarke White and Sir Thomas Lauder Brunton, by F. C. Shattuck; Cleveland Abbe, by R. DeC. Ward.

The following communication was presented:

Professor W. J. V. Osterhout, "The Relation of Life-processes to the Permeability of Protoplasm."

The meeting then adjourned.

### One thousand and sixty-fifth Meeting.

### MAY 9, 1917.— ANNUAL MEETING.

The Academy met at its House.

The PRESIDENT in the Chair.

There were thirty-four Fellows present.

The Corresponding Secretary presented a letter from J. L. Bremer, accepting Fellowship. The following biographical notices were also presented: William Watson, by C. R. Cross; D. I. Mendeléeff, by G. S. Forbes.

The following report of the Council was presented:—

Since the last report of the Council, there have been reported the deaths of seven Fellows: Emory McClintock, Eugene Waldemar Hilgard, Josiah Royce, Cleveland Abbe, Percival Lowell, Charles Pomeroy Parker, Edward Dyer Peters; and of five Foreign Honorary Members: Arthur Auwers, Elie Metchnikoff, Sir Victor A. H. Horsley, Sir Thomas Lauder Brunton, Arthur Sampson Napier.

Seventeen Fellows have been elected, of which number one has declined Fellowship, one has not yet accepted. Dr. Alexis Carrel elected in 1914, has not yet accepted.

One Foreign Honorary Member has been elected. Four previously elected, have not yet accepted.

The roll now includes 484 Fellows and 62 Foreign Honorary Members.

The annual report of the Treasurer was read, of which the following is an abstract:—

### General Fund.

*Receipts.*

| | | |
|---|---|---|
| Balance, April 1, 1916 | $2,259.60 | |
| Investments | 3,520.00 | |
| Assessments | 3,290.00 | |
| Admissions | 140.00 | |
| Sundries | 152.21 | $9,361.81 |

*Expenditures.*

| | | |
|---|---|---|
| Expense of Library | $2,301.44 | |
| Expense of House | 1,810.71 | |
| Treasurer | 404.29 | |
| General Expense of Society | 345.84 | |
| Printing Biog. Notices in Proc. 51 | 200.00 | |
| Interest on Bonds bought | 3.78 | |
| Income transferred to principal | 265.13 | $5,331.19 |
| Balance, April 1, 1917 | | 4,030.62 |
| | | $9,361.81 |

### Rumford Fund.

*Receipts.*

| | | |
|---|---|---|
| Balance, April 1, 1916 | $3,146.56 | |
| Investments | 3,164.27 | |
| Grant returned | 100.00 | 6,410.83 |

*Expenditures.*

| | | |
|---|---|---|
| Research | $2,519.88 | |
| Books, periodicals and binding | 72.94 | |
| Publication | 1,347.12 | |
| Sundries | 38.60 | |
| Interest on Mortgage, bought | 7.64 | |
| Income transferred to principal | 154.03 | $4,140.21 |
| Balance, April 1, 1917 | | 2,270.62 |
| | | $6,410.83 |

## C. M. Warren Fund.

*Receipts.*

| | | |
|---|---|---|
| Balance, April 1, 1916 | $1,902.23 | |
| Investments | 2,000.73 | 3,902.96 |

*Expenditures.*

| | | |
|---|---|---|
| Research | $1,277.00 | |
| Sundries | 3.00 | |
| Interest on Mortgage, bought | .77 | |
| Income transferred to principal | 29.92 | $1,310.69 |
| Balance April 1, 1917 | | $2,592.27 |
| | | $3,902.96 |

## Publication Fund.

*Receipts.*

| | | |
|---|---|---|
| Balance, April 1, 1916 | $1,456.95 | |
| Appleton Fund investments | 907.26 | |
| Centennial Fund investments | 2,384.27 | |
| Author's Reprints | 179.00 | |
| Sale of Publications | 164.05 | $5,091.53 |

*Expenditures.*

| | | |
|---|---|---|
| Publications | $2,995.31 | |
| Sundries | 10.00 | |
| Interest on Mortgage, bought | 4.96 | |
| Income transferred to principal | 159.43 | $3,169.70 |
| Balance, April 1, 1917 | | $1,921.83 |
| | | $5,091.53 |

## Francis Amory Fund.

*Receipts.*

| | | |
|---|---|---|
| Investments | $1,178.75 | $1,178.75 |

*Expenditures.*

| | | |
|---|---|---|
| Interest on Bonds and Mortgage, bought | 13.26 | |
| Sundries | 38.08 | |
| Income transferred to principal | 1,127.41 | $1,178.75 |

The following reports were also presented:—

## Report of the Library Committee.

The Librarian begs to submit the following report:—

During the year, 71 books have been borrowed from the Library by 24 persons including 16 Fellows and 2 libraries. Although no actual count has been kept, about 75 people have made use of the reading-room, consulting about 150 volumes. All books taken out have been satisfactorily accounted for.

The number of volumes on the shelves at the time of the last report was 34,681. 547 volumes have been added during the past year, making the number now on the shelves, 35,228. This includes 25 purchased from the income of the General Fund, 18 from that of the Rumford Fund, and 504 received by gift or exchange. The pamphlets added during the year number 1276.

The number of gifts of books and pamphlets was greatly augmented this year in response to the request for the publications of Fellows, sent out in November, 1916. It is hoped that still more of such publications may be received.

The expenses charged to the Library during the financial year are:—

| | |
|---|---|
| Salaries | $1,785.87 |
| Binding:— | |
| General Fund | 371.90 |
| Rumford Fund | 36.40 |
| Purchase of periodicals and books:— | |
| General Fund | 120.68 |
| Rumford Fund | 36.54 |
| Miscellaneous | 26.56 |
| Total | $2,377.95 |

While we have unfortunately been deprived through an illness of several months of the services of the assistant Librarian, Mrs. Holden, the Librarian takes pleasure in commending the manner in which the work has been carried on by Mrs. A. M. Smith.

A. G. Webster, *Librarian.*

May 9, 1917.

## REPORT OF THE RUMFORD COMMITTEE.

The Committee organized Oct. 11, 1916, choosing Messrs. Charles R. Cross, Chairman and Arthur G. Webster, Secretary.

During the present year, grants have been made in aid of research as fellows:—

| | |
|---|---|
| October 11, 1916. To Professor J. A. Parkhurst in aid of his investigations on the determination of the Photometric scale of stellar magnitude . . . . . . . . . . . . . . | $300 |
| Dec. 13, 1916. To Mr. E. T. King in aid of his researches on physical measurements of the color of pigments . . . . | 25 |
| To Professor Edward Kremers in aid of his research on the chemical action of light on organic compounds . . . . . . | 300 |
| Feb. 14, 1917. To Professor F. K. Richtmyer in aid of his research on the optical properties of thin films . . . . . . | 500 |
| To Professor Norton A. Kent in aid of his research on spectral lines. (Additional) . . . . . . . . . . . . . . . . . . | 400 |
| To Mr. Ancel St. John in aid of his research on the spectra of X-rays . . . . . . . . . . . . . . . . . . . . | 200 |
| May 9, 1917. To Mr. David L. Webster, for the salary of an assistant in connection with his research on the intensity of lines in X-ray spectra . . . . . . . . . . . . . . | 100 |
| To Professor Frederic Palmer Jr. in aid of his research on light of very short wave length. (Additional) . . . . . . . . | 100 |
| To Professor B. J. Spence in aid of his research upon a new Color Identity Pyrometer . . . . . . . . . . . . . | 75 |
| To Professor B. J. Spence in aid of his research upon a new and more sensitive form of radiometer . . . . . . . . . | 150 |
| To Professor R. C. Gibbs, in aid of his investigations on the absorption of organic and other solutions for ultra-violet, visible and infra-red rays . . . . . . . . . . . . . . . . | 500 |
| To Professor W. M. Baldwin in aid of his research on the character of chemical substances necessary to sensitize animal tissues to the influence of X-rays . . . . . . . . . . . | 125 |

Voted to refer to the Chairman the question of the award of $150 to Mr. Preston H. Edwards in aid of his research on solar radiation in India with power to act.

Mr. Everett T. King who, as already stated, received a grant on Dec. 13, 1916, to aid in the prosecution of a research on which he was then engaged, was shortly thereafter attacked with typhoid fever which resulted fatally. It is worthy of remark that Mr. King was the youngest person to whom a grant from the Rumford Fund has ever been made.

Reports of progress in their several researches have been received from the following persons:—

Messrs. C. G. Abbot, R. T. Birge, P. W. Bridgman, W. W. Campbell, A. L. Clark, H. Crew, F. Daniels, E. B. Frost, H. C. Hayes, H. P. Hollnagel, L. R. Ingersoll, N. A. Kent, F. E. Kester, L. V. King, C. A. Kraus, E. Kremers, G. M. Lewis (research finished), R. A. Millikan, C. L. Norton, F. Palmer, Jr., J. A. Parkhurst, H. M. Randall, T. W. Richards, F. K. Richtmyer, A. St. John, F. A. Saunders, W. O. Sawtelle, A. W. Smith, F. W. Very.

The following papers have been published with aid from the Rumford Fund in the Proceedings of the Academy, Vol. 52, since the last annual meeting.

No. 2. July 1916. The velocity of polymorphic changes between solids by P. W. Bridgman.

No. 3. July 1916. Polymorphism at high pressures, by P. W. Bridgman.

No. 9. Feb. 1917. The electrical resistance of metals under pressure by P. W. Bridgman.

No. 12. Due to appear shortly, the Joule-Thomson effect in superheated steam. I. An experimental study of heat leakage by H. M. Trueblood.

At its meeting of Feb. 14th, 1917, the Committee voted that the replica of the first Rumford Medal which, according to a previous vote, is to be presented to the Rumford Historical Society shall be of silver rather than of bronze.

It having occasionally occurred, especially in summer when the Committee is scattered, that important applications have been received for aid which could not be met in the usual manner, the Committee has endeavored to meet this exigency as indicated in the following action of Dec. 13, 1916: Voted: that in case of immediate need of an appropriation the chairman be authorized to communicate the case in writing to the members and, on receiving the written consent of a majority of the Committee, may make the appropriations thus authorized; provided that in case any member of the Committee presents serious objections and so requests the matter shall be reserved and presented to the Committee at their next regular meeting.

At the meeting of March 14, 1917, it was unanimously voted for the first time and at the meeting of April 11th, for the second time to recommend to the Academy that the Rumford Premium be awarded to Percy W. Bridgman for his Thermodynamical Researches at extremely high Pressures.

Charles R. Cross, *Chairman.*

May 9, 1917.

## Report of the C. M. Warren Committee.

The C. M. Warren Committee begs to submit the following report:

The unexpended balance of appropriations held by the Committee at the time of the last report was $1201.50. In March, 1917, a further appropriation of $1500 was made by the Academy.

During the past year the sum of $30 has been expended for reprinting of certain papers by Professor Charles F. Mabery, copies of which had been exhausted.

On December 14, 1916, a grant of $250 was made to Professor E. L. Mark to assist in carrying on an investigation of certain properties of sea water at the Bermuda Islands. This carried with it the understanding that so much of the apparatus purchased from this grant as might be of a permanent character should be considered the property of the Academy and subject to its directions at the close of the investigation.

On January 15, 1917, an additional grant of $150 was made to Professor R. F. Brunel for the purchase of chemicals needed for the continuation of his research on the relation between the constitution of aliphatic radicals and their chemical affinities.

The amount which remains at the disposal of the Committee at the present time is $2271.50. This sum, of course, is exceptionally large, and under ordinary conditions it would undoubtedly be wise to consider the investment of a part of this sum as an addition to the permanent fund. The Committee is of the opinion, however, that under the present conditions it is better to hold this sum in readiness for its possible use to promote researches which are necessary in connection with our national crisis.

During the past year Professor W. D. Harkins has published additional papers on Surface Energy in the March and April numbers of the Journal of the American Chemical Society.

Professor R. F. Brunel has also published results of research work aided by a grant from the Warren Fund on the "Reversible Replacement of Alcohols in Aldehydealcoholates," which appeared in the Journal of the American Chemical Society in September, 1916.

Respectfully submitted,

H. P. Talbot, *Chairman.*

May 9, 1917.

## Report of the Publication Committee.

The Committee of Publication submits the following report for the period from April 1, 1916 to April 1, 1917.

During this period, 1070 pages of the Proceedings have been issued, namely Nos. 11–14 of Vol. 51, and Nos. 1–9, of Vol. 52.

Several of these numbers, namely 51:12, 52:2, 52:3, 52:9, and part of 51:13, were paid for out of the funds of the Rumford Committee, the total charge against the Rumford Fund being $1324.39.

The accounts of the Committee of Publication stand as follows:

| | |
|---|---|
| Balance on hand April 1, 1916 | $1,408.55 |
| Appropriation for 1916–1917 | 3,000.00 |
| Additional appropriation, to cover the cost of 90 pages of biographical notices | 200.00 |
| Proceeds from the sale of publications | 164.05 |
| Total available funds | 4,772.60 |
| Expenses | 3,016.31 |
| Balance on hand April 1, 1917 | $1,756.29 |

During the present year, authors have ordered "extra" reprints, through the Committee, to the amount of $179.00.

Respectfully submitted,

Edward V. Huntington, *Chairman.*

May 9, 1917.

## Report of the House Committee.

The House Committee submits the following report for the year 1916–17.

The Committee had at its disposal at the beginning of the year a balance of $360.33. The appropriation by the Academy for the year was $1600, and there has been received for the use of the rooms,—from the Harvard-Technology Chemical Club $18, from the Colonial Society $15, from the Archaeological Society $5, from sale of chairs $4.25, making the total amount at the disposal of the committee $2002.58.

The total expenditure for the year was $1861.34, leaving at the close of the fiscal year an unexpended balance of $141.24. The expenditures may be summarized as follows:

| | | |
|---|---|---|
| Janitor | | $750.00 |
| Electricity | A. Light | 64.00 |
| | B. Power | 44.00 |
| Gas | | 11.04 |
| Water | | 8.00 |
| Telephone | | 50.95 |
| Coal | Furnace | 409.63 |
| | Water Heater | 45.50 |
| Ash tickets | | 7.00 |
| Care of elevator | | 21.46 |
| Ice | | 14.40 |
| Janitor's materials | | 16.72 |
| Furnishings | | 228.62 |
| Upkeep | | 188.02 |
| Sundries | | 2.00 |
| Total expenditure | | $1,861.34 |

Meetings have been held as follows:—

| | |
|---|---|
| American Academy of Arts and Sciences | 8 |
| Harvard Biblical Club | 6 |
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American Oriental Society held two all day sessions April 10 and 11, and one evening session, April 10.

Respectfully submitted,

Hammond V. Hayes, *Chairman.*

May 9, 1917.

On recommendation of the Rumford Committee, it was

Voted, To award Rumford Premium to Percy Williams Bridgman, of Cambridge, Mass., for his Thermodynamical Researches at Extremely High Pressures.

The annual election resulted in the choice of the following officers and committees:—

Charles P. Bowditch, *President.*
Elihu Thomson, *Vice-President for Class I.*
William M. Davis, *Vice-President for Class II.*
George F. Moore, *Vice-President for Class III.*
Harry W. Tyler, *Corresponding Secretary.*
Wm. Sturgis Bigelow, *Recording Secretary.*
Henry H. Edes, *Treasurer.*
Arthur G. Webster, *Librarian.*

*Councillors for Four Years.*

Henry Lefavour, *of Class I.*
William T. Sedgwick, *of Class II.*
Barrett Wendell, *of Class III.*

*Finance Committee.*

Henry P. Walcott, John Trowbridge,
George V. Leverett.

*Rumford Committee.*

Charles R. Cross, Arthur G. Webster,
Edward C. Pickering, Arthur A. Noyes,
Louis Bell, Elihu Thomson,
Theodore Lyman.

*C. M. Warren Committee.*

Henry P. Talbot, Charles L. Jackson,
Walter L. Jennings, Arthur D. Little,
Arthur A. Noyes, Gregory P. Baxter,
William H. Walker.

### *Publication Committee.*

Edward V. Huntington, *of Class I.*
Walter B. Cannon, *of Class II.*
Albert A. Howard, *of Class III.*

### *Library Committee.*

Harry M. Goodwin, *of Class I.*
Thomas Barbour, *of Class II.*
William C. Lane, *of Class III.*

### *House Committee.*

George T. Agassiz, Louis Derr,
Wm. Sturgis Bigelow.

### *Committee on Meetings.*

The President, William M. Davis,
The Recording Secretary, Edwin B. Wilson,
George F. Moore.

### *Auditing Committee.*

George R. Agassiz, John E. Thayer.

The following gentlemen were elected Fellows of the Academy:—

Raymond Clare Archibald, of Providence, to be a Fellow in Class I., Section 1.

Frank Morley, of Baltimore, to be a Fellow in Class I., Section 1.

Frederick Slocum, of Middletown, to be a Fellow in Class I., Section 1.

Gordon Ferrie Hull, of Hanover, to be a Fellow in Class I., Section 2.

John Zeleny, of New Haven, to be a Fellow in Class I., Section 2.

Bernard Arthur Behrend, of Boston, to be a Fellow in Class I., Section 4.

Charles Francis Brush, of Cleveland, to be a Fellow in Class I., Section 4.

Herbert Ernest Gregory, of New Haven, to be a Fellow in Class II., Section 1.

John Duer Irving, of New Haven, to be a Fellow in Class II., Section 1.

Frederic Brewster Loomis, of Amherst, to be a Fellow in Class II., Section 1.

Alexander George McAdie, of Readville, to be a Fellow in Class II., Section 1.

William John Miller, of Northampton, to be a Fellow in Class II., Section 1.

Louis Valentine Pirsson, of New Haven, to be a Fellow in Class II., Section 1.

Percy Edward Raymond, of Cambridge, to be a Fellow in Class II., Section 1.

William North Rice, of Middletown, to be a Fellow in Class II., Section 1.

Charles Willison Johnson, of Boston, to be a Fellow in Class II., Section 3.

Richard Swann Lull, of New Haven, to be a Fellow in Class II, Section 3.

John Broadus Watson, of Baltimore, to be a Fellow in Class II., Section 3.

Frederick John Foakes-Jackson, of New York, to be a Fellow in Class III., Section 1.

Arthur Lord, of Plymouth, to be a Fellow in Class III., Section 1.

Charles Edwards Park, of Boston, to be a Fellow in Class III., Section 1.

Edward Douglass White, of Washington, to be a Fellow in Class III., Section 1.

Eugene Watson Burlingame, of Albany, to be a Fellow in Class III., Section 2.

Joseph Clark Hoppin, of Boston, to be a Fellow in Class III., Section 2.

William Guild Howard, of Cambridge, to be a Fellow in Class III., Section 2.

Ralph Adams Cram, of Boston, to be a Fellow in Class III., Section 4.

Edward Waldo Emerson, of Concord, to be a Fellow in Class III., Section 4.

Horace Howard Furness, of Philadelphia, to be a Fellow in Class III., Section 4.

Chester Noyes Greenough, of Cambridge, to be a Fellow in Class III., Section 4.

Francis Barton Gummere, of Haverford, to be a Fellow in Class III., Section 4.

Allan Marquand, of Princeton, to be a Fellow in Class III., Section 4.

Richard Clipston Sturgis, of Boston, to be a Fellow in Class III., Section 4.

The following gentlemen were elected Foreign Honorary Members:—

Tullio Levi-Civita, of Padua, Italy, to be a Foreign Honorary Member in Class I., Section 1.

Frank Dawson Adams, of Montreal, to be a Foreign Honorary Member in Class II., Section 1.

Ramon Menendez Pidal, of Madrid, to be a Foreign Honorary Member in Class III., Section 2.

The following communication was presented:—

Professor Charles R. Lanman. "The Harvard Oriental Series; its Purpose and Set-backs and Progress."

The following papers were presented by title:

"Fraxinus in New Mexico and Arizona." By Alfred Rehder.

"The Algae of Bermuda." By Frank S. Collins and Alpheus P. Hervey.

On motion of Professor W. T. Sedgwick, it was

Voted, That the thanks of the Academy be extended to Dr. Henry Pickering Walcott on his retirement from the Presidency for his able and faithful administration of that office.

The meeting then adjourned.

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## CLEVELAND ABBE (1838–1916)

**Fellow in Class II, Section 1, 1884.**

To be known, for a quarter-century, as the dean of American meteorologists; to be recognized as the chief factor in bringing about the inauguration of our national system of weather forecasts; to be accepted as the mainstay of meteorology in the United States and as one of the world authorities in this science — these are no slight distinctions. They were accorded, and accorded ungrudgingly, to Cleveland Abbe, for nearly fifty years an active officer of our national weather service; a modest, careful scientific worker; a ready helper of all who came to him for advice and information; a veritable storehouse of meteorological facts; a devoted student who knew the literature of his science as few do, and as few have done. The death of Abbe, on Oct. 28, 1916, at the age of 77, has removed the last of the pioneers of the older school of American meteorologists. His work laid many of the foundations upon which our later progress has been built up.

Abbe early became a tremendous reader and his attention, even in boyhood, was directed toward the phenomena of the sky and of the air. His first work was astronomical. He studied at Ann Arbor (1858–60), and with Gould in Cambridge (1860–64). He was at the Pulkova Observatory, in Russia, for two years, under Struve; later at the United States Naval Observatory as aid, and in 1868 became director of the Cincinnati Observatory. Here his career as a meteorologist really began. On September 1, 1869, he there inaugurated a system of telegraphic daily weather reports and daily weather maps for the purposes of weather forecasting. Of this undertaking he wrote to his father: "I have started that which the country will not willingly let die." This pioneer venture was brought to the attention of the National Government, and on Feb. 4, 1870, Congress passed a joint resolution establishing a meteorological service under the jurisdiction of the Chief Signal Office of the Army.

On the invitation of Gen. A. J. Meyer, the Chief Signal Officer, Abbe entered the Government service in January, 1871, and was given charge of weather forecasting. For some time he did the lion's share of the work, and under him the early forecasters of the Signal Service were trained. For twenty years (1871–1891), Abbe was professor of meteorology and civilian assistant in the office of the Chief

Signal Officer and when, in 1891, the transfer of our Government meteorological work to the Department of Agriculture was accomplished, Abbe continued as professor of meteorology in the Weather Bureau. Through six changes of his administrative chiefs, until his resignation in August, 1916, Abbe kept on with his work, often under conditions which, to his sensitive nature, must have been trying in the extreme. Patient; uncomplaining; devoted to his studies, he continued faithfully at his post.

Abbe's remarkable knowledge of the literature of his science served him well in the preparation of his "Treatise on Meteorological Apparatus and Methods" (*Appendix 46, Annual Report of the Chief Signal Officer for 1887*), a practical and historical report of enduring value, and of "A First Report on the Relations between Climate and Crops" (*U. S. Weather Bureau, Bulletin 36*, 1905), as well as in his contributions to the "Bibliography of Meteorology" (*U. S. Signal Service,* 1891), a valuable but comparatively little known publication. The many duties of his position in Washington left Abbe little time for original investigation. As a partial substitute for such work on his part he brought before English and American readers an important, and indeed for working dynamical meteorologists an indispensable, collection of translations of important foreign memoirs ("Short Memoirs on Meteorological Subjects," first collection, *Annual Report of the Smithsonian Institution for 1877;* "The Mechanics of the Atmosphere"; second collection, *Smithsonian Miscellaneous Collections,* 843, 1891; third collection, Vol. 51, No. 4, 1910). By the publication of his "*Report on Standard Time*" (1879), Abbe became one of the leading promoters of the introduction of standard time in the United States.

It is as the editor, for many years, of the *Monthly Weather Review,* that Abbe will doubtless be longest and best known. In that position, his extraordinary grasp of meteorological literature, and his remarkable memory, made him invaluable. Not only did he himself write a very large number of articles, but he made frequent and helpful comments on the contributions of others, which added greatly to the value of the *Review* and made it one of the leading meteorological Journals of the world. In his capacity as editor, Abbe carried on an enormous correspondence, a good deal of it written with his own hand. He was always ready generously to assist all who came to him for information or for advice. Of peculiar interest to him were all questions that concerned meteorological education. He was keenly alive to every opportunity to advance and to improve meteorological teaching in the United States, and never weary of emphasizing the importance of sound training for meteorologists along mathematical and physical lines. Abbe held two academic positions. He was professor of meteorology at Columbian (now George Washington) University from 1886 to 1909, and lecturer in meteorology at Johns Hopkins University from 1896 to 1914. As a part of his work for meteorological education may be mentioned further his activity in connection with the International Meteorological Congress at Chicago (1893), and his editorship of the "Bulletin of the Mount Weather Observatory" (1909–1913). Through his articles on Meteorology in the *Encyclopaedia Britannica* he brought sound meteorological information to large numbers of general readers.

Abbe was a member of many scientific societies, both in the United States and abroad. He was elected a Fellow of the American Academy of Arts and Sciences in 1884. Two distinguished honors were awarded him abroad. He received the degree of LL.D. from the University of Glasgow in 1896, and the Symons Gold Medal of the Royal Meteorological Society (London) in 1912. The president of the Society, in presenting this medal, said of Abbe that he "has contributed to instrumental, statistical, dynamical, and thermodynamical meteorology, and forecasting," and "has, moreover, played throughout the part not only of an active contributor but also of a leader who drew others into the battle and pointed out the paths along which attacks might be successful." In April, 1916, Abbe was awarded the Marcellus Hartley gold medal of the National Academy of Sciences for "eminence in the application of science to the public welfare, in consideration of his distinguished service in inaugurating systematic meteorological observations in the United States."

During his lifetime, Abbe's modesty and self-depreciation to a large extent kept him from occupying the position of scientific prominence to which his learning and his service to meteorology entitled him. Now that he is dead, his work for the science to which he so faithfully devoted himself for fifty years is seen to have been far more important than even those who knew him best ever realized. Abbe's devotion to his work was an inspiration. His enthusiasm was a stimulus to all who came in contact with him. His cheerfulness and his patience were an example which could not fail to encourage his associates and his colleagues, everywhere.

ROBERT DEC. WARD.

## SIR THOMAS LAUDER BRUNTON (1844–1916)

**Foreign Honorary Member in Class II, Section 4, 1901.**

Thomas Lauder Brunton was born in Scotland, took his M. B. with honors at Edinburgh in 1866, his M. D. in 1868, with a gold medal for his thesis on Digitalis with Some Observations on the Urine. This thesis was based on experiments on himself. His friend, Sir David Ferrier, says in The Lancet,— "When experimenting on himself with digitalis he lived a life of penance for six months; and he told me that one of the greatest pleasures he ever experienced was when he felt at liberty to eat and drink without having to weigh and measure his ingesta and egesta." He was altogether too human to be an ascetic; but for the sake of science he could mortify the flesh. In his first Lettsomian lecture, 1885, the reading of which, as indeed of much of Brunton's writings, might be of service to preachers of the present day calories, alcohol-always-a-poison apostles, he says:

'The nerves of taste, like those of sight and hearing, are nerves of special sense, and are capable of education. But, while we usually regard the education of the senses of sight and hearing as a noble thing, we are too careless of the education of our taste, and look upon it rather as something degrading.

'Yet the education of the nerves of taste should be considered in the same light as that of the other special senses; and cookery has, I think, a perfect right to be ranked with music, painting, sculpture, and architecture as one of the fine arts. The difference between cookery and music, or painting, is, that while the objects which give rise to sight and sound remain outside the body, we are obliged to swallow the substances which excite sensations in our nerves of taste. It is not quite sufficient to turn them over in the mouth and put them out again, because the full sensation is only obtained just in the act of swallowing. For this reason devotees to the art of cookery must either be content with a moderate enjoyment of the pleasures of taste, or consent, like some of the Roman emperors of old or German students of the present day, to eject again the food or drink which they have already taken and enjoyed.

'Only rarely does one meet with a dinner which gives one the sense of high artistic perfection, although I remember having partaken of one such when enjoying the hospitality of a City company. Each course seemed to excite an appetite for the one which succeeded, and was accompanied by a wine so carefully selected that it gave zest to the food, while the food appeared to give additional flavour to the wine.

'This dinner was a revelation to me; it not only showed me that cookery might rank as one of the fine arts, but taught me that it might be a powerful moral agent. I went to the dinner exhausted with overwork, irritable in temper, and believing that City companies were wasteful bodies, who squandered money that might be employed for useful purposes, and that they should be abolished; I came away feeling strong and well, with an angelic temper, and firmly convinced that City companies had been established for the express purpose of giving dinners, and ought on no account to be interfered with. Nor was the good thus effected of a transitory nature; the irritability of temper, which had disappeared in the course of dinner, did not return; and the morning afterwards, instead of awaking with headache and depression, I awoke strong, well, and ready for work, and remained so for a considerable length of time. Nor do I think that mine is a solitary case. A succession of heavy dinners is, no doubt, injurious; but when the organism is exhausted, a good dinner, with abundance of wine, is sometimes of the greatest possible use. But there is one condition which must not be neglected, or otherwise the consequences will be anything but satisfactory; the dinner must be well cooked, and the wines must be thoroughly good.

'It is, as I have said, only occasionally that one meets with real high artistic cookery. But, even in the courses of an ordinary dinner, an order is adopted which is thoroughly physiological, and which shows that, whatever men may be in other things, they are not "mostly fools" in regard to the plan of their meals.' "

The above contains so much truth, and throws so much light on Brunton, his life work, and his brilliant success as a consultant, that no abstract would take its place.

He served for a year as resident in hospital, and then spent about two years in laboratory work in Vienna, Berlin, Amsterdam, and Leipsic. In the latter place he was busy on the action of nitrites, and introduced amyl nitrite as a remedy for angina, thus, according to Mitchell Bruce, being "the first to employ a remedy in disease because its pathological action was to act in an opposite manner to the pathological condition which he had discovered in the disease — angina pectoris — vis., rise in blood pressure." Rise in the general blood pressure is by no means invariable in angina — but let that pass.

He then, like many another canny Scot, settled in London, becoming casualty physician and lecturer on Materia Medica at Bartholomew's in 1871. This hospital connection lasted for thirty-three years, twenty as assistant, and nine as physician. He then resigned before he had reached the age limit, and became Consulting Physician and Governor to the Hospital. He acquired fame as a consultant, and among those who sought his advice were many Americans.

He was a prodigious worker. This is not the place for a full list of his publications and activities. At thirty he was elected a Fellow of the Royal Society. He was Goulstonian, Lettsomian, and Croonian lecturer, and Harveian orator. He was Fellow of the Royal College of Physicians, and for years one of the examiners; prominent in the British Medical Association, knighted in 1900, created Baronet in 1908. His writings, which are many and varied, deal mainly with Pharmacology and Therapeutics, based upon and colored by experimental physiology. In this line of work he was among the first. A hint of the breadth of his therapeutic horizon is afforded by his paper on "The Science of Easy Chairs," originally printed in Nature. His largest work was his text book of Pharmacology, Therapeutics, and Materia Medica, 1885. Several editions were called for, and also translations in several languages. Other books were "Disorders of Digestion," "Disorders of Assimilation," "Collected Papers on Circulation and Respiration," and "Therapeutics of the Circulation." The first edition of the latter appeared in 1908, a second March 14, 1914, "My 70th birthday," as he writes in a presentation and greatly valued copy to the writer of this inadequate sketch. It is interesting to note that this last named book was dedicated to Kronecker, a fellow worker in Leipsic under Ludwig, and a life-long friend. Brunton had actively supported all plans for furthering national health, school hygiene, and military training, and foresaw the inevitability of the present war. In August, 1915, Sir Douglas Haig writes him,—"You and I have often talked about the certainty of this war, and have done (each of us) our best to prepare in our own spheres for it."

He thought he had many valued friends among the German professors, and was not prepared for their attitude toward the war in general, and the British in particular. This attitude was a great grief to him. In October, 1915, his younger son was killed in action in France. His heart had for years given him trouble, at times serious, righted itself more than once, but finally gave out after distressing disability.

F. C. SHATTUCK.

## LUIGI PALMA DI CESNOLA (1832–1904)

Fellow in Class III, Section 4, 1881.

Luigi Palma di Cesnola was born at Rivarola June 29, 1832. After completing his education at the Royal Academy of Turin, he served with distinction in the Sardinian army during the revolution of 1848, and again in the Crimean war. The same love for liberty which led him into the war against Austrian rule brought him to the United States, where he was naturalized as a citizen in 1860. Enlisting in the army he was rapidly promoted till he became colonel of the fourth New York Cavalry. At the battle of Aldie he was severely wounded and taken prisoner, being held in Libby prison for nearly a year. Liberated in 1864 he served again till the close of the war and was mustered out with the rank of Brigadier General. From 1865 to 1876 he was Consul General of the United States in Cyprus. Here he soon became interested in antiquities, conducted extensive excavations, and gathered the largest collection of Cypriote antiquities that has ever been made. The acquisition of the main part of this collection by the Metropolitan Museum again brought Cesnola into relation with New York. In 1876 he resigned his post in Cyprus to become Secretary of the Metropolitan Museum, and in 1879 he was made Director, a position which he held till his death November 21, 1904. He held the degree of Doctor of Laws from Columbia and Princeton, in 1897 he received the medal of honor from Congress, and the King of Italy caused a gold medal to be struck in his honor besides granting him several knightly orders.

Although a student of art in his boyhood, his interest in the work for which he gained renown was largely the result of circumstances. When he came to Cyprus in 1865 his labors as consul for the United States and for Russia were not arduous, his British and French colleagues were already engaged in collecting antiquities, and he found abundant opportunity to engage in the same pursuit. His success in winning the friendship of the natives brought him early information of important discoveries, while his genius for organization proved invaluable in the excavations he undertook. General di Cesnola was not a scholar or an archaeologist, but a collector. In 1865, when he began his work, scientific excavations were as yet unknown, and relatively little was known of the complex problems raised by Cypriote antiquities. That Cesnola failed to develop scientific methods of excavation as they were developed by later excavators, has elicited criticism as bitter as it is unjust; that his records are incomplete and apparently sometimes inaccurate, is a just criticism on his work; that he was a most skillful excavator and a successful pioneer in bringing to the attention both of scholars and of the general public a vast mass of material illustrating ancient art in Cyprus, remains unquestioned. Of the charges brought against him by M. Gaston L. Feuardent and others in 1880–1882 that many of the antiquities were "faked," falsely restored, etc., it is enough to say that they were not substantiated. Recent careful examination of the objects in New York by trained scholars has revealed no trace of falsification. The great collection purchased for New York in 1872 is still the glory of the Metropolitan Museum of Art.

In his work as Director of this Museum General di Cesnola showed the same qualities of leadership that appeared in his work as excavator and collector. To blame him for not making the museum a great factor in popular education is, again, to judge him by the standards of a later age. The organizing genius that directed the work of the institution for nearly thirty years, built up its collections, arranged its exhibitions, and laid the foundation for its recent growth has won the respect and admiration of all who are familiar with what Cesnola accomplished there. He was a successful pioneer in this field, as in the field of Cypriote antiquities.

In the minute of the trustees of the Metropolitan Museum on the occasion of his death, the following estimate of his work is given:

"His fidelity, his minute attention to his duties and his capacity for work during his long career of service, merit great praise. Other distinctions and other interests in life, if not forgotten, were permanently laid aside, and the welfare and growth of the Museum became his single interest and absorbing occupation. His military training, when joined to his public experience, gave him distinguished powers of administration; and, while critics are never wanting, his capacity to administer the Museum and adequately to exhibit its contents has not been questioned."

ARTHUR FAIRBANKS.

## SAMUEL FRANKLIN EMMONS (1841–1911)

**Fellow in Class II, Section 1, 1903.**

Samuel Franklin Emmons, one of the most eminent students and investigators of economic geology, was born in Boston, March 29, 1841, and died at his home in Washington, March 28, 1911, thus lacking only one day to complete his seventieth year. Since 1867 he was connected with the federal scientific surveys, first as a member of the Geological Exploration of the Fortieth Parallel under Clarence King and later, since 1879, as a geologist of the U. S. Geological Survey. For both organizations he performed most important scientific work both of a purely geological and practical nature and his name will always be prominently associated with the development of geology in America.

Emmons obtained his degree of Bachelor of Arts at Harvard and afterward studied three years at Ecole des Mines in Paris and at the Bergakademie at Freiberg. Returning to the United States he joined the Fortieth Parallel Survey in 1867, a work admirably planned for obtaining the greatest efficiency and speediest results in geological reconnaissance. A large part of the geological results of this survey is due to the painstaking and exact work of Emmons.

When the U. S. Geological Survey was created in 1879 Emmons was selected to take charge of investigations in economic geology. The first important work completed was a monograph on the Leadville district in Colorado, a region presenting most intricate problems of mining geology. Emmons' monograph has been of the greatest value and importance to the miners and it may be said that it is the most monumental evidence of the value of geology to the mining industry. In this and in the many later reports published by Emmons the importance of "replacement" as a process in ore formation was strongly emphasized and it is one of his principal merits to have made the mining engineer acquainted with this mode of nature's operation, by which, for instance, ore bodies of galena take the place, molecule for molecule, as it were, of strata of limestone.

Space does not suffice to mention all the investigations and reports which occupied his time. Among other mining districts he examined may be mentioned Butte, Aspen, Mercur, Bingham, the Black Hills, and Cananea.

In 1900 he contributed a most important paper to the Institute of

Mining Engineers, entitled "Secondary Enrichment of Ore Deposits" in which for the first time attention was drawn to the rich sulphides just below the water level which owe their origin to the descending surface waters. Emmons' contributions to geological literature were contained in nearly 100 monographs, reports and papers.

Emmons' name, as a mining geologist, is known all over the world. Thoroughness, efficiency and good judgment characterized his work throughout. His kindly and unselfish personality endeared him to all who had the privilege of his acquaintance and he was a potent influence in the work of younger geologists in the organization in which he for many years directed the investigations in mining geology.

WALDEMAR LINDGREN.

## DANIEL COIT GILMAN (1831-1908)

**Fellow of Class III, Section 2, 1875.**

Daniel Coit Gilman was born July 6, 1831, at Norwich, Connecticut, and died in the town of his birth, October 13, 1908. He was the son of William Charles Gilman (1795-1863) and his wife, Eliza Coit (1796-1868), and connected with many of the best-known families immigrant from England to New England in the seventeenth century.[1] The facts of his life are briefly and admirably told by his brother, William Charles Gilman, in the *Johns Hopkins University Circular* for December, 1908; and more fully in *The Life of Daniel Coit Gilman* by Fabian Franklin, New York, 1910. But nowhere, now that he is gone, can one get a better idea of his character and personality than from his own public addresses, especially the collection made by him, and published at New York in 1906 under the title, *The Launching of a University, and other papers: a sheaf of remembrances.*

Mr. Gilman received the bachelor's degree at Yale in 1852 and spent the following year there as a resident graduate. "On the whole," he says, "I think that the year was wasted." And his experience at Harvard in the autumn of 1853 was similar. So far as his then immediate purpose was concerned, his estimate of these two experiences is doubtless true. But nothing could be further from the truth, if we consider his experience of the scantiness of opportunity for advanced non-professional study at Yale and Harvard in its bearing upon the great problems that were to confront him twenty-two years later. We may well deem it the most happily fruitful failure of his whole life.

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[1] See *The Gilman Family*, by Arthur Gilman, Albany, N. Y., 1869; *The Coit Family*, by F. W. Chapman, Hartford, Conn., 1874.

His public service began even in that "wasted year." For in August, 1853, the first annual convention of American Librarians was held, largely as the result of his efforts. In December, 1853, he and his life-long friend, Andrew Dickson White, sailed for Europe as attachés of the American Legation at St. Petersburg. Here his official position gave him uncommon opportunities for learning about libraries and schools and other institutions for public welfare, an admirable preparation for the work of his life as a leader in educational and social progress. On returning, he became librarian at Yale (1856–1865), and then professor of physical and political geography in the Sheffield Scientific School, and indeed virtually its chief executive, improving its working-plans and strengthening its finances. From 1872 to 1875 he served as president of the University of California, and, in the face of most discouraging obstacles, succeeded in placing it upon a much securer foundation. Then came the call to organize the new institution created by Johns Hopkins at Baltimore. The Trustees, a group of enlightened and devoted men, sought the advice of President Eliot of Harvard and President White of Cornell and President Angell of Michigan, invited them to come to Baltimore to give it by word of mouth, and wrote to each of them after their return home asking whom they would suggest for the office of president. They all with one accord, and without any previous conference on the matter, replied that "the one man" was Daniel C. Gilman.

Now that university education in America has grown to be what in large measure Mr. Gilman's initiative and example have made it, it is hard to realize what the problem then was. The will of Johns Hopkins left the utmost freedom to the Trustees. Should they, as was suggested, "raise an architectural pile that shall be a lasting memorial of its founder"? should they establish one more college? At his first meeting with the Trustees, Mr. Gilman urged them to create a "means of promoting scholarship of the first order," something, as he himself says, that should be "more than a local institution" and that should "aim at national influence." Those whose privilege it was to hear the testimony of such men as President Eliot or Francis

A. Walker in the early eighties, were abundantly assured of the fact that President Gilman's broad and noble ideals were indeed becoming realities, and realities of great influence as examples the country over. On the occasion of Mr. Gilman's retirement after twenty-five years of service, Mr. Eliot's address specifies three great achievements: the creation of a school of graduate studies, the prodigious advancement of medical teaching, and the promotion of scientific investigation. The first overtures looking to Mr. Gilman's appointment as president of the Carnegie Institution came some six months after his resignation of the presidency of the university. Although past the limit of three score and ten, he served the Institution for three years at the critical time when the fundamental purposes of so novel an undertaking were yet to be determined.

Of the amount and variety of Mr. Gilman's public service outside the sphere of strictly official duty, it is not feasible in a brief paragraph to give an adequate idea. The very important position of Superintendent of the Schools of New York City he felt obliged in 1896 to decline. But as member of the Board of School Commissioners of Baltimore, as president of the Slater Fund for the Education of Freedmen, as member of the General Education Board, as a trustee of the Russell Sage Foundation, as president of the National Civil Service Reform League, and of the American Oriental Society (with whose early history and most eminent members he was intimately acquainted), he was a fellow-worker of amazing constancy and faithfulness,— "a fellow-worker," for he always thought and spoke of his associates, not as subordinates, but as colleagues.

As one looks back on Mr. Gilman's presidency at Baltimore, it seems as if he could not have fitted himself better for it, even if he had known that just *that* was to be the main business of his life. His personal acquaintance with men eminent in science and letters, men like Huxley and Cayley and Lord Kelvin, men like Lowell and Child and Whitney, his wide and studious observation of great technical schools and his experience in the building up of the one at Yale, his realization of the unity of knowledge, his intelligent comprehension of the aims of the most diverse fields of study,— these were factors of his equipment for the work of "naturalizing in America the idea of a true university." Many men of equal industry and force have failed because they did not realize as he did "the inanity of rivalry, the pettiness of jealousy, and the joyfulness of association for the good of mankind." The informing principle of his life was service to others. As an old-time New Englander, he was brought up in obedience to

"Duty, stern daughter of the voice of God." But to him the paths of duty were also ways of pleasantness. For it was a supreme delight to him to see his pupils and associates and colleagues become men of distinguished usefulness to their fellow-men. It was, I believe, one of the greatest secrets of his success as president of the university that he made his associates feel sure that he took a genuine and sympathetic interest in what they were doing. He was wont to quote Emerson's saying, "Nothing great was ever achieved without enthusiasm." With Mr. Gilman, enthusiasm was a divine gift, and from his living flame he was able to kindle the sacred torch in the hand of others. The belief that "the things which are not seen are eternal" was part of his very life, and sustained his courage in the absence of showy results for which many were hoping. To few lives do the words of St. Paul at Antioch more fitly apply: He "fell on sleep, after he had served his own generation by the will of God."

Charles R. Lanman.

## FRIEDRICH KOHLRAUSCH (1840–1910)

Foreign Honorary Member in Class I, Section 2, 1900.

Friedrich Kohlrausch was born in Rinteln, Oct. 14, 1840, and died in Marburg, Jan. 17, 1910.

There is a commonly accepted belief that successive generations of the same family do not attain great distinction. To this the family of Kohlrausch is an exception. Rudolf Kohlrausch, the father of Friedrich, was a distinguished physicist and professor in the University of Göttingen, well known because of his determination, with Wilhelm Weber, of the relation between the electrostatic and electromagnetic unit of current, which forms one of the great landmarks in the history of our science. His grandfather, it may be noted, was a historian, also of national reputation, whose history of Germany, in two volumes, ran through sixteen editions at a time when such occurrences were less common than now.

The father died when the son was but eighteen (1858), leaving Friedrich to pursue his studies alone. He obtained his doctor's degree at Göttingen with his father's colleague, Weber, in 1863, and after a short service as Docent in the laboratory of the Physikalischer Verein in Frankfurt, entered upon his academic career as professor extraordinarius in 1866 in the same university. After four years of service in Göttingen, he was appointed Ordinarius at Zürich in 1870. The fear of annexation, widely current in Switzerland during the Franco-Prussian war, aroused some bitterness even in academic circles, and Kohlrausch never looked back upon his residence in Zürich with pleasure. He returned to Germany (Darmstadt Polytechnicum) the following year. It was in the University of Würzburg, to which he was appointed in 1875 and where he remained for thirteen years, that his most fruitful work, namely, that on the conductivity of solutions, was begun. These studies of conductivity were continued with occasional interruptions throughout his active career.

In 1888, Kohlrausch was chosen to the professorship of physics in Strassburg, and in 1894 he was invited to become the successor of Kundt in the great University of Berlin. This latter position, to occupy which was to be accredited the dean of the German university physicists, Kohlrausch declined, partly for reasons of health and partly because of the tremendous volume of administrative work required of the occupant of this chair. But a few months elapsed, however, before he was called again to Berlin, this time as President of the Physikalisch-technische Reichsanstalt in succession to Helmholtz. In this position Kohlrausch proved a veritable inspiration to the considerable group of young men (of which the writer had the good fortune to be one) which had been gathered together for research in connection with the fundamental standards of physical measurement, and it was during his administration that the Reichsanstalt took its position at the head of institutions of its kind throughout the world. The English National Physical Laboratory, established at Teddington during this period, and the Bureau of Standards at Washington, were modeled directly from the Reichsanstalt.

The burden of this responsibility, however, proved to be too great for health which had never been rugged, and he laid it down amid universal regret and retired to private life in 1905. Even in his retirement a considerable number of papers testified to the continued fertility of this extraordinary brain, and at the time of his death in 1910 he had just completed the eleventh edition of his Lehrbuch der praktischen Physik, a book which has proved indispensable to every laboratory worker in physics during our time.

Since Kohlrausch's death his papers have been gathered together and published in two large volumes of "Gesammelte Abhandlungen," which, with the Lehrbuch above referred to, form a monumental record of scientific activity. To pass this in review is hardly practicable within the space of a short memorial sketch. He has made contributions to many branches of physical research, and through the "Lehrbuch" and the great variety of apparatus which bears his name, his relation to his chosen science, both in his own country and abroad, has been of the most intimate kind. Rather by way of illustration than with any purpose of presenting an adequate review of his scientific work, certain of his papers may be cited.

In the category of precise physical measurements we may notice first of all his determination of the ohm and a considerable number of papers upon the measurement of the earth's magnetic constants. In the course of these latter studies new methods and new apparatus were developed many of which are still fully equal to the most exacting requirements of modern quantitative science. The determination, in association with his brother Wilhelm, of the electrochemical equivalent of silver also finds a place in this period of his activity.

The name of Kohlrausch is perhaps most familiarly associated with the work on electrical conductivity in solutions which owes its foundation (the applicability of Ohm's Law to conductors of the second class) and much of its modern development (the theory of polarization, dissociation, and ionic conductivity) to the work of Kohlrausch and his pupils, among whom were included Arrhenius, Barus, Nernst, and many others hardly less distinguished.

ARTHUR L. DAY.

## ERASMUS DARWIN LEAVITT (1836–1916)

**Fellow of Class I, Section 4, 1878.**

Erasmus Darwin Leavitt was born at Lowell, Mass. on October 27, 1836, and died in Cambridge, Mass. on March 11, 1916. He was named for his father, who was given the name because of his father's admiration for Darwin's grandfather.

The public schools of Lowell furnished young Leavitt his education up to the age of fourteen. After that he educated himself for the profession of mechanical engineer, supplementing his work in various machine shops with long and patient study far into the night. To have become, under such conditions, one of the foremost men of the day in his profession, speaks volumes for his grit, pertinacity and ability.

After serving three years as apprentice in the Lowell Machine Shops, he entered the employ of Corliss & Nightingale. In 1858 he served as assistant foreman in the works of Harrison Loring at South Boston, where he had charge of constructing the engine of a United States man-of-war. The next three years found him in Providence, filling the position of chief draftsman for Thurston, Gardner & Co., celebrated at that time as builders of high grade steam engines.

At the outbreak of the Civil War, Leavitt enlisted in the engineers' department of the Navy. He served for nearly three years on the gunboat Sagamore, attached to the Eastern Gulf Squadron; where he reached the grade of second assistant engineer. After this, he was transferred to construction work in several of the United States Navy Yards. In 1865, he was detailed as instructor of steam engineering at Annapolis. Two years later he resigned from the Navy, and opened an office for the practice of mechanical engineering.

One of his first pieces of work, designed for the Plymouth Cordage Co., was a simple condensing walking beam engine, which is still in working order. After the construction of this engine, Mr. Leavitt remained the consulting engineer of the Company until his retirement from business.

It was not long before Mr. Leavitt's engines began to attract wide attention. He had for some time been interested in the economy of pumping engines, and his fame as an engineer may be said to date from the installation of an engine for the Lynn Water Works, embracing all his ideas of efficiency and economy.

Without entering into the details of the construction of this pump, it is sufficient to state that it set a world wide standard for pumping engines. It was officially tested by a number of the best experts of the day, and developed a duty of over 103,900,000 foot pounds per hundred pounds of picked Lackawanna anthracite.

Shortly after this, Mr. Leavitt designed a pair of similar, but somewhat larger pumping engines for Lawrence, Mass.

By 1874 his fame was so well established, that when Mr. Alexander Agassiz was looking about for an engineer to take charge of designing the equipment of the Calumet and Hecla Mine, the latter's choice fell on Mr. Leavitt.

This opened up a wide field for a mechanical engineer. For Mr. Agassiz was looking ahead to the future great development of the mine, and it was his policy to install an equipment for the coming years, and to install it in duplicate, to avoid the loss that would otherwise arise from any delays caused by accidents to the plant.

Mr. Leavitt stood for low speed engines of high economy, and to this end he used a complicated valve gear. The initial cost of such engines was high, but he always justly maintained, and in this he was upheld by Mr. Agassiz, that the cost of any engine was of slight account compared to its efficiency. Many experts have doubted if this holds true for hoisting engines, on account of the irregularity of the load. It is only of recent years that it has been generally acknowledged that high efficiency hoisting engines justified their expense.

The "Superior," installed in 1883, was the largest hoisting engine Mr. Leavitt built for the Calumet and Hecla. It was an inverted compound beam engine with cylinders 40′ and 70′ in diameter. The fly wheel and the belt wheel were both 36 ft. in diameter. This engine was designed to hoist six four ton skips from a depth of four thousand feet, and also to run four Rand compressors. As this was vastly in excess of the needs of the day, the engine was looked on as a foolish monster by many people. In 1911 it was hoisting five ton skips from a depth of six thousand feet, and it is still doing efficient work to-day.

The steam stamps that Mr. Leavitt designed for the Calumet and Hecla marked a very distinct advance in the construction of such machinery. The distinctive feature of these stamps is a steam cylinder with two pistons, one with steam on top to give the blow, while under the other is a constantly applied reduced pressure to lift the stamp, the steam being forced back to the boiler with each blow.

Mr. Leavitt was not inclined to talk of his achievements. Once when he was asked how he got the idea of the Leavitt stamp, he replied: "One day in the Calumet mine, I stepped on the man engine at the twentieth level to go to the surface. I had no thought how to meet the problem at the moment. But when I stepped off, about twenty minutes later, the whole scheme was clear in my mind." To properly appreciate this feat, one should realize that the man engine, now obsolete, offered an absorbing and somewhat hazardous method of transportation, which fully occupied the attention of most people making use of it.

One of Mr. Leavitt's pumping engines for the Calumet and Hecla mills, had a capacity of sixty million gallons in twenty-four hours. It remained the largest pump of its day, until superseded by an engine of seventy-five million gallons per twenty-four hours, designed by Mr. Leavitt for the sewer works of the City of Boston.

His designs for Calumet included sand wheels of metal, instead of wood, the construction previously used in Northern Michigan. Two such wheels, built like bicycle wheels, had diameters of fifty and sixty feet.

Until ill health caused his retirement from business in 1904, Mr. Leavitt continued to act as consulting mechanical engineer for the Calumet and Hecla Mine. But his activities were by no means confined to work for that company.

He acted as consulting engineer for Henry R. Worthington; developed their high duty pumping engine for the Dickson Manufacturing Co.; and assisted the Bethlehem Steel Co. in modernizing their plant and in introducing hydraulic forging.

He designed three huge pumping engines for the Boston sewage department; and pumping engines for the following city water works: Cambridge, Mass., New Bedford, Boston, and Louisville. That for the last named city broke all previous records for economy in consumption of steam.

A pair of engines which he designed for the Washington Mills at Lawrence, have been steadily at work since 1887. He designed the equipment of the El Callao Mining Co. of Venezuela. The first engines of the cable railway of the Brooklyn Bridge came from the drawing boards of his remarkably efficient and well organized office in Central Square, Cambridgeport.

Mr. Leavitt's fame as a mechanical engineer was international, and during his frequent visits abroad he established a wide circle of professional friends. He was on intimate terms with the Krupps, and was on board their yacht Rona at the opening of the Kiel Canal.

A roomy old fashioned house on the slope of the hill on Harvard Street, Cambridgeport, was Mr. Leavitt's home for many years. There he delighted to entertain the prominent members of his profession, and many eminent European engineers enjoyed his hospitality. He was an ardent church man, a man of broadly charitable instincts, and widely known for the liberality of his gifts.

He was distinguished for an exceptional command of English, which was evident to a casual acquaintance. Although of a retiring character, his affectionate disposition and inborn geniality endeared him to all who were fortunate enough to penetrate his reserve.

Mr. Leavitt was one of the thirty founders of the American Society of Mechanical Engineers, and served as its president in 1882–83. He was a foreign member of several European engineering and scientific societies, and for many years was on the visiting committees of the Engineering Department, and of the Observatory of Harvard

University. He was elected a Fellow of the American Academy of Arts and Sciences in 1878, and long served on its Rumford Committee.

The last years of his life were passed in a house that he built for himself on Garden Street, Cambridge. Mr. Leavitt had worked so hard that he never learned to play; and although fondly cared for by his family, to whom he was devoted, and notwithstanding his interest in civic affairs, it is to be feared that time hung heavily on his hands, when he felt that his health demanded his retirement from business.

In 1867 he married Anne Elizabeth Pettit of Philadelphia who died in 1889. They had five children, the survivors being Mrs. Walter Wesselhoeft, Mrs. Paul Van Daell, and Miss Margaret A. Leavitt.

G. R. AGASSIZ.

## PERCIVAL LOWELL (1855–1916)

**Fellow in Class I, Section 1, 1892.**

On both his maternal and paternal sides Percival Lowell came of stock prominent in the development of New England. The cities of Lowell and Lawrence were named for his ancestors. His father, Augustus Lowell, was a Vice President of the American Academy, his mother Katharine Lawrence was the daughter of a former Minister to Great Britian.

Lowell was born in Boston on March 13th, 1855, and fitted for college at "Noble's" school. He took honors in mathematics at Harvard and graduated *cum laude* in 1876. The elder Pierce spoke of him as one of the two most brilliant mathematicians whom he had seen at Harvard.

After leaving college he entered business in Boston. Unlike most devotees of pure science he possessed a marked gift for business matters, and became a force in the business world, where he occupied various positions of responsibility and trust.

From 1883 to 1893 he devoted his life chiefly to literature and travel. Much of this time was spent in the far East, chiefly in Japan, where for some years he made his headquarters in a charming native house in Tokio. He was appointed foreign secretary of the Special Mission from Korea to the United States, and conducted its travels through this country. On their return to Korea he remained for some time as the guest of the government. The result of this experience was a volume entitled "Chosön — The Land of the Morning Calm." The work is full of imagination and charm and infused with the light touch and true literary gift which never deserted him, and was as carefully fostered in his scientific work as elsewhere. His writings during this decade include the "Soul of the Far East," which Janet, the French psychologist, has cited as showing a remarkable insight into the Oriental mind; also a treatise on some hitherto little known aspects of Shintoism; and "Noto," a delightful account of his travels in an out-of-the-way corner of Japan.

He had always taken a keen interest in Schiaparelli's work on Mars. In the early nineties that distinguished astronomer's eyesight had so far failed that it was evident his observing days were over. Then Lowell determined to take up Schiaparelli's work where the latter had left it.

Before establishing an observatory, with characteristic thoroughness, he searched diligently for the best available spot — his investigations including sites in America, France, Algiers, the Mexican Plateau, and a station in the Andes.

One of the results of these investigations was to show that the "seeing" on an elevated plateau is much better than on a mountain top. While no place has been found better than the site chosen at Flagstaff, Arizona, it seems probable that even better results would have been obtained could the Observatory have been set back a mile or two from the edge of the table land on which it stands.

The Observatory founded in 1894 was intended chiefly for a study of the planets, especially Mars. But the investigations at the Observatory have been by no means confined to this field, much valuable work having been done on the constitution of comets, and the spectra and velocities of nebulae; while many refinements in stellar photography have been perfected there. Here the rotation of Venus and Uramus were both determined by the Doppler effect. Two of Lowell's mathematical investigations are of special interest. His "Memoir on a Trans-Neptunian Planet" gives the results of many years of painstaking labor, by himself and a staff of computors. The analysis of the disturbances produced on the outer planets by this unknown body was conducted by methods of celestial mechanics, differing considerably from those employed by Adams or Leverrier. It has not as yet been possible to verify the results either visually or photographically. A "Memoir on Saturn's Rings" is a most ingenious investigation of the probable internal constitution of the planet, deduced from the relation of the position of the divisions of the rings to the satellites.

His work and theories on Mars are most widely known. His observations on that planet have accumulated an amount of data greatly in excess of the total results of all other observers combined. If the theory, which he deduces from this data, that Mars is inhabited, seems fanciful to many, it should at least be borne in mind that it is deduced from observed facts by logical reasoning. Furthermore, no other satisfactory explanation of the facts have ever been offered. His theory would doubtlessly have made much more headway in the scientific world, had it been less dogmatically presented.

The publications of the Observatory up to the time of Lowell's death include fully 2,100 quarto pages and 2,500 octavo pages. He was besides prolific in more popular works, the chief of which were; "Mars" (1895): "The Solar System" (1903): "Mars and its Canals" (1906): "Mars as the Abode of Life" (1909): "The Evolution of Life" (1909): "The Genesis of the Planets" (1916).

Lowell died suddenly of apoplexy at his Observatory on November 12, 1916, shortly after a most successful lecture tour in the West. He lies, fittingly, close to the dome of his telescope. His entire fortune, with a certain life interest for his wife, was left to maintain the Lowell Observatory as a separate institution. It is thought that its income will eventually be at least twice that of the Harvard Observatory.

G. R. AGASSIZ.

## DMITRI IVANOVITSCH MENDELÉEFF (1834–1907)

Foreign Honorary Member in Class I, Section 3, 1889.

> "Strecker, De Chancourtois, and Newlands stood foremost in the way towards the discovery of the periodic law, and...they merely wanted the boldness to place the whole question at such a height that its reflection on the facts could be clearly seen."
>
> Mendeléeff's Faraday Lecture.

In such modest fashion this distinguished Foreign Honorary Member of the American Academy of Arts and Sciences characterized his greatest contribution to the sum of human knowledge.

Dmitri Ivanovitsch Mendeléeff, the youngest of fourteen children, was born to his parents, Ivan Pavlovitsch and Maria Dmitrievna, in Tobolsk, Siberia, on January 27th, (O. S.) 1834. His father, the Director of the College at Tobolsk, was a man of social preëminence and splendid education. It is his mother, however, who in greater measure excites our wonder and admiration. Under a regime which gave little opportunity for the higher education of women, she began to acquire the knowledge which she craved by repeating the lessons of her brother Vassili. "Books are the best friends of my life," she later assures us, "and it would be hard for me to exist only for the needs of the body, and to have no moments free for the heart, the mind, and the soul." Nor were her abilities limited to things intellectual, for when her husband lost his eyesight, and was compelled to resign, she administered with success and profit her brother's glass factory, and thus secured the means to bring up her numerous children.

Enrolled in the Gymnasium at Tobolsk at the early age of seven, Dmitri became deeply interested in science and mathematics. But he incurred the displeasure of his masters because of his distaste for languages, especially Latin — indeed he was always an inveterate foe of classicism in education. The closing years of his course brought with them his father's death, and the destruction of the glass factory by fire. In 1849, his mother wound up her affairs at Tobolsk, and brought her favorite son to Moscow. She intended to make him a student of medicine, but this career was closed to him when he collapsed at the sight of a corpse. In spite of failing strength and resources, she struggled on to St. Petersburg, where he was at last admitted to the Pedagogic Institute of the University and granted a scholarship. In the autumn of this year the mother died, with the injunction: — "Refrain from illusions, insist on work and not on words. Patiently search divine and scientific truth."

At the University, in spite of a dangerous weakness of the lungs, he "found himself" from the mental standpoint, and astonished all his instructors by his zeal and ability. His dissertation on isomorphism was ready in 1855, and the brilliancy of his final examination was recognized by the award of a gold medal. Although the condition of his health necessitated his immediate departure for the Crimea, he was soon able to go back to St. Petersburg. In 1859 he studied with Regnault in Paris, and later with Bunsen in Heidelberg. Upon his second return to St. Petersburg he attained the doctorate, and was soon made Professor in the Technological Institute. In 1866 he became Professor of General Chemistry in the University, a position which he held till 1890, when he resigned in consequence of friction with the authorities; in 1893 he was appointed Director of the Bureau of Weights and Measures, a position which he held until his death on January 20 (O. S.), 1907.

Mendeléeff was twice married, first in 1863, to a lady named Lestshoff, by whom he had a son Vladimir and a daughter Olga. Divorced from her, he married, in 1881, Anna Ivanovna Popova, an artist of ability, who bore him four other children, Lioubov, Ivan, and the twins Maria and Vassili.

His interests were by no means confined to pure science. The tremendous development of the petroleum industry in Russia is closely associated with the name of Mendeléeff, as is also the exploitation of some of her largest coal fields. He made an intimate study of the tariff question and urged reasonable protection for Russian industries, a policy which had far-reaching consequences. Agriculture, art, astronomy, education, and philosophy all received attention for him. In 1875 spiritualism gained many adherents in St. Petersburg, and Mendeléeff suggested a commission to investigate it by scientific methods. After a thorough study of famous mediums, this body, of which he was a member, concluded that all such phenomena resulted from unconscious movements or deliberate deception. His last work (1906) "Information about Russia," which ran through four editions in the six months following its initial publication, discussed his country from racial, religious, economic, industrial and educational standpoints.

Mendeléeff's travels were extensive. He had a first-hand acquaintance with the greater part of Russia; France and Germany he visited during his student years; in 1889 he delivered the Faraday Lecture in London, and again in 1891 he visited France and England to investigate the manufacture of smokeless powder. In 1893 he came to America, and attended the World's Fair in Chicago, which he did not fail to describe in an article published after his return.

Mendeléeff was a typical Russian; tall in figure, broad shouldered, with a head of unusual size, crowned with a remarkable abundance of hair. His full beard was blond; and his blue eyes deep set and piercing. His voice was deep, his gait rapid, and he was given to nervous and rapid movements of the hands. Restless, imperious, and brusque at times, nicknamed "the lion in his den," he was still affable, democratic, and deeply beloved by students and common people alike. He was a lifelong adherent of the Orthodox Church, and a ready friend to the discouraged and downtrodden. Such was the personality of one of Russia's greatest sons.

Mendeléeff's works, published between 1854 and 1907, include two hundred and sixty-six titles, according to Walden's compilation. They give evidence of the wide scope of his interests and activities, as already mentioned above. Among the purely scientific subjects which claimed his most careful attention were: — the densities of liquids, the relation between the volume of liquids and temperature, the formation of compounds between solvent and solute, and the compressibility of gases under reduced pressures. Of lasting influence upon instruction in the science is his masterly "Principles of Chemistry," which appeared in eight Russian and three English editions. Its most striking feature is found in the voluminous notes, appended to the text proper. These notes enlarge upon debatable subjects and theoretical questions, portraying Mendeléeff's personal opinions and habits of thought. In the Preface he remarks: —

"Knowing how contented, free, and joyful is life in the realms of science, one fervently wishes that many would enter their portals. On this account many pages of this treatise are unwittingly stamped with the earnest desire that the habits of chemical contemplation which I have endeavored to instil into the minds of my readers will incite them to the further study of science. Science will then flourish in them and by them, on a fuller acquaintance not only with the little that is enclosed within the narrow limits of my work, but with the further learning which they must imbibe in order to make themselves masters of our science and partakers in its further advancement."

His name will always be most closely associated with the announcement of the Periodic Law. In March, 1869, before the Russian Chemical Society, he set forth his conclusions as follows:—

"1. The elements, if arranged according to their atomic weights, exhibit an evident *periodicity* of properties.

"2. Elements which are similar as regards their chemical properties have atomic weights which are either of nearly the same value (e. g., platinum, iridium, osmium) or which increase regularly (e. g., potassium, rubidium, caesium).

"3. The arrangement of the elements, or of groups of elements in the order of their atomic weights corresponds to their so-called *valencies* as well as, to some extent, to their distinctive chemical properties — as is apparent among other series in that of lithium, beryllium, boron, carbon, nitrogen, oxygen and fluorine.

"4. The elements which are the most widely diffused have *small* atomic weights.

"5. The *magnitude* of the atomic weight determines the character of the element just as the magnitude of the molecule determines the character of a compound body.

"6. We must expect the discovery of many yet *unknown* elements, for example, elements analogous to aluminium and silicon, whose atomic weight would be between 65 and 75.

"7. The atomic weight of an element may sometimes be amended by a knowledge of those of the contiguous elements. Thus, the atomic weight of tellurium must lie between 123 and 126, and cannot be 128.

"8. Certain characteristic properties of the elements can be foretold from their atomic weights."

In his Faraday Lecture, June 4th, 1889, he describes, in dramatic fashion, the verification of his predictions:—

"Before the promulgation of this law the chemical elements were mere fragmentary, incidental facts in Nature; there was no special reason to expect the discovery of new elements, and the new ones which were discovered from time to time appeared to be possessed of quite novel properties. The law of periodicity first enabled us to perceive undiscovered elements at a distance which formerly was inaccessible to chemical vision; and long ere they were discovered new elements appeared before our eyes possessed of a number of well-defined properties. We now know three cases of elements whose existence and properties were foreseen by the instrumentality of the periodic law. I need but mention the brilliant discovery of *gallium*, which proved to correspond to eka-aluminium of the periodic law, by

Lecoq de Boisbaudran; of *scandium*, corresponding to eka-boron, by Nilson; and of *germanium*, which proved to correspond in all respects to eka-silicon, by Winkler. When, in 1871, I described to the Russian Chemical Society the properties, clearly defined by the periodic law, which such elements ought to possess, I never hoped that I should live to mention their discovery to the Chemical Society of Great Britain as a confirmation of the exactitude and the generality of the periodic law. Now, that I have had the happiness of doing so, I unhesitatingly say that although greatly enlarging our vision, even now the periodic law needs further improvements in order that it may become a trustworthy instrument in further discoveries."

Since this time two hitherto unsuspected groups of elements have been discovered and proved capable of inclusion within the Periodic System: — the inert gases, affording a natural transition between the halogens and the alkali metals; and the radio-active elements, which for the most part long defied classification, but which now, thanks to Fajans and Soddy, constitute a further proof of the universality of the law. His attempt to introduce into the system the hypothetical element coronium with an atomic weight of 0.4, and the "ether" of the physicist with the atomic weight of 0.000000000053 have scarcely found acceptance, especially since Moseley's calculation of atomic numbers has indicated that hydrogen has the smallest atomic weight of any element.

Two of his fundamental ideas in connection with the Periodic System remain open to question: First, his repugnance to the doctrine of the unity of matter; second, his contention that the elements, if arranged periodically, must follow inexorably in the order of the atomic weights. He lived to see many atomic weights revised so as to conform to this principle, and died with the conviction that tellurium, for instance, must have a lower atomic weight than iodine. The discovery of isotopy makes this outcome still a possibility.

Mendeléeff, given up by the doctors to die at an early age, lived in activity and usefulness to the age of seventy-three. The summer before his death, weakened by influenza, he began to put his affairs in order, but he retained his vigor of mind and his interest in current events up to the last. The Orthodox Church, the Czar, and the educational institutions of St. Petersburg rendered his memory extraordinary honors as he was borne out to be laid beside his mother and his favorite son. Count Witte well characterized his services when he said: — "In him Russia lost her pride, the great scholar and the upright patriot; industry lost its best adviser; the government a notable helper, and we, his acquaintances and admirers, a loyal friend and the best of men."

Acknowledgement is made of the use of P. Walden's biography, *Ber. d. deutsch. chem. Ges.*, *41*, 4719 (1908) and of W. A. Tilden's Memorial Lecture, *Jour. Chem. Soc.*, *95*$^2$, 2077 (1909). These papers contained material otherwise inaccessible.

George Shannon Forbes.

## SIR JOHN MURRAY (1841-1914)

Foreign Honorary Member in Class II, Section 1, 1900.

Sir John Murray, the son of Scotch settlers in Canada, was born at Coburg, Ontario, on March 3d, 1841. There he passed the first seventeen years of his life. In the primitive conditions of a new community the natural robustness of his nature found a free development in congenial soil.

In 1858 he came to Edinburgh where he prepared for its University at the Stirling High School. His career at the University appears to have been stamped by some of the qualities that distinguished him in after life. Impatient of dogmatic authority, he was somewhat scornful of inherited tradition, and treated his prescribed studies with a cheerful sans gêne. For even in those days he desired to find out things for himself, and delve for knowledge independently. The capacity of clear and original thought, with a genius of disentangling the heart of a subject from its enveloping details, was as characteristic of the youth as of the man. From the small circle of scientific men who then made Edinburgh famous, he gathered, during his student days, what was most worth having, and went his way. That one of the facets of his personality drew him into a friendship with Louis Stevenson, offers a suggestive glimpse into a by-way of his character.

After continuing his scientific training for a period of several years at Bridge of Allen; he undertook a hazardous voyage to Spitzburgen, in a Peterhead whaler in 1868, to study the Arctic Sea. This was the initial exploit that marked him as a pioneer in Oceanography. With the history of the development of this science his name is inextricably bound as a recognized leader. The work of Pourtalès, in 1867–1869, off the Florida coast in the Corwin and Bibb, had stimulated among scientific men the interest in deep sea exploration. This was further aroused by several English expeditions under the joint charge of Thomson, Carpenter and Jeffreys.

When, in December, 1872, the Challenger set out on her famous voyage, under the leadership of Sir Wyville Thomson, to explore the oceans of the world, Murray was appointed one of the three principal assistants. On the return of the Challenger from her cruise of nearly four years, he was made chief assistant in the colossal labor of publishing the Reports of the expedition. At the death of Sir Wyville Thomson in 1882, it was freely predicted that the work would never be finished. But Murray was appointed editor, rose superior to all obstacles and vicissitudes, and finally brought the enterprise to a successful conclusion by issuing the last of the fifty volumes in 1895.

He will probably be best remembered by his work in connection with the Challenger Expedition. The labor of editing the Reports was one of which the difficulty has perhaps not been fully realized. It could never have been completed without first class powers of organization and great determination of purpose. And it required skill and tact of the highest order to keep in hand the small army of specialists who were working on the reports in every quarter of the globe. Not the least of his troubles were his constant struggles to extract money from a grudging Treasury, that felt its patience sorely tried by the length and expense of the undertaking. At one stage of the proceedings Murray forced the Government to produce the necessary funds by threatening to finish the work at his own personal expense.

Murray used to say that he was the only man who had read every word of all the volumes. To carefully read all the page proof was in itself no light task.

With the assistance of Renard of the University of Ghent, he himself studied the deep sea deposits collected by the expedition. The result of this work was published as one of the volumes of the Report. This gave to science the first minute description of the deposits on the bed of the ocean, and disclosed the extreme slowness with which some of them are accumulating.

His active mind gave him a wide sympathy for many scientific activities. Among the several fields in which his services to science were important, should be mentioned his bathymetrical survey of the fresh-water locks of Scotland. This work he conducted for many years with a capable corps of observers. These investigations were published in a series of six volumes, finished in 1910. This is probably the most complete work of its kind in existence.

Chiefly for the purpose of testing in deep water various new apparatus which had lately been used in shallow seas, Murray organized an oceanographic expedition to the North Atlantic in 1910, under the auspices of the Norwegian Government. He financed this enterprise himself, with the exception of the salaries of the government assistants, who were in charge of Dr. Johan Hjort. In his capacity of promoter and advisor of the cruise, Murray was cooped up and tossed about for several months, when nearly seventy, in the uncomfortable little steamer Michael Sars; a hardship that he made light of, for he loved the ocean which he knew so well.

In 1912 Murray and Dr. Hjort collected the results of the voyage in a volume entitled "The Depths of the Ocean." This publication, a valuable reference-book on thalassography, contains a complete summary of oceanography, it treats of the apparatus, the manner of its use and the ends reached in this science; while it brings the whole subject up to date with a description of the work accomplished by the Michael Sars.

To commemorate the memory of a close friendship, Murray gave a fund to the National Academy at Washington, establishing the Alexander Agassiz Medal, which is to be awarded occasionally for distinguished work in Oceanography. On the occurrence of its first award in 1913, the Academy adopted the following course. They selected Dr. Hjort for the honor, and sent a replica of the Medal to Murray.

It is hoped that at the end of the present war, a similar tribute can be offered through The Royal Society, which will establish a Sir John Murray Medal.

The Zoölogical stations on the Firth of Forth and on the Firth of Clyde were founded by him. It was in part due to his efforts that the meteorological observatory on Ben Nevis was created.

He took a keen interest in Polar Exploration, and made a journey to Norway for the express purpose of seeing Nansen start on his attempt to reach the North Pole. He first suggested the idea that the land around the South Pole is one continuous continent, which the explorations of Scott and Amundsen have done much to substantiate. The stimulus that Antarctic research received from Murray's enthusiastic support, was a powerful factor in materializing at least one of those expeditions.

Murray was the authority on deep sea deposits. Many of the numerous explorers who, since the days of the Challenger, have probed the depths of the ocean, placed their collections of muds and slimes at his disposal for study and description. His familiarity with this subject led him to think there are no rocks on continental areas that could have been formed from such deposits as the red clays, the pteropod and the Globigerina oozes, which cover vast areas of the ocean's floor, where they have been accumulating for long periods of geological time. This led him to the firm belief that the ocean basins have remained fixed since the early ages of geology, and to a disbelief in those lost Atlantes and elevated pathways called on to explain the geographical distribution of land flora and fauna. Nor did he admit that Australia, India, Africa, South America and Antarctica had ever formed a single continent.

Murray very naturally considered that the pendulum and geodetic. observations of late years, as well as measurements of gravity over the ocean, attest the permanence of the ocean basins. "For," as he wrote to a friend not long before his death, "it is extremely improbable that there could be such a shifting of materials in the deeper parts of the crust as to cause sub-oceanic heaviness to give place to sub-continental lightness — such as now is found to exist."

He insisted that abyssal Radiolarian ooze was a different deposit from those that have formed Radiolarian rocks. Although Molengraaff, in his recent papers on the Danau formation, dissents from this view, he believes in the permanence of continents and ocean basins. For he considers that the theory is supported by the rarity of the Radiolarites, and the fact of their being limited to the geosynclinals; that is to the more mobile portions of the earth's crust, which in broader or narrower strips separate the great stable areas.

In common with most naturalists who since Dana's day have examined coral reefs in the field, Murray returned from the voyage of the Challenger convinced that Darwin's theory of subsidence did not satisfactorily explain the formation of coral atolls and barrier reefs. Murray's theory lays special stress on the building up of marine platforms, by the gradual deposit of the remains of marine organisms, to a suitable height for the growth of reef building corals; and to the seaward growth of corals on the talus, broken from the living reef and rolled down its outer slope. The formation of the lagoons of atolls and the passages between barrier reefs and the land he attributed to the solvent action of sea water.

When Murray, then a comparatively young man, first suggested his theory, he was advised not to publish anything hastily. This delayed its appearance for about two years. The Duke of Argyll, learning of this fact, wrote accusing the scientific world of a deliberate attempt to suppress the truth for fear of injuring the prestige of Darwin. This called forth the indignant protest of Huxley. The controversy, which created a considerable commotion among the scientific men of that day, was known as the "Conspiracy of Silence."

Murray maintained that the famous coral boring on the Atoll of Funafuti in the Ellice Islands, made under the auspices of the Royal Society of London, supported his views. In fact he predicted that the diamond drill would penetrate into a talus. It might have been inferred from this prophecy that the core taken from Funafuti would lead to a discussion of what it actually revealed. A site for the hole should have been selected where, if, as many believe, the theory of subsidence is mistaken, the drill would have encountered only a comparatively thin stratum of coral rock. Such a site might be found at some point a short distance from the centre of a lagoon, but even there the evidence would not be conclusive if the atoll happened to rest on a foundation of limestone. The situation chosen for the Funafuti bore, on the rim of a large atoll, was unfortunate, and the work instead of proving anything has complicated the subject; for eminent men have drawn very different conclusions from the results of the undertaking. Distinguished supporters of Darwin's theory of subsidence have held that the drill pierced a continuous coral reef. Murray believed it "passed through a portion of the talus produced by the fragments torn from the growing face of the reef, and on which it had proceeded seawards." While Alexander Agassiz was inclined to think that the drill passed in part through Tertiary limestones, and in part through a talus of modern material.

The theories of Murray, Agassiz, and Gardiner differ in the amount of work that they attribute to modern corals, and the relative values they assign to such agencies as organic deposits, erosion, solution, the trade winds, and the scouring force of the ocean. But they all agree in asserting that Darwin's theory of subsidence does not offer a satisfactory solution of the method of formation of atolls and barrier reefs.

One episode in Murray's life furnishes a good example of the unexpected practical benefits that may result from the pursuit of pure science. While crusing in the regions adjacent to the island of Java, the nets of the Challenger collected some bits of phosphate. A careful examination of these objects convinced Murray that they must have been formed on land. Subsequent search for their origin, under Murray's auspices, led to the discovery of the phosphate deposits of

Christmas Island. The island was annexed to Great Britain, and a company under Murray's presidency developed a highly prosperous mine. Some years before his death the company had already paid in royalties, for the protection of the English flag, more than the entire cost of the Challenger expedition!

This enterprise made Murray rich, and while he accepted the opportunities which the possession of wealth offers to an intelligent man, it in no way affected his interest in the pursuit of science. One of the chief projects of his last years was to equip a vessel on the lines of the Prince of Monaco's "Princesse Alice," and set out in her for a protracted cruise around the world in the interest of oceanography.

Murray was elected a Foreign Honorary Member of the American Academy in 1900. Among the many other honors that came to him in recognition of his scientific work, he received the Prussian order 'Pour le Mérite. Punch celebrated the event with a cartoon, which always delighted Murray. As the final decision in the award rests with the King of Prussia, the picture represents the Kaiser who has called for the publications of the candidate. Vistas of lackeys are staggering in loaded with the mighty volumes of the Challenger Report, while the astonished monarch asks in amazement why the name of this prolific author had not been previously suggested.

Under a somewhat brusque manner, Murray could not conceal a genial kindliness, and deep human sympathy and interest. His devotion to research was combined with a strength of will and a steadfastness of purpose, that rendered him singularly efficient in anything he undertook, whether scientific or practical; for he had an unusually clear and steady vision in worldly affairs, uncommon in the devotee of pure science.

His connection with the Challenger Reports began a wide acquaintance among scientific men; his business interests in Christmas Island, Canada, and the United States threw him in broad touch with a different world. Accustomed to meet many varieties of people, the readiness with which his keen and active mind struck fire in contact with other men, made him, wherever he went, a commanding figure.

Murray had little sympathy for those whom he termed the hod carriers of science. Men whose mental activities seem to be satisfied in collecting undigested facts. Not that he undervalued facts, but that he strove to fit them into the body of human knowledge. He never lost sight of the aim of science, a deeper insight into Nature, and a broader outlook on the Universe.

In 1889 Murray married Isabel Henderson, daughter of Thomas

Henderson the shipowner, and brought his wife back to Edinburgh, where their home became one of its intellectual centres. For many years of his later life, Sir John and Lady Murray, with their family of two boys and three girls, lived in a roomy house on the outskirts of Edinburgh, which he had christened "Challenger Lodge." It was characteristic of the man that his unfailing insight enabled him to established a most sympathetic relation with his children, and caused him to use original methods, based on great independence and liberty, to develop them into efficient and self reliant personalities.

Turning into his own avenue, on March 16, 1914, Murray's automobile skidded and capsized, killing him instantly. Such an end, always wished for by him, came as a shock to his friends in many lands, whose admiration for the naturalist was only exceeded by their love of a very human fellow-man.

G. R. Agassiz.

## ANDREW HOWLAND RUSSELL (1846–1915)

Fellow in Class I, Section 4, 1892.

Andrew Howland Russell was born in Plymouth, Massachusetts, on the 24th of December, 1846. His father, Andrew Leach Russell, and his mother, Hannah White Davis, were both of the old Pilgrim stock. He was educated first in the public schools of Plymouth, then spend two or three years at Philips Exeter Academy. In 1865 he was one of the first class to enter the Massachusetts Institute of Technology, but did not complete the course because in 1867 he received an appointment to the Military Academy from which he was graduated fourth in his class in 1871.

He was then promoted to Second Lieutenant in the Third Cavalry, in 1876 to First Lieutenant of Ordnance, in 1888 to Captain, 1901 to Major, 1895 to Lieutenant Colonel, 1907 to Colonel. From July to November, 1898, he held a Volunteer Commission as Major and Chief Ordnance Officer; from 1901 to 1904, he was Chief Ordnance Officer of the Division of the Philippines with the rank of Lieutenant Colonel.

As a Cavalry Officer, in 1871–1872, he served with his regiment in Arizona and Nebraska; in 1873–1874 on the Wheeler Expedition for Surveys west of the 100th Meridian, in New Mexico, Colorado and

Arizona. In 1874–1876, at the Military Academy at West Point as instructor in Natural Philosophy, Astronomy, Ordnance Mineralogy and Geology.

As an Ordnance Officer, he served from time to time at Watertown, Rock Island, Benicia, Fort Union and Frankfort Arsenals, at Vancouver Ordnance Depot; as Inspector of Ordnance in Boston, Providence and St. Paul, as Chief Ordnance Officer of the Department of the Columbia and of the Philippines, and as Assistant to the Chief of Ordnance in Washington, D. C. At the Centennial Exposition of 1887–1888 at Cincinnati, he had charge of the War Department Exhibit, in that of 1892–1893 at Chicago, of the Ordnance Exhibit.

All of this service was in many respects congenial, and favorable to the natural bent of his disposition. His transfer to the Ordnance gave him a good opportunity to exercise his ingenuity; and there was scarcely a branch of the work of that department in which he did not suggest useful improvements, some of which were adopted by the government at the time, and others after their value had been demonstrated in action by the armies of foreign nations.

In 1875, while still a Lieutenant of Cavalry, he invented an hydraulic buffer for checking the recoil of a gun on its carriage, afterwards known throughout the world under the name of Vavasseur. Colonel Russell not only antidated Vavasseur in this matter, but appears to be the pioneer in the field of modern gun carriage recoil systems. He obtained patents at home and abroad for a great number of ingenious devices relating to guns and their auxiliary appliances, which make the artillery of to-day so much more effective than before.

But the object to which he devoted the most labor and study was the improvement of small arms. His ingenuity suggested devices by which one musket could be made to do the work of several. In 1876, he invented devices for loading and firing rapidly, and made wooden models to illustrate their action; but they found little favor with "practical" military men who regarded them as more curious than useful, and most objectionable from sound military considerations. Soon after, he met Capt. W. R. Livermore who showed him designs and models so nearly like his own that at Russell's suggestion they decided to combine their efforts.

In 1878, a Board was convened at Springfield to test Magazine Guns. By that time the prejudice against magazines was so far modified that many officers were willing to try them provided the magazine was reserved for the final charge. The Hotchkiss Gun operated by a bolt, and with a tubular magazine in the breech was most favored. Russell and Livermore presented to the Board a wooden model of their device as applied to a Hotchkiss Magazine, and at the same time, prepared drawings of their own devices which dispensed with the tube and had a fixed box magazine under the receiver of the gun. In each case the cartridges were placed side by side in the box magazine into which they could be loaded either singly or all together, by a single motion. Five or six cartridges were carried side by side in a clip for this purpose. The clips were to be carried in the belt or in the cartridge box. The inventors explained how, for the most rapid fire, the magazine could be replaced by a belt and the piece fired like a machine gun, and how the principles could be applied to guns of all calibres.

Edward W. Byrn, describing "The Progress of Invention in the Nineteenth Century," says,

"This idea was subsequently developed by Livermore and Russell in Patent No. 230, 823, August 3, 1880, and this feature, viewed in the light of the importance subsequently attained by the "clip" in the Mauser and Mannlicher guns, may be fairly considered the pioneer of this idea of grouping cartridges in made-up packets for bolt guns. Its great advantage is the large number of shot that may be fired in a short space of time without an excessive weight in the gun itself." "Before the United States Army Gun Board of 1882, Livermore and Russell submitted a completed gun for trial in which the magazine was placed at the side of the receiver, extending downward, and was arranged to be filled through a side gate at the top from a cartridge package or "clip" grasped in the hand, and applied to the mouth of the magazine for stripping the cartridges from the clip into the magazine. This system also contemplated the use of a clip with a central as well as with a side magazine." . . . .

The gun with some changes was tested before the Army Board of 1892 and the Navy Board of 1895. When the inventors explained that they had fired sixty aimed shots from their musket in a minute, a member of one of the Boards said that that alone was enough to condemn it, as even with muzzle-loaders soldiers often exhausted their ammunition.

Russell and Livermore also invented guns with straight pulling bolts and with automatic action. The United States Government adopted the Clip System in the construction of the musket now in use, although not until many other nations following the lead of Germany had already adopted it.

General Bernhardi writing a few years ago upon how Germany makes war (p. 58) says:—

"With the adoption of small calibre and clip-magazine, as well as with the introduction of smokeless powder, and of pointed projectiles, the development seems to have reached a certain climax and to have come to a finish for the time being.... The character of fighting has altogether changed."

In 1908, having served over forty years he applied to be placed on the retired list. In approving his application, the Chief of Ordnance spoke in highest terms of his ability, good judgment and devotion to duty, especially while acting as Chief of Ordnance for several months, saying that his reports of the Ordnance Exhibits at the Cincinnati and Chicago Exhibitions had been valuable contributions to the service, and adding;—

"Colonel Russell has also a very substantial claim to the inception and first presentation of the modern clip system of loading magazine guns, almost universally applied to the small arms of to-day. The original gun, embodying this feature, presented by him and Col. W. R. Livermore, U. S. A. before the United States Magazine Gun Board of 1882 is in this office. Several other inventions of Col. Russell's have been embodied in Ordnance Constructions, but without pecuniary compensation to him. The Department and the Army are indebted to him for efficient services."

After retiring from active service he travelled in Europe for about a year and then moved back to Plymouth where he died on the 16th of June, 1915.

It was a bitter disappointment that his own country had not been the first to adopt his inventions; but on his death bed was gratified to realize that the great war waging in Europe had demonstrated beyond question the truth of the principles for which he had fought so long; that one nation after another had adopted the system of guns of which he was a recognized pioneer, that the effect of their fire was all that had been claimed, and that warfare had taken the form which had been predicted.

His name will long be remembered in the history of firearms and especially of their development during the past forty years.

W. R. Livermore.

## THOMAS DAY SEYMOUR (1848–1907)

Fellow of Class III, Section 2, 1900.

Thomas Day Seymour was born in Hudson, Ohio, on April 1, 1848, and died in New Haven on December 31, 1907. He felt just pride in his ancestry. His father, Nathan Perkins Seymour, was the sixth descendant in direct line of Richard Seymour, a Devonshire man, who emigrated and settled in Hartford in 1639, and became the ancestor of many distinguished men in New York and Connecticut. His mother, Elizabeth, was the daughter of Thomas Day of Hartford, for twenty-six years Secretary of the State of Connecticut, and niece of President Jeremiah Day of Yale College.

Seymour's father graduated from Yale in 1834 and was tutor there for four years. In 1840 he accepted the professorship of Greek and Latin in Western Reserve College, then in Hudson. Here the younger Seymour passed his boyhood, was fitted for college, spent four years as an undergraduate, and, after two years in Europe, taught for eight years. In 1874 he married Sarah, daughter of Henry L. Hitchcock, then President of Western Reserve. His widow and a son and two daughters still survive him.

While in Hudson, Seymour had free access to his father's library, which contained between two and three thousand carefully selected volumes, and is said to have been, at one time, the best library west of the Alleghanies. The ties that bound father and son were intimate and tender. The elder Seymour was a man of refined and gentle nature, an excellent classical scholar, and possessed also of a knowledge of the German, French and Italian languages that was then unusual. The son was a quiet and reserved, but happy, boy, who went singing and whistling about the house. It is related that he was "a great worker, with a passion for accuracy." He entered college in the autumn of 1866, maintained the rank of first scholar, and at graduation was valedictorian of his class, but he found leisure for other interests. "He was no more a recluse then," a classmate writes, "than subsequently. Nobody was in closer touch with the whole body of students."

The elder Seymour resigned his professorship in Western Reserve College in 1870, and Thomas Seymour was then elected professor of Greek there, with leave of absence for two years. He went to Europe and studied in Leipsic and Berlin for a year and a half. In the spring of 1872 he was in Italy and Greece. During his first semester abroad he came, with rare independence, to the grave decision not to stand for the doctorate in philosophy. He could not spare time, he said, to make special investigations, embody them in a thesis, and prepare himself for examination in certain subjects that he did not think it was profitable for him to study. Later in life he was honored by great universities with the degree of Doctor of Laws: Western Reserve in 1894, Glasgow in 1901, and Harvard in 1906. He was elected to membership in the American Academy of Arts and Sciences (Class III, Section 2) in May, 1900.

In 1880 he was called to Yale, and in 1884, on the death of Professor Lewis Packard, he was elected Hillhouse Professor of Greek. The range of his teaching during his twenty-seven years in New Haven was remarkable. Undergraduates read with him in elective courses Homer, Pindar and the lyric fragments, Greek Tragedy, Thucydides, Plato and Aristotle, Demosthenes and Isocrates, Theocritus, and the Septuagint and the New Testament. The subjects offered to graduate students were epic poetry, lyric poetry, the Greek historians, the drama, Plato, the orators, the bucolic poets, the Greek dialects, Greek inscriptions, and the history and encyclopedia of Greek studies. Aeschylus engaged his interest deeply, Plato was his constant companion. He carried some part of the text of Plato with him when he travelled and read him wherever he happened to be. His studies in Greek oratory were quickened by the investigations of Friedrich Blass, his intimate friend for more than twenty years. The two scholars were singularly alike in many ways: unostentatious in their lives; unwearied in study; impatient of error; accurate, learned, and fruitful. Seymour, like his father was a student of the Bible. This was his other constant companion. In teaching it he applied, with due regard to the change in period, precisely the canons of interpretation that he had found valid in his study of the Greek orators. He was an indefatigable worker. One year he taught twenty-four hours each week, and the hours for one of the courses were from ten o'clock until midnight. The five graduate students in this course eventually succumbed, and he reluctantly changed the time to eight o'clock. When the students withdrew at ten, he cheerily bade them good-night and turned to other occupations. One of his colleagues speaks of his "joyous industry." The tale is current that he never refused service on a committee — and that, too, although member of a faculty that has the envied reputation of initiating and executing policies of its own. Nor was he idle in the summer time — he was never idle. He gave two summer courses of lectures in Chautauqua, one in Chicago, another in California.

Notwithstanding his devotion to his college duties, he found much time for writing. He presented fifteen papers at sessions of the American Philological Association. When president of the Association, he chose as the subject of his address 'Philological Study in America.' He was one of the editors of the Classical Review, published in London, and an editor, also, of the College Series of Greek Authors. He wrote three of the volumes in this series, revised another, and put twelve others through the press. He published his first book in 1882, an excellent edition of Select Odes of Pindar. Men always interested him, ancient or modern, and in 1888 he published in the *Chautauquan* a series of studies of nine characters illustrious in the annals of Greece. He was the best Homeric scholar that America has produced. His contributions to the study and interpretation of Homer were numerous and diverse, editions of parts of the poems for use in school and college, an introduction to the language and verse of Homer, reviews and original articles in journals, and finally his *Life in the Homeric Age*, published shortly before his death, his largest single contribution to knowledge, and that on which his fame as scholar and expositor will chiefly and securely rest.

The introduction of elective studies in American colleges compelled sharp attention to methods of teaching in all departments of knowledge. On none was the effect more immediate than on that of the classics. It soon became apparent that the best and broadest provision of training for teachers in this subject must include study in Greece and Italy. The Archaeological Institute of America was founded by Charles Norton in 1879, the American School of Classical Studies in Athens in 1881. Professor Seymour became the second chairman of the managing committee of the school in 1887, and held this influential but arduous position fourteen years. During his administration, the building occupied by the School was finished, the endowment was increased, the principle of a permanent directorship and of annual professorships was established, five volumes of papers were published, fellowships were founded, and important excavations were conducted; but the worthiest monument of his devotion to this cause is one hundred men and women that studied at the School during his time and are now nearly all teachers of the classics. He resigned the chairmanship of the managing committee of the School to become in 1903 the fourth president of the Archaeological Institute. This also is an arduous position, but Seymour had acquired intimate knowledge of the history of the Institute and its affairs, and was personally acquainted with many scholars in all parts of the country. He had a rare gift for friendship. His administration of the Institute was eminently successful and he had large plans for the promotion of its growth and efficiency. He had expected to attend its annual meeting held in Chicago during the Christmas holidays of 1907, but he fell ill and died, while the Institute was in session, on the last day of the old year.

Thomas Seymour's life is an inspiring example of noble service and high achievement. Its controlling impulse was an ardent desire for knowledge, yet his activity was remarkably varied. He was not only a learned man who spoke with recognized authority, but also an earnest teacher, a wise adviser in college councils, a writer and editor of distinction, and an able administrator of important interests. His influence as a scholar steadily widened and strengthened as he grew older, and enhanced the reputation of Yale University as a great seat of learning.

He belonged to the finer and gentler type of scholars. He avoided fruitless controversy, but never shirked a duty. And thus it was that all who knew him loved him — for his candor, his modesty, his considerateness, his unselfishness, his unswerving devotion to truth.

John Williams White.

## WILLIAM ROBERT WARE (1832–1915)

Fellow in Class III, Section 4, 1866.

In two important fields Professor Ware performed services of almost unique importance to his chosen profession of architecture. In the educational field he laid firmly the foundations of architectural training in this country; and in the field of active professional work he was largely instrumental in putting architectural competition on a dignified and secure basis which commanded the respect both of the building public and the profession. In both these fields he was a pioneer. That he was able to perform such signal service was due on the one hand to his attainments and character, and on the other to the fortunate circumstance that his active life fell in the formative period of professional architectural growth in America, when his talents could most avail.

How well he builded in laying out the plan of architectural study at the Massachusetts Institute of Technology, when in 1865 he was called upon to organize what was practically the first school of architecture in the United States, is shown by the fact that this scheme, as shaped and modified in his hands, has stood the test of time, has shown itself admirably adapted to American needs, and is still the basis of the American method of architectural education. His wise foresight is revealed in the way this plan has shaped itself to meet the larger development of professional life which has come with the growth of the country.

One of the most difficult problems of professional ethics which American architects have been called upon to solve has been that of competition among themselves: how to avoid the injustice, and the waste of professional time and talent, which was the inevitable result of the informal, unregulated and uncompensated submission of competitive designs. Very early the American Institute of Architects (founded in 1857) and its affiliated professional bodies attacked this difficult question. Gradually, for the irregular and demoralizing scramble which was formerly common if not usual, has come to be substituted the formal, paid competition, subject to definite rules, controlled by a professional adviser and impartially decided by this adviser or in its later form by a professional jury. The submission of competitive sketches under other conditions is now regarded as unprofessional.

In the development and gradual improvement of this scheme Professor Ware's good judgment, far-sighted wisdom and absolute and universally recognized impartiality was invaluable. He was more often called upon to act as professional adviser in competitions than any other man, indeed nearly all the important competitions during his period of fullest activity came under his control, and he did more than any other one architect in securing the general recognition of, and the confidence of the building public in this form of regulation. His lucid reports and his fair mindedness and impartiality made the advantages of proper regulation and control so clear that his activity greatly tended toward the steady reduction in the number of badly regulated or unregulated competitions. At the same time while he recognized the advantage which the competition in certain cases offers to the owner and to the public and often on that account advocated it, his influence was always thrown against the competition when it seemed unnecessary or inadvisable. His services to his profession in this field brought what was perhaps the most distinguished public recognition which came to him: his appointment in 1906 to represent America on the international jury of architects which was called upon to decide the world-competition for the Peace Palace at the Hague.

William Robert Ware was born at Cambridge, Massachusetts, on May 27th, 1832. He was the son of the Rev. Henry Ware, Jr. Ten years later the family moved to Framingham and in April, 1844, to Milton, to the home where on June 9th, 1915, he passed peacefully away. Here he attended the Milton Academy of which in later life he was one of the Trustees; but his health not being vigorous his mother sent him to England to the care of her cousin. He went alone and was gone six months. The journal which he then kept shows, even in the lad of fifteen, his taste and his independence of mind. It seems probable that this journey had its influence in turning him ultimately toward architecture, a career which gave scope both to the scientific bent of his mind and to his interest in the fine arts. On his return he went to the Phillips Exeter Academy and from there entered Harvard College as a member of the Class of 1852, and was elected to the Phi Beta Kappa. On graduation he taught school for two years in New York to support himself, and then entered the Lawrence Scientific School, graduating in engineering in 1856. Horace Porter, Prof. T. H. Safford, Prof. F. W. Putnam, Alexander Agassiz and Dr. William Watson, Secretary of the American Academy of Arts and Sciences, were among the students in the Lawrence Scientific School at this time. After that he studied his profession in the office of Mr. Richard M. Hunt of New York, one of the first American graduates of the Paris École des Beaux Arts. Mr. Hunt made of his office a sort of Atelier, and here young Ware found himself in the company of Henry Van Brunt, who was later to be his partner, of George B. Post and others who later attained prominence as architects. Mr. Hunt's office was certainly the first American 'atelier' and might almost be called the first American school of architecture. Later Mr. Ware entered in Boston the office of Mr. Edward C. Cabot whose scholarly and conscientious work, such as the Boston Athenaeum and the Boston Theatre, was distinctly the best then being done. When Mr. Ware was twenty-six or twenty-seven years of age he formed a partnership with Mr. Edward L. Philbrick and began his independent professional career. Together they carried out the railroad station at Worcester, architecturally one of the most important projects of the kind that had so far been built in this country. Its architectural form was, of course, wholly due to Mr. Ware. For long it remained architecturally one of the most successful of American railroad stations and bore comparison with any of similar size that came into being much later. It has only recently been removed to give way to the present larger and more ambitious, but certainly, as a work of art, not more successful station. The dignified tower of Mr. Ware's Worcester station still stands. In 1860 Mr. Philbrick went to Europe and the brief partnership came to an end. Shortly thereafter Mr. Ware formed a partnership with his life-long friend Mr. Henry Van Brunt, a connection which lasted until 1881 when Mr. Ware went to New York. It is impossible to separate the part of the two friends in the many buildings which the firm carried out. Mr. Van Brunt's share in these designs tended to increase as Mr. Ware found his time more and more engrossed by educational work. Among the more important buildings which resulted from this partnership, Memorial Hall and Sanders Theatre in Cambridge, the First Church in Boston, and St. John's Church in Cambridge may be singled out. In 1865 Mr. Ware was invited to take charge of the Department of Architecture of the recently founded Massachusetts Institute of Technology and to formulate a course of professional study. In preparation for this unprecedented undertaking, Mr. Ware stipulated that he should first spend a year in Europe, examining schools of architecture and preparing himself for his new work. His association with Mr. Hunt naturally led him to think of the École des Beaux Arts in Paris, to which American students of architecture, in part through Mr. Hunt's influence, were already finding their way. But Mr. Ware found in another more modest Paris institution, the École Centrale d'Architecture, then conducted by its founder Mr. Trélat, a model which seemed to offer, especially in its association of liberal studies with professional training, suggestions better adapted to the needs of American students and to the conditions which had to be met at the Institute of Technology. But the advantages of the École des Beaux Arts were not overlooked and in 1871 Professor Ware secured the appointment of one of its distinguished graduates, Mr. Eugene Létang, to take charge of the work in Design. Mr. Létang proved a most sympathetic associate and continued to direct the work in architectural design at the Institute until his death in 1892. Meanwhile, Professor Ware had been called, in 1881, under most favorable conditions, to New York, to found a school of architecture at Columbia University, where he remained until 1903. He thus directly founded two of the prominent schools of architecture of the country, and as others came into being his advice was eagerly sought and always generously given. There thus came to exist the pleasantest personal relations between Professor Ware, the Dean of architectural education in this country, and those in charge of other schools, several of whom had been his pupils. Constantly they went to him for advice, and he seemed to take as much interest in their schools as in his own, and was always eager to exchange experiences and to discuss plans. The generous devotion of his time to the interests of his friends and especially of his pupils, in whom his interest always continued, and with many of whom he kept in personal touch throughout their later careers, his ready sympathy, and his high character made him greatly beloved by all those who were privileged to come under his influence. His keen wit, a peculiar and very individual humor, and brilliant powers of conversation gave great charm to his companionship. His influence on his pupils was perhaps even more valuable in the upbuilding of character than in directly professional instruction.

Busy as Mr. Ware was both as teacher and practitioner, he still found time for a wide interest and activity in educational matters, especially where the Fine Arts were concerned. From 1875 until he went to New York in 1887 he was one of the trustees of the Museum of Fine Arts in Boston and was on the managing committee of the School of Drawing and Painting. He was similarly active at the Metropolitan Museum after he went to New York, and, for several years acted as Secretary of the Trustees of the American Academy of Fine Arts in Rome founded by Mr. McKim. The vacations nearly always found him at the old home in Milton with his sisters, and here he found time to write his exhaustive treatise on Perspective, and another on the theory of Shades and Shadows. Both of these were first published in "The American Architect." After his retirement in 1903, the latter was rewritten and enlarged for the Scranton correspondence schools. He also wrote and compiled an illustrated treatise on the orders for the use of schools of architecture, which he entitled "The American Vignola." The breadth of his interests is shown by his devising after his retirement an ingenious and entertaining method of Teaching Latin, which he conceived might be more directly taught than through the grammar. This he put into book form, but it was never published.

In 1883 he made a journey to Italy with his friend Mr. Wilder Bancroft, and in the year 1889 to 1890, with his sister Harriet, he visited Egypt stopping on the way at Gibraltar, Naples, Sicily and Malta. In the spring of 1890, the brother and sister spent some time in Greece, visiting also Constantinople and the Dardanelles, travelling a part of the time with Professor and Mrs. Goodwin, and Professor (now President) and Mrs. A. Lawrence Lowell, returning home through Italy, France and England.

After his retirement he received the title of Emeritus Professor from Columbia University and settled permanently at Milton. In the spring of 1906, he was absent for eight weeks on his journey to the Hague as the American representative on the jury of the competition for the Peace Palace. On this journey also Miss Harriet Ware accompanied him. As his health declined he and his sister went south during two winters to avoid the coldest weather; but after 1910 such long journeys proved impossible, though he was still able to go away for change during the summer.

He was a Fellow of the American Institute of Architects and Honorary Corresponding Member of the Royal Institute of British Architects. In 1896 the high value of his work as pioneer in architectural education was recognized by the bestowal of the LL.D. degree from Harvard University.

H. L. WARREN.

## WILLIAM WATSON (1884–1915)

Fellow in Class I, Section 4, 1864, Recording Secretary, 1884–1915.

William Watson was born at Nantucket, Mass., January 19, 1834. His parents were William and Mary (Macy) Watson.

He graduated from the Lawrence Scientific School, Harvard University with the degree S. B. in Engineering in 1857 and prolonged his study of mathematics during the following year. Throughout his undergraduate course he was distinguished for his mathematical ability and won the Boyden Prize in mathematics. He served as an instructor in the Calculus in Harvard from 1857 to 1859. Shortly thereafter he began a course of study at the University of Jena where he received the degree of Ph. D. in 1862. This was succeeded by further engineering study at the École des Ponts et Chaussées, Paris. In the years immediately following he made an extended examination of European technical schools his knowledge of which proved highly serviceable in connection with the laying out of the engineering courses in the Massachusetts Institute of Technology then in process of organization.

He was elected to membership in this Academy February 9th, 1864 and in 1884 was chosen to fill the office of Recording Secretary. This he continued to hold up to the time of his death which occurred on September 30, 1915.

He was a staunch friend of the Academy and devoted to its interests. He realized very fully the desirability of a more general personal acquaintance among its members than formerly existed and was anxious to remove the frigidity which characterized its sessions in earlier times. To him was chiefly due the institution of the social features of the monthly meetings which have proved so successful.

Mr. Watson was one of the original professors in the Massachusetts Institute of Technology at its beginning, having in his charge the instruction in mechanical engineering together with descriptive geometry and stereotomy. Upon him devolved the planning of the Course in Mechanical Engineering under the conditions demanded for it in this country and also most of the teaching in its professional subjects as well, so few in number was the instructing force, a serious task for any man. To this work he devoted himself most earnestly sparing no pains to make his subject clear to classes of rather insufficiently prepared students. The lack of text-books in mechanical engineering based upon American practice hampered him greatly. His special interest, however, was in descriptive geometry and its applications of which he possessed a wide knowledge. To illustrate these he secured for the Institute what was for that time a remarkable collection of models of various surfaces. He also gave for the first time in the United States laboratory instruction in the practical applications of stereotomy, the students of which were required to construct actual models in plaster from their drawings. He retained his professorship until 1873 when he resigned to devote himself more exclusively to study. In the same year he married Miss Margaret Fiske of Boston who died a number of years later.

Professor Watson contributed much in an informal way to advance the interests of the many instructors in mathematics and physics in Harvard and Technology as a very active member of the Mathematical and Physical Club, or M. P. Club as it was colloquially called, an organization which from its beginning in the early eighties for over thirty years played a large part in bringing the older and the younger instructors at these institutions together for scientific discussion and friendly intercourse.

Professor Watson held many offices in connection with various engineering Congresses, among which were the Vienna Exposition of 1873, and the Paris Exposition of 1878. He was Honorary President of the Paris Congress of Architects and Vice President of the International Congress of Hygiene in 1878 and Honorary President of the Engineering Section of the French Association for the Advancement of Science for several terms. He was a member of the French Society of Civil Engineers, the French National Academy of Cherbourg, the American Society of Civil Engineers, the American Society of Mechanical Engineers and various other scientific and engineering organizations.

He was the author of several works on engineering subjects and of many technical papers.

Charles R. Cross.

## JAMES CLARKE WHITE (1833–1916)

Fellow in Class II, Section 3, 1866.

Dr. White was of Scotch-Irish stock, founders of Londonderry, New Hampshire, some of whom, moving to the Maine coast and mindful of their origin, called their place of settlement Belfast, another important Ulster town. Here, in 1833, our friend was born. One would not suspect that one of his great grandmothers was a Viennese, so characteristically Scotch-Irish were his qualities. It is perhaps well that he was one of a family of seven children, an education in itself. His father was shipbuilder, ship owner, manufacturer, bank president, a leader in all the activities of the town and the country round about.

James, fifth child and eldest son, took his A. B. at Harvard in 1853, member of a class prolific in professors, Charles W. Eliot, Justin Winsor, James Mills Pierce, Elbridge G. Cutler, Adams S. Hill, all of Harvard, John Quincy Adams, Fellow of the University. It is noteworthy that unconsciously he fitted himself during his boyhood and college years for the study of medicine. Without having decided as to his profession, he devoted himself to those preparatory studies now required of students entering that of medicine. In his undergraduate diary he wrote at the end of his junior year,— "I have done much work outside the curriculum in natural history — botany and ornithology especially — fascinating studies under such teachers as Gray and Wyman." During vacations he shot and stuffed birds for the college Natural History Society. May 15, 1853, he wrote in his diary, "There came to me this afternoon in church the sudden conviction that I would choose medicine as my life work."

While a student in the Medical School he took special interest in chemistry, analyzed the Warren collection of urinary calculi, and wrote an essay based on that work which received the Boylston Society's prize.

In 1855 he served a year as medical house pupil at the Massachusetts General Hospital, and in August '56, went to Europe, choosing Vienna instead of Paris partly at the suggestion of Professor Calvin Ellis who had lately visited that city and recognized the advantages offered by the group of remarkable men then there active,— Oppolzer, Skoda, Rokitansky, and Hebra. In this step he showed a characteristic trait, that of doing his own thinking. Paris was then living on the medical glamour of the past. White was among the first to separate glamour from fact. After a year in Europe, a year which all who knew him are sure was filled with diligent purpose alike in purely professional and in general improvement, he settled in general practice in Boston. In dermatology and medical chemistry he had qualified himself especially. His character, his knowledge, and his readiness to use them fully wherever service could be rendered, met with prompt recognition. The memory of those Vienna days was kept alive by a club of six men who had studied there together,— Drs. Hay, H. K. Oliver, B. J. Jeffries, Hasket Derby, F. P. Sprague and J. C. White. They dined together regularly, and a photograph of the group occupied a prominent place on Dr. White's office wall.

In 1858 he was appointed instructor in chemistry and in 1866 adjunct professor thereof, often appearing in court as a medico-legal expert. He made it a rule to appear only for the government, a practice which, in combination with his obvious sincerity and competence, enhanced respect for the impartiality of his evidence. Meantime he was doing general practice and was visiting physician at the Massachusetts General Hospital, all the time increasing his knowledge, which in dermatology was greater than that of any contemporary in Boston. He soon found that there was ample exercise for his faculties in this branch alone. As thorough a man as he could not slur work, and he had to reconcile or decide between the rival claims of general practice, medical chemistry, and dermatology. To all of these no one man, even fifty years ago could do justice. He chose dermatology, and he was appointed professor in '71, and cut himself off from all other practice. It required great courage in those days to specialize in dermatology; indeed, in anything except ophthalmology. But courage is a quality of which Dr. White had, at least, his share, and his integrity of character was so well known that all understood there was no sham in his adoption of a specialty. It seems a pity that so many present day specialists start as such, instead of growing into specialism as did White. Perhaps it is unavoidable, so intensively and extensively has knowledge opened up, above all in the last quarter of a century. The body is one, although its parts are many. General practice may be compared with the low power of the microscope. Dr. Owen Wister, of Philadelphia, remarked to the writer many years ago that "it takes a mighty big man to be a specialist." Dr. White was not what his great teacher, Hebra, used to call a "specialist by the grace of God." His specialism was based on a wide, general experience.

For a brief time he had a ward for skin cases at the Hospital, and toward the latter part of his life, a few beds for such cases. He built up a large out-patient clinic, frequented by many attracted by his reputation rather than that of the Hospital.

As a teacher, he was clear, practical, concise, convincing.

In the Faculty of the Harvard Medical School he rendered great service as a protagonist in the reform of Medical education in which Harvard led the van. It was not only in the Faculty that Dr. White pleaded the cause of improvement in medical education. It was the subject of his address opening the winter course of lectures in 1870, and again in his anniversary oration before the Massachusetts Medical Society, 1878.

He was one of the founders of the Boston Society of Natural History, serving as its curator of comparative anatomy for ten years. He was an original member and first President of the American Dermatological Association, and a very constant attendant and active participant in its meetings. He was again its President twenty years later. Proof of the recognition of his services and attainments is found in the fact that he was President of the International Dermatological Congress held in New York in 1907. In the Massachusetts Medical Society he was Anniversary Chairman, Orator, and President. He was for some years Editor of the Boston Medical and Surgical Journal. His contributions to literature were mainly papers, especially on subjects related to diseases of the skin.

His book on "Dermatitis Venenata," published in 1887, covered ground hitherto but little cultivated.

After his retirement from the Hospital and Medical School, he published privately "Sketches from my Life," containing his diaries while at Cambridge and in the Medical School. He also printed privately a sketch of the Clarke-White family.

He was a methodical and very industrious man, well read generally, a connoisseur in food, wine, and china. Rarely sleeping after six, he read for an hour before rising. Every summer he made a list of birds seen, and, during a visit to the writer in the Adirondacks, a list of all the berry bearing plants he encountered in the woods.

The last years of his life he passed the months of June to October at Islesboro in Penobscot Bay. His white house was on the crest of a ridge one hundred feet above the water, and contained collections of books, china, furniture, and pictures, which gave him great enjoyment, alike in collecting and in owning. The cupola, to which he liked to lead the way, commanded both east and west bays, and more than twenty towns, among them Belfast, his birthplace, some ten miles away. On his west porch he passed much time, delighting in the everchanging views of the Camden Hills, and the activities of Gilkey Harbor. No yacht entered or left unnoted by him. Many friends will cherish many memories of this porch, and regret that memories alone remain for them.

Of sentiment he rarely talked, but he had it abundantly.

Some we call good are so negatively, rather than positively. Not so Dr. White. Virile, fearless, aggressive, he was a good fighting man, good man of medicine, good citizen, good friend. An unusual degree of these by no means synonymous forms of goodness was happily and rarely blended in him.

He was a knight, sans peur et sans reproche. In life he stood for all that is best. Is there any better preparation for death, anything which one of us could more wish said of him? Let us honor him by striving to follow his example.

F. C. Shattuck.

## CHARLES OTIS WHITMAN (1843–1910)

Fellow in Class II, Section 3, 1890.

Charles Otis Whitman was born December 12th, 1843[2] at Pinhook, town of Woodstock, Maine, the son of Joseph and Marcia (Leonard) Whitman. He early showed a love of natural history and was especially interested in birds. He graduated from Bowdoin College (B.A.) July, 1868, having been obliged to teach meantime to secure funds for his education. He was for four years principal of the Academy at Westford, Massachusetts and, in September 1872, was appointed submaster at the English High School in Boston where his uncle, George F. Leonard, had been for some years master. He came under the influence of Louis Agassiz in 1873 and entered the laboratory at Penikese. There he met Professor E. S. Morse who was struck by his ability. He went to the Naples Laboratory and studied at Leipzig under Leuckhart, graduating (Ph.D.) in 1878. His doctor's thesis on "The Embryology of Clepsine" introduced new principles, as well as facts, into embryological science, and was beautifully illustrated by his own drawings.

Returning to America he was invited in the summer of 1879 by Professor Morse to take up the work Morse was laying down at the Imperial University, Japan. Accordingly Professor Whitman taught zoölogy at Tokyo until the summer of 1881. Here he trained four investigators, all of whom became professors of zoölogy at the university. Becoming estranged from the University officials because he could not adapt himself to their ideas of official control of intellectual property, he left Japan in August 1881. He went to Naples where he studied from November, 1881 to May, 1882. Here he worked out the embryology, life history and classification of Dicyemids, using the newest methods of microscopical research. Returning to America in the autumn of 1882 he was appointed Assistant in Zoölogy at the Museum of Comparative Zoölogy of Harvard University. Here he worked in coöperation with Alexander Agassiz on the development of pelagic fish eggs. Two papers were published on this subject and his own book on "Methods of Research in Microscopical Anatomy and Embryology" appeared at this time. Whitman was put in charge of a private laboratory for biology and related research, founded by

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[2] The date December 14, 1842 is also given.

Mr. Edward Phelps Allis, Jr., on the lake at Milwaukee, Wisconsin. While here he launched the Journal of Morphology, characterized by the scholastic and artistic excellence of its contributions. In 1888 Professor Whitman accepted the invitation of the Trustees of the newly organized Marine Biological Laboratory to become its director. This laboratory he developed with extraordinary success during 21 years. It was during the early years of the laboratory that the technical scientific society now called the American Society of Zoölogists was founded, largely through his initiative. In 1889 Whitman was called to Clark University as professor of zoölogy. He removed in 1892 to the new University of Chicago where he and his associates developed a large school of zoölogical research. For a period of fifteen years Whitman bred pigeons to get at an understanding of the evolution of their color markings. He paid particular attention to the phylogeny of pigeons, instinct and animal behavior, infertility and the nature of sex. Caring for his pigeons he contracted a heavy cold and died suddenly of pneumonia on March sixth, 1910, at the age of 67. His principal biographer records 67 titles of publications of which 7 are his annual reports. The others are brief notices of technical methods, a few are polemical, 9 are of a semi-popular sort relating to the work and aims of the biological laboratory. A number are brief essays chiefly upon philosophical-biological matters, such as "The seat of formative and regenerative energy," 1887; "The naturalist's occupation," 1891; "The inadequacy of the cell theory of development," 1893; "General physiology and its relation to morphology," 1893; "Evolution and epigenesis," 1895; "Bonnet's theory of evolution; a system of negations," also "The palingenesia and the germ doctrine of Bonnet," 1895; "Animal behavior," 1899; "Myths in animal psychology," 1899. The more strictly investigational papers fall into three periods; 1. The invertebrate period devoted chiefly to the leech and to Dicyemids. 2. The period of vertebrate embryology, including especially work on pelagic fish eggs, on amphibian eggs and the ganoid fish, Amia. 3. The period of genetics, foreshadowed in his note "Artificial production of variation in types," 1892, and continued with the pigeons to the end, 1910,— in all 18 years. While the quantity of his published work is not great it is mostly characterized by fine literary style, scientific accuracy and philosophic insight.

CHAS. B. DAVENPORT.

# American Academy of Arts and Sciences

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| | *Terms expire* 1919. | |
| Gregory P. Baxter, | William M. Wheeler, | Archibald C. Coolidge. |
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The President,
The Recording Secretary,
William M. Davis, Edwin B. Wilson, George F. Moore.

# LIST
## OF THE
# FELLOWS AND FOREIGN HONORARY MEMBERS.

(Corrected to July 1, 1917.)

## FELLOWS.— 498.

(Number limited to six hundred.)

### CLASS I.— *Mathematical and Physical Sciences.*— 181.

### SECTION I.— *Mathematics and Astronomy.*— 40.

| | |
|---|---|
| George Russell Agassiz | Boston |
| Raymond Clare Archibald | Providence, R. I. |
| Solon Irving Bailey | Cambridge |
| Edward Emerson Barnard | Williams Bay, Wis. |
| George David Birkhoff | Cambridge |
| Charles Leonard Bouton | Cambridge |
| Ernest William Brown | New Haven, Conn. |
| Sherburne Wesley Burnham | Williams Bay, Wis. |
| William Elwood Byerly | Cambridge |
| William Wallace Campbell | Mt. Hamilton, Cal. |
| Julian Lowell Coolidge | Cambridge |
| George Cary Comstock | Madison, Wis. |
| Leonard Eugene Dickson | Chicago Ill. |
| Philip Fox | Evanston, Ill. |
| Fabian Franklin | New York, N. Y. |
| Edwin Brant Frost | Williams Bay, Wis. |
| Frank Lauren Hitchcock | Belmont |
| Edward Vermilye Huntington | Cambridge |
| Dunham Jackson | Cambridge |
| Edward Skinner King | Cambridge |
| Carl Otto Lampland | Flagstaff, Ariz. |

| | |
|---|---|
| Joel Hastings Metcalf | Winchester |
| Clarence Lemuel Elisha Moore | Watertown |
| Eliakim Hastings Moore | Chicago, Ill. |
| Edward Charles Pickering | Cambridge |
| William Henry Pickering | Cambridge |
| Charles Lane Poor | New York, N. Y. |
| Roland George Dwight Richardson | Cambridge |
| Arthur Searle | Cambridge |
| George Mary Searle | Washington, D. C. |
| Vesto Melvin Slipher | Flagstaff, Ariz. |
| Frederick Slocum | Middletown, Conn. |
| John Nelson Stockwell | Cleveland, O. |
| William Edward Story | Worcester |
| Henry Taber | Worcester |
| Harry Walter Tyler | Newton |
| Robert Wheeler Willson | Cambridge |
| Edwin Bidwell Wilson | Brookline |
| Frederick Shenstone Woods | Newton |
| Paul Sebastian Yendell | Dorchester |

### CLASS I., SECTION II.—*Physics.*—55.

| | |
|---|---|
| Joseph Sweetman Ames | Baltimore, Md. |
| Carl Barus | Providence, R. I. |
| Louis Agricola Bauer | Washington, D. C. |
| Alexander Graham Bell | Washington, D. C. |
| Louis Bell | Newton |
| Clarence John Blake | Boston |
| Percy Williams Bridgman | Cambridge |
| Henry Andrews Bumstead | New Haven, Conn. |
| George Ashley Campbell | New York, N. Y. |
| Emory Leon Chaffee | Belmont |
| Daniel Frost Comstock | Brookline |
| William David Coolidge | Schenectady, N. Y. |
| Henry Crew | Evanston, Ill. |
| Charles Robert Cross | Brookline |
| Harvey Nathaniel Davis | Cambridge |
| Arthur Louis Day | Washington, D. C. |
| Louis Derr | Brookline |
| William Johnson Drisko | Winchester |
| William Duane | Boston |
| Alexander Wilmer Duff | Worcester |

| | |
|---|---|
| Arthur Woolsey Ewell | Worcester |
| Harry Manley Goodwin | Brookline |
| George Ellery Hale | Pasadena, Cal. |
| Edwin Herbert Hall | Cambridge |
| Hammond Vinton Hayes | Boston |
| William Leslie Hooper | Somerville |
| John Charles Hubbard | New York, N. Y. |
| Charles Clifford Hutchins | Brunswick, Me. |
| James Edmund Ives | Worcester |
| William White Jacques | Boston |
| Norton Adams Kent | Cambridge |
| Frank Arthur Laws | Boston |
| Henry Lefavour | Boston |
| Theodore Lyman | Brookline |
| Richard Cockburn Maclaurin | Boston |
| Thomas Corwin Mendenhall | Ravenna, O. |
| Ernest George Merritt | Ithaca, N. Y. |
| Albert Abraham Michelson | Chicago, Ill. |
| Dayton Clarence Miller | Cleveland, O. |
| Robert Andrews Millikan | Chicago, Ill. |
| Harry Wheeler Morse | Los Angeles, Cal. |
| Edward Leamington Nichols | Ithaca, N. Y. |
| Ernest Fox Nichols | Hanover, N. H. |
| Charles Ladd Norton | Boston |
| George Washington Pierce | Cambridge |
| Michael Idvorsky Pupin | New York, N. Y. |
| Wallace Clement Sabine | Boston |
| Frederick Albert Saunders | Poughkeepsie, N. Y. |
| John Stone Stone | New York, N. Y. |
| Maurice deKay Thompson | Brookline |
| Elihu Thomson | Swampscott |
| John Trowbridge | Cambridge |
| Arthur Gordon Webster | Worcester |
| Charles Herbert Williams | Milton |
| Robert Williams Wood | Baltimore, Md. |

### Class I., Section III.—*Chemistry.*—43.

| | |
|---|---|
| Wilder Dwight Bancroft | Ithaca, N. Y. |
| Gregory Paul Baxter | Cambridge |
| Marston Taylor Bogert | New York, N. Y. |

| | |
|---|---|
| Bertram Borden Boltwood | New Haven, Conn. |
| William Crowell Bray | Berkeley, Cal. |
| Russell Henry Chittenden | New Haven, Conn. |
| Arthur Messinger Comey | Chester, Pa. |
| Charles William Eliot | Cambridge |
| Henry Fay | Boston |
| George Shannon Forbes | Cambridge |
| Frank Austin Gooch | New Haven, Conn. |
| Lawrence Joseph Henderson | Cambridge |
| Charles Loring Jackson | Cambridge |
| Walter Louis Jennings | Worcester |
| Elmer Peter Kohler | Cambridge |
| Charles August Kraus | Worcester |
| Arthur Becket Lamb | Cambridge |
| Gilbert Newton Lewis | Berkeley, Cal. |
| Warren Kendall Lewis | Boston |
| Arthur Dehon Little | Brookline |
| Charles Frederic Mabery | Cleveland, O. |
| Forris Jewett Moore | Boston |
| George Dunning Moore | Worcester |
| Edward Williams Morley | West Hartford, Conn. |
| Harmon Northrop Morse | Baltimore, Md. |
| Samuel Parsons Mulliken | Boston |
| Charles Edward Munroe | Washington, D. C. |
| James Flack Norris | Brookline |
| Arthur Amos Noyes | Boston |
| William Albert Noyes | Urbana, Ill. |
| Thomas Burr Osborne | New Haven, Conn. |
| Samuel Cate Prescott | Brookline |
| Ira Remsen | Baltimore, Md. |
| Robert Hallowell Richards | Jamaica Plain |
| Theodore William Richards | Cambridge |
| Martin André Rosanoff | Pittsburgh, Pa. |
| Stephen Paschall Sharples | Cambridge |
| Miles Standish Sherrill | Brookline |
| Alexander Smith | New York, N. Y. |
| Julius Oscar Stieglitz | Chicago, Ill. |
| Henry Paul Talbot | Newton |
| William Hultz Walker | Boston |
| Willis Rodney Whitney | Schenectady, N. Y. |

### CLASS I., SECTION IV.— *Technology and Engineering*.— 43.

| | |
|---|---|
| Henry Larcom Abbot | Cambridge |
| Comfort Avery Adams | Cambridge |
| Bernard Arthur Behrend | Boston |
| William Herbert Bixby | Washington, D. C. |
| Francis Tiffany Bowles | Boston |
| Charles Francis Brush | Cleveland, O. |
| William Hubert Burr | New Canaan, Conn. |
| Alfred Edgar Burton | Newton |
| John Joseph Carty | New York, N. Y. |
| Eliot Channing Clarke | Boston |
| Harry Ellsworth Clifford | Newton |
| Desmond FitzGerald | Brookline |
| John Ripley Freeman | Providence, R. I. |
| George Washington Goethals | New York, N. Y. |
| John Hays Hammond | New York, N. Y. |
| Rudolph Hering | New York, N. Y. |
| Ira Nelson Hollis | Worcester |
| Henry Marion Howe | New York, N. Y. |
| Hector James Hughes | Cambridge |
| Alexander Crombie Humphreys | New York, N. Y. |
| Frederick Remsen Hutton | New York, N. Y. |
| Dugald Caleb Jackson | Cambridge |
| Lewis Jerome Johnson | Cambridge |
| Arthur Edwin Kennelly | Cambridge |
| Gaetano Lanza | Philadelphia, Pa. |
| William Roscoe Livermore | Boston |
| Lionel Simeon Marks | Cambridge |
| Edward Furber Miller | Newton |
| Hiram Francis Mills | Lowell |
| Charles Francis Park | Boston |
| William Barclay Parsons | New York, N. Y. |
| Cecil Hobart Peabody | Boston |
| Harold Pender | Philadelphia, Pa. |
| Albert Sauveur | Cambridge |
| Peter Schwamb | Arlington |
| Henry Lloyd Smyth | Cambridge |
| Charles Milton Spofford | Brookline |
| Frederic Pike Stearns | Boston |
| Charles Proteus Steinmetz | Schenectady, N. Y. |

| | |
|---|---|
| George Fillmore Swain | Cambridge |
| George Chandler Whipple | Cambridge |
| Robert Simpson Woodward | Washington, D. C. |
| Joseph Ruggles Worcester | Boston |

## CLASS II.— *Natural and Physiological Sciences.*— 164.

### SECTION I.— *Geology, Mineralogy, and Physics of the Globe.*— 47.

| | |
|---|---|
| Wallace Walter Atwood | Cambridge |
| Joseph Barrell | New Haven, Conn. |
| George Hunt Barton | Cambridge |
| Isaiah Bowman | Washington, D. C. |
| Thomas Chrowder Chamberlin | Chicago, Ill. |
| William Bullock Clark | Baltimore, Md. |
| John Mason Clarke | Albany, N. Y. |
| Henry Helm Clayton | Canton |
| Herdman Fitzgerald Cleland | Williamstown |
| William Otis Crosby | Jamaica Plain |
| Reginald Aldworth Daly | Cambridge |
| Edward Salisbury Dana | New Haven, Conn. |
| Walter Gould Davis | Cordova, Arg. |
| William Morris Davis | Cambridge |
| Benjamin Kendall Emerson | Amherst |
| Grove Karl Gilbert | Washington, D. C. |
| James Walter Goldthwait | Hanover, N. H. |
| Louis Caryl Graton | Cambridge |
| Herbert Ernest Gregory | New Haven, Conn. |
| Ellsworth Huntington | Milton |
| Oliver Whipple Huntington | Newport, R. I. |
| Robert Tracy Jackson | Peterborough, N. H. |
| Thomas Augustus Jaggar | Honolulu, H. I. |
| Douglas Wilson Johnson | New York, N. Y. |
| Alfred Church Lane | Cambridge |
| Andrew Cowper Lawson | Berkeley, Cal. |
| Charles Kenneth Leith | Madison, Wis. |
| Waldemar Lindgren | Brookline |
| Alexander George McAdie | Readville |
| Charles Palache | Cambridge |
| John Elliott Pillsbury | Washington, D. C. |
| Louis Valentine Pirsson | New Haven, Conn. |

| | |
|---|---|
| Raphael Pumpelly | Newport, R. I. |
| William North Rice | Middletown, Conn. |
| Robert Wilcox Sayles | Cambridge |
| Charles Schuchert | New Haven, Conn. |
| William Berryman Scott | Princeton, N. J. |
| Hervey Woodburn Shimer | Watertown |
| Charles Richard Van Hise | Madison, Wis. |
| Thomas Wayland Vaughan | Washington, D. C. |
| Charles Doolittle Walcott | Washington, D. C. |
| Robert DeCourcy Ward | Cambridge |
| Charles Hyde Warren | Auburndale |
| Herbert Percy Whitlock | Albany, N. Y. |
| Bailey Willis | Palo Alto, Cal. |
| Samuel Wendell Williston | Chicago, Ill. |
| John Eliot Wolff | Cambridge |
| Jay Backus Woodworth | Cambridge |
| Frederick Eugene Wright | Washington, D. C. |

### Class II., Section II.—*Botany*.—31.

| | |
|---|---|
| Oakes Ames | North Easton |
| Irving Widmer Bailey | Cambridge |
| Liberty Hyde Bailey | Ithaca, N. Y. |
| Douglas Houghton Campbell | Palo Alto, Cal. |
| George Perkins Clinton | New Haven, Conn. |
| Frank Shipley Collins | North Eastham |
| John Merle Coulter | Chicago, Ill. |
| Edward Murray East | Jamaica Plain |
| Alexander William Evans | New Haven, Conn. |
| William Gilson Farlow | Cambridge |
| Charles Edward Faxon | Jamaica Plain |
| Merritt Lyndon Fernald | Cambridge |
| George Lincoln Goodale | Cambridge |
| Robert Almer Harper | New York, N. Y. |
| John George Jack | East Walpole |
| Edward Charles Jeffrey | Cambridge |
| Fred Dayton Lambert | Tufts College |
| Burton Edward Livingston | Baltimore, Md. |
| George Richard Lyman | Washington, D. C. |
| Winthrop John Vanleuven Osterhout | Cambridge |
| Alfred Rehder | Jamaica Plain |
| Lincoln Ware Riddle | Wellesley |

| | |
|---|---|
| Benjamin Lincoln Robinson | Cambridge |
| Charles Sprague Sargent | Brookline |
| William Albert Setchell | Berkeley, Cal. |
| Arthur Bliss Seymour | Cambridge |
| Erwin Frink Smith | Washington, D. C. |
| John Donnell Smith | Baltimore, Md. |
| William Codman Sturgis | Boston |
| Roland Thaxter | Cambridge |
| William Trelease | Urbana, Ill. |

### Class II., Section III.— *Zoölogy and Physiology.*— 53.

| | |
|---|---|
| Glover Morrill Allen | Boston |
| Joel Asaph Allen | New York, N. Y. |
| John Wallace Baird | Worcester |
| Thomas Barbour | Boston |
| Francis Gano Benedict | Boston |
| Henry Bryant Bigelow | Concord |
| Robert Payne Bigelow | Brookline |
| John Lewis Bremer | Boston |
| William Brewster | Cambridge |
| Charles Thomas Brues | Boston |
| Hermon Carey Bumpus | Tufts College |
| Walter Bradford Cannon | Cambridge |
| William Ernest Castle | Belmont |
| Charles Value Chapin | Providence, R. I. |
| Samuel Fessenden Clarke | Williamstown |
| Edwin Grant Conklin | Princeton, N. J. |
| William Thomas Councilman | Boston |
| Joseph Augustine Cushman | Sharon |
| William Healey Dall | Washington, D. C. |
| Charles Benedict Davenport | Cold Spring Harbor, N. Y. |
| Gilman Arthur Drew | Woods Hole |
| Otto Knut Olof Folin | Brookline |
| Alexander Forbes | Milton |
| Samuel Henshaw | Cambridge |
| Leland Ossian Howard | Washington, D. C. |
| Herbert Spencer Jennings | Baltimore, Md. |
| Charles Willison Johnson | Brookline |
| Charles Atwood Kofoid | Berkeley, Cal. |
| Frederic Thomas Lewis | Waban |
| Ralph Stayner Lillie | Worcester |

| | |
|---|---|
| Jacques Loeb | New York, N. Y. |
| Franklin Paine Mall | Baltimore, Md. |
| Edward Laurens Mark | Cambridge |
| Ernest Gale Martin | Palo Alto, Cal. |
| Albert Davis Mead | Providence, R. I. |
| Edward Sylvester Morse | Salem |
| Herbert Vincent Neal | Tufts College |
| Henry Fairfield Osborn | New York, N. Y. |
| George Howard Parker | Cambridge |
| John Charles Phillips | Wenham |
| James Jackson Putnam | Boston |
| Herbert Wilbur Rand | Cambridge |
| William Emerson Ritter | La Jolla, Cal. |
| William Thompson Sedgwick | Boston |
| Percy Goldthwait Stiles | Newtonville |
| John Eliot Thayer | Lancaster |
| Addison Emory Verrill | New Haven, Conn. |
| Arthur Wisswald Weysse | Boston |
| William Morton Wheeler | Boston |
| Harris Hawthorne Wilder | Northampton |
| Edmund Beecher Wilson | New York, N. Y. |
| Frederick Adams Woods | Brookline |
| Robert Mearns Yerkes | Cambridge |

### CLASS II., SECTION IV.—*Medicine and Surgery.*—31.

| | |
|---|---|
| Edward Hickling Bradford | Boston |
| Henry Asbury Christian | Boston |
| Harvey Cushing | Boston |
| David Linn Edsall | Boston |
| Harold Clarence Ernst | Jamaica Plain |
| Simon Flexner | New York, N. Y. |
| William Stewart Halsted | Baltimore, Md. |
| Reid Hunt | Brookline |
| Abraham Jacobi | New York, N. Y. |
| Elliott Proctor Joslin | Boston |
| William Williams Keen | Philadelphia, Pa. |
| Frank Burr Mallory | Brookline |
| Samuel Jason Mixter | Boston |
| Edward Hall Nichols | Boston |
| Sir William Osler | Oxford, Eng. |
| Theophil Mitchell Prudden | New York, N. Y. |

| | |
|---|---|
| William Lambert Richardson | Boston |
| Milton Joseph Rosenau | Boston |
| Frederick Cheever Shattuck | Boston |
| Theobald Smith | Princeton, N. J. |
| Elmer Ernest Southard | Boston |
| Richard Pearson Strong | Boston |
| Ernest Edward Tyzzer | Boston |
| Frederick Herman Verhoeff | Boston |
| Henry Pickering Walcott | Cambridge |
| John Collins Warren | Boston |
| William Henry Welch | Baltimore, Md. |
| Francis Henry Williams | Boston |
| Simeon Burt Wolbach | Boston |
| Horatio Curtis Wood | Philadelphia, Pa. |
| James Homer Wright | Boston |

## CLASS III.— *Moral and Political Sciences.*— 153.

### SECTION I.— *Theology, Philosophy and Jurisprudence.*— 40.

| | |
|---|---|
| Simeon Eben Baldwin | New Haven, Conn. |
| Joseph Henry Beale | Cambridge |
| Melville Madison Bigelow | Cambridge |
| James De Normandie | Roxbury |
| Frederic Dodge | Belmont |
| Edward Staples Drown | Cambridge |
| William Harrison Dunbar | Cambridge |
| Timothy Dwight | New Haven, Conn. |
| William Wallace Fenn | Cambridge |
| Frederick Perry Fish | Brookline |
| Frederick John Foakes-Jackson | New York, N. Y. |
| George Angier Gordon | Boston |
| John Wilkes Hammond | Cambridge |
| Alfred Hemenway | Boston |
| Marcus Perrin Knowlton | Springfield |
| William Lawrence | Boston |
| George Vasmer Leverett | Boston |
| Arthur Lord | Plymouth |
| William Caleb Loring | Boston |
| Nathan Matthews | Boston |
| Samuel Walker McCall | Winchester |
| Edward Caldwell Moore | Cambridge |

| | |
|---|---|
| George Herbert Palmer | Cambridge |
| Charles Edwards Park | Boston |
| George Wharton Pepper | Philadelphia, Pa. |
| John Winthrop Platner | Cambridge |
| Roscoe Pound | Belmont |
| Elihu Root | New York, N. Y. |
| James Hardy Ropes | Cambridge |
| Arthur Prentice Rugg | Worcester |
| Henry Newton Sheldon | Boston |
| Moorfield Storey | Boston |
| William Howard Taft | New Haven, Conn. |
| William Jewett Tucker | Hanover, N. H. |
| William Cushing Wait | Medford |
| Williston Walker | New Haven, Conn. |
| Eugene Wambaugh | Cambridge |
| Edward Henry Warren | Boston |
| Samuel Williston | Belmont |
| Woodrow Wilson | Washington, D. C. |

### CLASS III., SECTION II.—*Philology and Archæology.*—45.

| | |
|---|---|
| Francis Greenleaf Allinson | Providence, R. I. |
| William Rosenzweig Arnold | Cambridge |
| Maurice Bloomfield | Baltimore, Md. |
| Franz Boas | New York, N. Y. |
| Charles Pickering Bowditch | Jamaica Plain |
| Franklin Carter | Williamstown |
| George Henry Chase | Cambridge |
| Roland Burrage Dixon | Cambridge |
| William Curtis Farabee | Cambridge |
| Jesse Walter Fewkes | Washington, D. C. |
| Jeremiah Denis Mathias Ford | Cambridge |
| Basil Lanneau Gildersleeve | Baltimore, Md. |
| Charles Hall Grandgent | Cambridge |
| Louis Herbert Gray | Boston |
| Charles Burton Gulick | Cambridge |
| William Arthur Heidel | Middletown, Conn. |
| Bert Hodge Hill | Athens, Greece |
| Edward Washburn Hopkins | New Haven, Conn. |
| Joseph Clark Hoppin | Boston |
| Albert Andrew Howard | Cambridge |
| William Guild Howard | Cambridge |

| | |
|---|---|
| Aleš Hrdlička | Washington, D. C. |
| Carl Newell Jackson | Cambridge |
| Hans Carl Gunther von Jagemann | Cambridge |
| James Richard Jewett | Cambridge |
| Alfred Louis Kroeber | Berkeley, Cal. |
| Kirsopp Lake | Cambridge |
| Henry Roseman Lang | New Haven, Conn. |
| Charles Rockwell Lanman | Cambridge |
| David Gordon Lyon | Cambridge |
| Clifford Herschel Moore | Cambridge |
| George Foot Moore | Cambridge |
| Hanns Oertel | New Haven, Conn. |
| Bernadotte Perrin | New Haven, Conn. |
| Edward Kennard Rand | Cambridge |
| George Andrew Reisner | Cambridge |
| Edward Robinson | New York, N. Y. |
| Fred Norris Robinson | Cambridge |
| Edward Stevens Sheldon | Cambridge |
| Herbert Weir Smyth | Cambridge |
| Franklin Bache Stephenson | Claremont, Cal. |
| Charles Cutler Torrey | New Haven, Conn. |
| Alfred Marston Tozzer | Cambridge |
| Andrew Dickson White | Ithaca, N. Y. |
| James Haughton Woods | Cambridge |

### CLASS III., SECTION III.—*Political Economy and History.*—34.

| | |
|---|---|
| Henry Adams | Washington, D. C. |
| Charles Jesse Bullock | Cambridge |
| Thomas Nixon Carver | Cambridge |
| John Bates Clark | New York |
| Archibald Cary Coolidge | Boston |
| Richard Henry Dana | Cambridge |
| Andrew McFarland Davis | Cambridge |
| Davis Rich Dewey | Cambridge |
| Edward Bangs Drew | Cambridge |
| Ephraim Emerton | Cambridge |
| Henry Walcott Farnam | New Haven, Conn. |
| Irving Fisher | New Haven, Conn. |
| Worthington Chauncey Ford | Cambridge |
| Edwin Francis Gay | Cambridge |
| Frank Johnson Goodnow | Baltimore, Md. |

| | |
|---|---|
| Arthur Twining Hadley | New Haven, Conn. |
| Albert Bushnell Hart | Cambridge |
| Charles Homer Haskins | Cambridge |
| Henry Cabot Lodge | Nahant |
| Abbott Lawrence Lowell | Cambridge |
| Roger Bigelow Merriman | Cambridge |
| Samuel Eliot Morison | Boston |
| William Bennett Munro | Cambridge |
| James Ford Rhodes | Boston |
| William Mulligan Sloane | New York, N. Y. |
| Charles Card Smith | Boston |
| Henry Morse Stephens | Berkeley, Cal. |
| John Osborne Sumner | Boston |
| Frank William Taussig | Cambridge |
| William Roscoe Thayer | Cambridge |
| Frederick Jackson Turner | Cambridge |
| Thomas Franklin Waters | Ipswich |
| George Grafton Wilson | Cambridge |
| George Parker Winship | Providence, R. I. |

### Class III., Section IV.— *Literature and the Fine Arts.*— 34.

| | |
|---|---|
| George Pierce Baker | Cambridge |
| Arlo Bates | Boston |
| James Phinney Baxter | Portland, Me. |
| William Sturgis Bigelow | Boston |
| Le Baron Russell Briggs | Cambridge |
| Ralph Adams Cram | Boston |
| Samuel McChord Crothers | Cambridge |
| Wilberforce Eames | New York, N. Y. |
| Henry Herbert Edes | Cambridge |
| Edward Waldo Emerson | Concord |
| Arthur Fairbanks | Cambridge |
| Arthur Foote | Brookline |
| Kuno Francke | Cambridge |
| Daniel Chester French | Stockbridge |
| Horace Howard Furness | Philadelphia, Pa. |
| Robert Grant | Boston |
| Chester Noyes Greenough | Cambridge |
| Francis Barton Gummere | Haverford, Pa. |
| Henry Lee Higginson | Boston |
| James Kendall Hosmer | Minneapolis, Minn. |

| | |
|---|---|
| Mark Antony DeWolfe Howe | Boston |
| George Lyman Kittredge | Cambridge |
| William Coolidge Lane | Cambridge |
| Allan Marquand | Princeton, N. J. |
| Albert Matthews | Boston |
| William Allan Neilson | Cambridge |
| Herbert Putnam | Washington, D. C. |
| Denman Waldo Ross | Cambridge |
| John Singer Sargent | London, Eng. |
| Ellery Sedgwick | Boston |
| Richard Clipston Sturgis | Boston |
| Barrett Wendell | Boston |
| Owen Wister | Philadelphia, Pa. |
| George Edward Woodberry | Beverly |

## FOREIGN HONORARY MEMBERS.— 64.

(Number limited to seventy-five).

### Class I.— *Mathematical and Physical Sciences.*— 22.

#### Section I.— *Mathematics and Astronomy.*— 6.

| | |
|---|---|
| Johann Oskar Backlund | Petrograd |
| Felix Klein | Göttingen |
| Tullio Levi-Civita | Padua |
| Sir Joseph Norman Lockyer | London |
| Émile Picard | Paris |
| Charles Jean de la Vallée Poussin | Louvain |

#### Class I., Section II.— *Physics.*— 9.

| | |
|---|---|
| Svante August Arrhenius | Stockholm |
| Oliver Heaviside | Torquay |
| Sir Joseph Larmor | Cambridge |
| Hendrik Antoon Lorentz | Leyden |
| Max Planck | Berlin |
| Augusto Righi | Bologna |
| Sir Ernest Rutherford | Manchester |
| John William Strutt, Baron Rayleigh | Witham |
| Sir Joseph John Thomson | Cambridge |

#### Class I., Section III.— *Chemistry.*— 4.

| | |
|---|---|
| Adolf, Ritter von Baeyer | Munich |
| Emil Fischer | Berlin |
| Fritz Haber | Berlin |
| Wilhelm Ostwald | Leipsic |

### Class I.— Section IV.— *Technology and Engineering.*— 3.

| | |
|---|---|
| Heinrich Müller Breslau | Berlin |
| Vsevolod Jevgenjevic Timonoff | Petrograd |
| William Cawthorne Unwin | London |

## Class II.— *Natural and Physiological Sciences.*— 18.

### Section I.— *Geology, Mineralogy, and Physics of the Globe.*— 7.

| | |
|---|---|
| Frank Dawson Adams | Montreal |
| Waldemar Christofer Brögger | Christiania |
| Sir Archibald Geikie | Haslemere, Surrey |
| Viktor Goldschmidt | Heidelberg |
| Julius Hann | Vienna |
| Albert Heim | Zürich |
| Johan Herman Lie Vogt | Trondhjem |

### Class II., Section II.— *Botany.*— 6.

| | |
|---|---|
| John Briquet | Geneva |
| Adolf Engler | Berlin |
| Wilhelm Pfeffer | Leipsic |
| Hermann, Graf zu Solms-Laubach | Strassburg |
| Ignatz Urban | Berlin |
| Eugene Warming | Copenhagen |

### Class II.— Section III. —*Zoölogy and Physiology.*— 2.

| | |
|---|---|
| Sir Edwin Ray Lankester | London |
| Magnus Gustav Retzius | Stockholm |

### Class II., Section IV.— *Medicine and Surgery.*— 3.

| | |
|---|---|
| Emil von Behring | Marburg |
| Angelo Celli | Rome |
| Adam Politzer | Vienna |

## Class III.— *Moral and Political Sciences.*— 24.

### Section I.— *Theology, Philosophy and Jurisprudence.*— 4.

| | |
|---|---|
| Arthur James Balfour | Prestonkirk |
| Heinrich Brunner | Berlin |
| Albert Venn Dicey | Oxford |
| Sir Frederick Pollock, Bart | London |

### Section II.— *Philology and Archæology.*— 8.

| | |
|---|---|
| Friedrich Delitzsch | Berlin |
| Hermann Diels | Berlin |
| Wilhelm Dörpfeld | Athens |
| Henry Jackson | Cambridge |
| Hermann Georg Jacobi | Bonn |
| Sir Gaston Camille Charles Maspero | Paris |
| Alfred Percival Maudslay | Hereford |
| Eduard Seler | Berlin |

### Section III.— *Political Economy and History.*— 6.

| | |
|---|---|
| Viscount Bryce | London |
| Adolf Harnack | Berlin |
| Alfred Marshall | Cambridge |
| John Morley, Viscount Morley of Blackburn | London |
| Sir George Otto Trevelyan, Bart. | London |
| Pasquale Villari | Florence |

### Section IV.— *Literature and the Fine Arts.*— 6.

| | |
|---|---|
| Georg Brandes | Copenhagen |
| Thomas Hardy | Dorchester |
| Jean Adrien Aubin Jules Jusserand | Paris |
| Rudyard Kipling | Burwash |
| Sir Sidney Lee | London |
| Sir James Augustus Henry Murray | Oxford |

# STATUTES AND STANDING VOTES

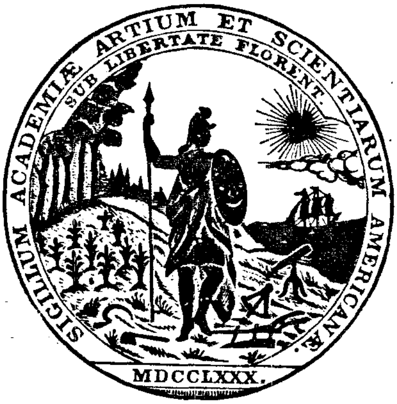
## STATUTES

*Adopted November* 8, 1911: *amended May* 8, 1912, *January* 8, *and May* 14, 1913, *April* 14, 1915, *April* 12, 1916.

### CHAPTER I

### The Corporate Seal

Article 1. The Corporate Seal of the Academy shall be as here depicted:


SIGILLUM ACADEMIÆ ARTIUM ET SCIENTIARUM AMERICANÆ.
SUB LIBERTATE FLORENT.
MDCCLXXX.

Article 2. The Recording Secretary shall have the custody of the Corporate Seal.

*See* Chap. v. art. 3; chap. vi. art. 2.

## CHAPTER II

### Fellows and Foreign Honorary Members and Dues

Article 1. The Academy consists of Fellows, who are either citizens or residents of the United States of America, and Foreign Honorary Members. They are arranged in three Classes, according to the Arts and Sciences in which they are severally proficient, and each Class is divided into four Sections, namely:

Class I. *The Mathematical and Physical Sciences*
Section 1. Mathematics and Astronomy
Section 2. Physics
Section 3. Chemistry
Section 4. Technology and Engineering

Class II. *The Natural and Physiological Sciences*
Section 1. Geology, Mineralogy, and Physics of the Globe
Section 2. Botany
Section 3. Zoölogy and Physiology
Section 4. Medicine and Surgery

Class III. *The Moral and Political Sciences*
Section 1. Theology, Philosophy, and Jurisprudence
Section 2. Philology and Archaeology
Section 3. Political Economy and History
Section 4. Literature and the Fine Arts

Article 2. The number of Fellows shall not exceed Six hundred, of whom not more than Four hundred shall be residents of Massachusetts, nor shall there be more than Two hundred in any one Class.

Article 3. The number of Foreign Honorary Members shall not exceed Seventy-five. They shall be chosen from among citizens of foreign countries most eminent for their discoveries and attainments in any of the Classes above enumerated. There shall not be more than Twenty-five in any one Class.

Article 4. If any person, after being notified of his election as Fellow, shall neglect for six months to accept in writing and to pay his Admission Fee (unless he be absent from the Commonwealth at the time of his notification) his election shall be void; and if any Fellow resident within fifty miles of Boston shall neglect to pay his Annual Dues for six months after they are due, provided his attention shall have been called to this Article of the Statutes in the meantime, he shall cease to be a Fellow; but the Council may suspend the provisions of this Article for a reasonable time.

With the previous consent of the Council, the Treasurer may dispense (*sub silentio*) with the payment of the Admission Fee or of the Annual Dues or both whenever he shall deem it advisable. In the case of officers of the Army or Navy who are out of the Commonwealth on duty, payment of the Annual Dues may be waived during such absence if continued during the whole financial year and if notification of such expected absence be sent to the Treasurer. Upon similar notification to the Treasurer, similar exemption may be accorded to Fellows subject to Annual Dues, who may temporarily remove their residence for at least two years to a place more than fifty miles from Boston.

If any person elected a Foreign Honorary Member shall neglect for six months after being notified of his election to accept in writing, his election shall be void.

*See* Chap. vii. art. 2.

ARTICLE 5. Every Fellow hereafter elected shall pay an Admission Fee of Ten dollars.

Every Fellow resident within fifty miles of Boston shall, and others may, pay such Annual Dues, not exceeding Fifteen dollars, as shall be voted by the Academy at each Annual Meeting, when they shall become due, except in the case of Fellows elected at the January meetings, who shall be obliged to pay but one half of such Annual Dues in the year in which they are elected; but any Fellow shall be exempt from the annual payment if, at any time after his admission, he shall pay into the treasury Two hundred dollars in addition to his previous payments.

All Commutations of the Annual Dues shall be and remain permanently funded, the interest only to be used for current expenses.

Any Fellow not previously subject to Annual Dues who takes up his residence within fifty miles of Boston, shall pay to the Treasurer within three months thereafter Annual Dues for the current year, failing which his Fellowship shall cease; but the Council may suspend the provisions of this Article for a reasonable time.

Only Fellows who pay Annual Dues or have commuted them may hold office in the Academy or serve on the Standing Committees or vote at meetings.

ARTICLE 6. Fellows who pay or have commuted the Annual Dues and Foreign Honorary Members shall be entitled to receive gratis one copy of all Publications of the Academy issued after their election.

*See* Chap. x. art. 2.

ARTICLE 7. Diplomas signed by the President and the Vice-President of the Class to which the member belongs, and countersigned by the Secretaries, shall be given to all the Fellows and Foreign Honorary Members.

ARTICLE 8. If, in the opinion of a majority of the entire Council, any Fellow or Foreign Honorary Member shall have rendered himself unworthy of a place in the Academy, the Council shall recommend to the Academy the termination of his membership; and if three fourths of the Fellows present, out of a total attendance of not less than fifty, at a Stated Meeting, or at a Special Meeting called for the purpose, shall adopt this recommendation, his name shall be stricken from the Roll.

*See* Chap. iii.; chap. vi. art. 1; chap. ix. art. 1, 7; chap. x. art. 2.

## CHAPTER III

### ELECTION OF FELLOWS AND FOREIGN HONORARY MEMBERS

ARTICLE 1. Elections of Fellows and Foreign Honorary Members shall be by ballot, and only at the Stated Meetings in January and May. Three fourths of the ballots cast, and not less than twenty, must be affirmative to effect an election.

ARTICLE 2. Nominations to Fellowship or Foreign Honorary Membership in any Section must be signed by two Fellows in that Section. These nominations shall be sent to the Corresponding Secretary accompanied by statements of qualifications and brief biographical data, and shall be retained by him until the first of the following October or February, as the case may be. All nominations then in his hands shall be sent in printed form to every Fellow having the right to vote, with the names of the proposers in each case, and with a request to send to the Corresponding Secretary written comments on these names not later than the fifth of November or the fifth of March respectively.

All the nominations, with the comments thereon, received up to the fifth of November or the fifth of March shall be referred at once to the appropriate Class Committees, which shall report their decisions to the Council at a special meeting to be called to consider nominations, not later than two days before the meeting of the Academy in December or April respectively.

Notice shall be sent to every Fellow having the right to vote, not later than the fifteenth of September or January, of each year, calling attention to the fact that the limit of time for sending nominations to the Corresponding Secretary will expire on the first of the following month.

ARTICLE 3. Not later then the fourth Wednesday of December and April, the Corresponding Secretary shall send in print to every Fellow having the right to vote all nominations that have been approved by the Council, with a brief account of each nominee.

*See* Chap. ii.; chap. vi. art. 1; chap. ix. art. 1.

## CHAPTER IV

### OFFICERS

ARTICLE 1. The Officers of the Academy shall be a President (who shall be Chairman of the Council), three Vice-Presidents (one from each Class), a Corresponding Secretary (who shall be Secretary of the Council), a Recording Secretary, a Treasurer, and a Librarian, all of whom shall be elected by ballot at the Annual Meeting, and shall hold their respective offices for one year, and until others are duly chosen and installed.

There shall be also twelve Councillors, one from each Section of each Class. At each Annual Meeting three Councillors, one from each Class, shall be elected by ballot to serve for the full term of four years and until others are duly chosen and installed. The same Fellow shall not be eligible for two successive terms.

The Councillors, with the other officers previously named, and the Chairman of the House Committee, *ex officio*, shall constitute the Council.

*See* Chap. x. art. 1.

ARTICLE 2. If any office shall become vacant during the year, the vacancy may be filled by the Council in its discretion for the unexpired term.

ARTICLE 3. At the Stated Meeting in March, the President shall appoint a Nominating Committee of three Fellows having the right to vote, one from each Class. This Committee shall prepare a list of nominees for the several offices to be filled, and for the Standing Committees, and file it with the Recording Secretary not later than four weeks before the Annual Meeting.

*See* Chap. vi. art. 2.

ARTICLE 4. Independent nominations for any office, if signed by at least twenty Fellows having the right to vote, and received by the Recording Secretary not less than ten days before the Annual Meeting, shall be inserted in the call therefor, and shall be mailed to all the Fellows having the right to vote.

*See* Chap. vi. art. 2.

ARTICLE 5. The Recording Secretary shall prepare for use in voting at the Annual Meeting a ballot containing the names of all persons duly nominated for office.

## CHAPTER V

### THE PRESIDENT

ARTICLE 1. The President, or in his absence the senior Vice-President present (seniority to be determined by length of continuous fellowship in the Academy), shall preside at all meetings of the Academy. In the absence of all these officers, a Chairman of the meeting shall be chosen by ballot.

ARTICLE 2. Unless otherwise ordered, all Committees which are not elected by ballot shall be appointed by the presiding officer.

ARTICLE 3. Any deed or writing to which the Corporate Seal is to be affixed, except leases of real estate, shall be executed in the name of the Academy by the President or, in the event of his death, absence, or inability, by one of the Vice-Presidents, when thereto duly authorized.

*See* Chap. ii. art. 7; chap. iv. art. 1, 3; chap. vi. art. 2; chap. vii. art. 1; chap. ix. art. 6; chap. x. art. 1; 2; chap. xi. art. 1.

## CHAPTER VI

### THE SECRETARIES

ARTICLE 1. The Corresponding Secretary shall conduct the correspondence of the Academy and of the Council, recording or making an entry of all letters written in its name, and preserving for the files all official papers which may be received. At each meeting of the Council he shall present the communications addressed to the Academy which have been received since the previous meeting, and at the next meeting of the Academy he shall present such as the Council may determine.

He shall notify all persons who may be elected Fellows or Foreign Honorary Members, send to each a copy of the Statutes, and on their acceptance issue the proper Diploma. He shall also notify all meetings of the Council; and in case of the death, absence, or inability of the Recording Secretary he shall notify all meetings of the Academy.

Under the direction of the Council, he shall keep a List of the Fellows and Foreign Honorary Members, arranged in their several Classes and Sections. It shall be printed annually and issued as of the first day of July.

> *See* Chap. ii. art. 7; chap. iii. art. 2, 3; chap. iv. art. 1; chap. ix. art. 6; chap. x. art. 1; chap. xi. art. 1.

ARTICLE 2. The Recording Secretary shall have the custody of the Charter, Corporate Seal, Archives, Statute-Book, Journals, and all literary papers belonging to the Academy.

Fellows borrowing such papers or documents shall receipt for them to their custodian.

The Recording Secretary shall attend the meetings of the Academy and keep a faithful record of the proceedings with the names of the Fellows present; and after each meeting is duly opened, he shall read the record of the preceding meeting.

He shall notify the meetings of the Academy to each Fellow by mail at least seven days beforehand, and in his discretion may also cause the meetings to be advertised; he shall apprise Officers and Committees of their election or appointment, and inform the Treasurer of appropriations of money voted by the Academy.

After all elections, he shall insert in the Records the names of the Fellows by whom the successful nominees were proposed.

He shall send the Report of the Nominating Committee in print to every Fellow having the right to vote at least three weeks before the Annual Meeting.

> *See* Chap. iv. art. 3.

In the absence of the President and of the Vice-Presidents he shall, if present, call the meeting to order, and preside until a Chairman is chosen.

> *See* Chap. i.; chap. ii. art. 7; chap. iv. art. 3, 4, 5; chap. ix. art. 6; chap. x. art. 1, 2; chap. xi. art. 1, 3.

ARTICLE 3. The Secretaries, with the Chairman of the Committee of Publication, shall have authority to publish such of the records of the meetings of the Academy as may seem to them likely to promote its interests.

## CHAPTER VII

### The Treasurer and the Treasury

Article 1. The Treasurer shall collect all money due or payable to the Academy, and all gifts and bequests made to it. He shall pay all bills due by the Academy, when approved by the proper officers, except those of the Treasurer's office, which may be paid without such approval; in the name of the Academy he shall sign all leases of real estate; and, with the written consent of a member of the Committee on Finance, he shall make all transfers of stocks, bonds, and other securities belonging to the Academy, all of which shall be in his official custody.

He shall keep a faithful account of all receipts and expenditures, submit his accounts annually to the Auditing Committee, and render them at the expiration of his term of office, or whenever required to do so by the Academy or the Council.

He shall keep separate accounts of the income of the Rumford Fund, and of all other special Funds, and of the appropriation thereof, and render them annually.

His accounts shall always be open to the inspection of the Council.

Article 2. He shall report annually to the Council at its March meeting on the expected income of the various Funds and from all other sources during the ensuing financial year. He shall also report the names of all Fellows who may be then delinquent in the payment of their Annual Dues.

Article 3. He shall give such security for the trust reposed in him as the Academy may require.

Article 4. With the approval of a majority of the Committee on Finance, he may appoint an Assistant Treasurer to perform his duties, for whose acts, as such assistant, he shall be responsible; or, with like approval and responsibility, he may employ any Trust Company doing business in Boston as his agent for the same purpose, the compensation of such Assistant Treasurer or agent to be fixed by the Committee on Finance and paid from the funds of the Academy.

Article 5. At the Annual Meeting he shall report in print all his official doings for the preceding year, stating the amount and condition of all the property of the Academy entrusted to him, and the character of the investments.

ARTICLE 6. The Financial Year of the Academy shall begin with the first day of April.

ARTICLE 7. No person or committee shall incur any debt or liability in the name of the Academy, unless in accordance with a previous vote and appropriation therefor by the Academy or the Council, or sell or otherwise dispose of any property of the Academy, except cash or invested funds, without the previous consent and approval of the Council.

*See* Chap. ii. art. 4, 5; chap. vi. art. 2; chap ix. art. 6; chap. x. art. 1, 2, 3; chap. xi. art. 1.

## CHAPTER VIII

### THE LIBRARIAN AND THE LIBRARY

ARTICLE 1. The Librarian shall have charge of the printed books, keep a correct catalogue thereof, and provide for their delivery from the Library.

At the Annual Meeting, as Chairman of the Committee on the Library, he shall make a Report on its condition.

ARTICLE 2. In conjunction with the Committee on the Library he shall have authority to expend such sums as may be appropriated by the Academy for the purchase of books, periodicals, etc., and for defraying other necessary expenses connected with the Library.

ARTICLE 3. All books procured from the income of the Rumford Fund or of other special Funds shall contain a book-plate expressing the fact.

ARTICLE 4. Books taken from the Library shall be receipted for to the Librarian or his assistant.

ARTICLE 5. Books shall be returned in good order, regard being had to necessary wear with good usage. If any book shall be lost or injured, the Fellow to whom it stands charged shall replace it by a new volume or by a new set, if it belongs to a set, or pay the current price thereof to the Librarian, whereupon the remainder of the set, if any, shall be delivered to the Fellow so paying, unless such remainder be valuable by reason of association.

Article 6. All books shall be returned to the Library for examination at least one week before the Annual Meeting.

Article 7. The Librarian shall have the custody of the Publications of the Academy. With the advice and consent of the President, he may effect exchanges with other associations.

*See* Chap. ii. art. 6; chap. x. art. 1, 2.

## CHAPTER IX

### The Council

Article 1. The Council shall exercise a discreet supervision over all nominations and elections to membership, and in general supervise all the affairs of the Academy not explicitly reserved to the Academy as a whole or entrusted by it or by the Statutes to standing or special committees.

It shall consider all nominations duly sent to it by any Class Committee, and present to the Academy for action such of these nominations as it may approve by a majority vote of the members present at a meeting, of whom not less than seven shall have voted in the affirmative.

With the consent of the Fellow interested, it shall have power to make transfers between the several Sections of the same Class, reporting its action to the Academy.

*See* Chap. iii. art. 2, 3; chap. x. art. 1.

Article 2. Seven members shall constitute a quorum.

Article 3. It shall establish rules and regulations for the transaction of its business, and provide all printed and engraved blanks and books of record.

Article 4. It shall act upon all resignations of officers, and all resignations and forfeitures of Fellowship; and cause the Statutes to be faithfully executed.

It shall appoint all agents and subordinates not otherwise provided for by the Statutes, prescribe their duties, and fix their compensation.

They shall hold their respective positions during the pleasure of the Council.

Article 5. It may appoint, for terms not exceeding one year, and prescribe the functions of, such committees of its number, or of the Fellows of the Academy, as it may deem expedient, to facilitate the administration of the affairs of the Academy or to promote its interests.

Article 6. At its March meeting it shall receive reports from the President, the Secretaries, the Treasurer, and the Standing Committees, on the appropriations severally needed for the ensuing financial year. At the same meeting the Treasurer shall report on the expected income of the various Funds and from all other sources during the same year.

A report from the Council shall be submitted to the Academy, for action, at the March meeting, recommending the appropriation which in the opinion of the Council should be made.

On the recommendation of the Council, special appropriations may be made at any Stated Meeting of the Academy, or at a Special Meeting called for the purpose.

*See* Chap. x. art. 3.

Article 7. After the death of a Fellow or Foreign Honorary Member, it shall appoint a member of the Academy to prepare a Memoir for publication in the Proceedings.

Article 8. It shall report at every meeting of the Academy such business as it may deem advisable to present.

*See* Chap. ii. art. 4, 5, 8; chap. iv. art. 1, 2; chap. vi. art. 1; chap. vii. art. 1; chap. xi. art. 1, 4.

## CHAPTER X

### Standing Committees

Article 1. The Class Committee of each Class shall consist of the Vice-President, who shall be chairman, and the four Councillors of the Class, together with such other officer or officers annually elected as may belong to the Class. It shall consider nominations to Fellowship in its own Class, and report in writing to the Council such as may receive at a Class Committee Meeting a majority of the votes cast, provided at least three shall have been in the affirmative.

*See* Chap. iii. art. 2.

ARTICLE 2. At the Annual Meeting the following Standing Committees shall be elected by ballot to serve for the ensuing year:

(i) *The Committee on Finance,* to consist of three Fellows, who, through the Treasurer, shall have full control and management of the funds and trusts of the Academy, with the power of investing the funds and of changing the investments thereof in their discretion.

*See* Chap. iv. art. 3; chap. vii. art. 1, 4; chap. ix. art. 6.

(ii) *The Rumford Committee,* to consist of seven Fellows, who shall report to the Academy on all applications and claims for the Rumford Premium. It alone shall authorize the purchase of books publications and apparatus at the charge of the income from the Rumford Fund, and generally shall see to the proper execution of the trust.

*See* Chap. iv. art. 3; chap. ix. art. 6.

(iii) *The Cyrus Moors Warren Committee,* to consist of seven Fellows, who shall consider all applications for appropriations from the income of the Cyrus Moors Warren Fund, and generally shall see to the proper execution of the trust.

*See* Chap. iv. art. 3; chap. ix. art. 6.

(iv) *The Committee of Publication,* to consist of three Fellows, one from each Class, to whom all communications submitted to the Academy for publication shall be referred, and to whom the printing of the Proceedings and the Memoirs shall be entrusted.

It shall fix the price at which the Publications shall be sold; but Fellows may be supplied at half price with volumes which may be needed to complete their sets, but which they are not entitled to receive gratis.

Two hundred extra copies of each paper accepted for publication in the Proceedings or the Memoirs shall be placed at the disposal of the author without charge.

*See* Chap. iv. art. 3; chap. vi. art. 1, 3; chap. ix. art. 6.

(v) *The Committee on the Library,* to consist of the Librarian, *ex officio,* as Chairman, and three other Fellows, one from each Class, who shall examine the Library and make an annual report on its condition and management.

*See* Chap. iv. art. 3; chap. viii. art. 1, 2; chap. ix. art. 6.

(vi) *The House Committee,* to consist of three Fellows, who shall have charge of all expenses connected with the House, including the general expenses of the Academy not specifically assigned to the care of other Committees or Officers.

*See* Chap. iv. art. 1, 3; chap. ix. art. 6.

(vii) *The Committee on Meetings,* to consist of the President, the Recording Secretary, and three other Fellows, who shall have charge of plans for meetings of the Academy.

*See* Chap. iv. art. 3; chap. ix. art. 6.

(viii) *The Auditing Committee,* to consist of two Fellows, who shall audit the accounts of the Treasurer, with power to employ an expert and to approve his bill.

*See* Chap. iv. art. 3; chap. vii. art. 1; chap. ix. art. 6.

ARTICLE 3. The Standing Committees shall report annually to the Council in March on the appropriations severally needed for the ensuing financial year; and all bills incurred on account of these Committees, within the limits of the several appropriations made by the Academy, shall be approved by their respective Chairmen.

In the absence of the Chairman of any Committee, bills may be approved by any member of the Committee whom he shall designate for the purpose.

*See* Chap. vii. art. 1, 7; chap. ix. art. 6.

## CHAPTER XI

### MEETINGS, COMMUNICATIONS, AND AMENDMENTS

ARTICLE 1. There shall be annually eight Stated Meetings of the Academy, namely, on the second Wednesday of October, November, December, January, February, March, April and May. Only at these meetings, or at adjournments thereof regularly notified, or at Special Meetings called for the purpose, shall appropriations of money be made or amendments of the Statutes or Standing Votes be effected.

The Stated Meeting in May shall be the Annual Meeting of the Corporation.

Special Meetings shall be called by either of the Secretaries at the request of the President, of a Vice-President, of the Council, or of ten

Fellows having the right to vote; and notifications thereof shall state the purpose for which the meeting is called.

A meeting for receiving and discussing literary or scientific communications may be held on the fourth Wednesday of each month, excepting July, August, and September; but no business shall be transacted at said meetings.

ARTICLE 2. Twenty Fellows having the right to vote shall constitute a quorum for the transaction of business at Stated or Special Meetings. Fifteen Fellows shall be sufficient to constitute a meeting for literary or scientific communications and discussions.

ARTICLE 3. Upon the request of the presiding officer or the Recording Secretary, any motion or resolution offered at any meeting shall be submitted in writing.

ARTICLE 4. No report of any paper presented at a meeting of the Academy shall be published by any Fellow without the consent of the author; and no report shall in any case be published by any Fellow in a newspaper as an account of the proceedings of the Academy without the previous consent and approval of the Council. The Council, in its discretion, by a duly recorded vote, may delegate its authority in this regard to one or more of its members.

ARTICLE 5. No Fellow shall introduce a guest at any meeting of the Academy until after the business has been transacted, and especially until after the result of the balloting upon nominations has been declared.

ARTICLE 6. The Academy shall not express its judgment on literary or scientific memoirs or performances submitted to it, or included in its Publications.

ARTICLE 7. All proposed Amendments of the Statutes shall be referred to a committee, and on its report, at a subsequent Stated Meeting or at a Special Meeting called for the purpose, two thirds of the ballot cast, and not less than twenty, must be affirmative to effect enactment.

ARTICLE 8. Standing Votes may be passed, amended, or rescinded at a Stated Meeting, or at a Special Meeting called for the purpose, by a vote of two thirds of the members present. They may be suspended by a unanimous vote.

*See* Chap. ii. art. 5, 8; chap. iii.; chap. iv. art. 3, 4, 5; chap v. art. 1; chap. vi. art. 1, 2; chap. ix. art. 8.

# STANDING VOTES

1. Communications of which notice has been given to either of the Secretaries shall take precedence of those not so notified.

2. Fellows may take from the Library six volumes at any one time, and may retain them for three months, and no longer. Upon special application, and for adequate reasons assigned, the Librarian may permit a larger number of volumes, not exceeding twelve, to be drawn from the Library for a limited period.

3. Works published in numbers, when unbound, shall not be taken from the Hall of the Academy without the leave of the Librarian.

4. There may be chosen by the Academy, under the same rules by which Fellows are now chosen, one hundred Resident Associates. Not more than forty Resident Associates shall be chosen in any one Class.

The election of Resident Associates shall be for a term of three years with eligibility for reëlection.

Resident Associates shall be entitled to the same privileges as Fellows, in the use of the Academy building, may attend meetings and present papers, but they shall not have the right to vote. They shall pay no Admission Fee, and their Annual Dues shall be one-half that of Fellows residing within fifty miles of Boston.

The Council and Committees of the Academy may ask one or more Resident Associates to act with them in an advisory or assistant capacity.

# RUMFORD PREMIUM

In conformity with the terms of the gift of Sir Benjamin Thompson, Count Rumford, of a certain Fund to the American Academy of Arts and Sciences, and with a decree of the Supreme Judicial Court of Massachusetts for carrying into effect the general charitable intent and purpose of Count Rumford, as expressed in his letter of gift, the Academy is empowered to make from the income of the Rumford Fund, as it now exists, at any Annual Meeting, an award of a gold and a silver medal, being together of the intrinsic value of three hundred dollars, as a Premium to the author of any important discovery or useful improvement in light or heat, which shall have been made and published by printing, or in any way made known to the public, in any part of the continent of America, or any of the American Islands; preference always being given to such discoveries as, in the opinion of the Academy, shall tend most to promote the good of mankind; and, if the Academy sees fit, to add to such medals, as a further Premium for such discovery and improvement, a sum of money not exceeding three hundred dollars.

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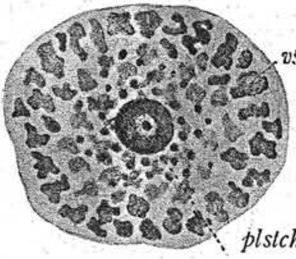

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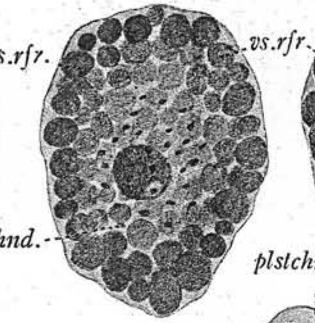

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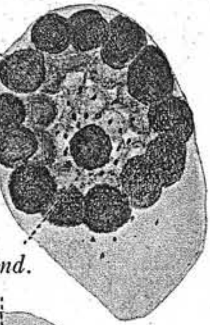
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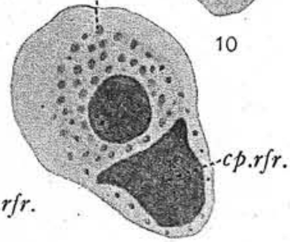

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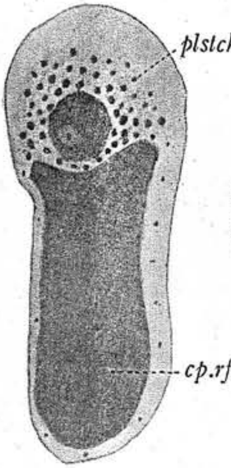
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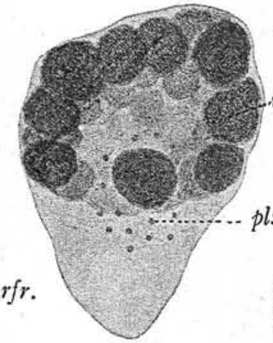
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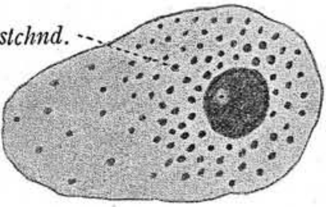

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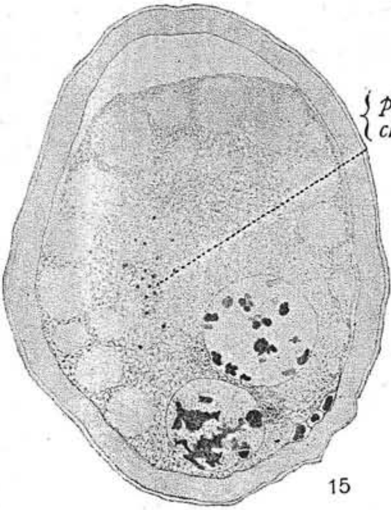

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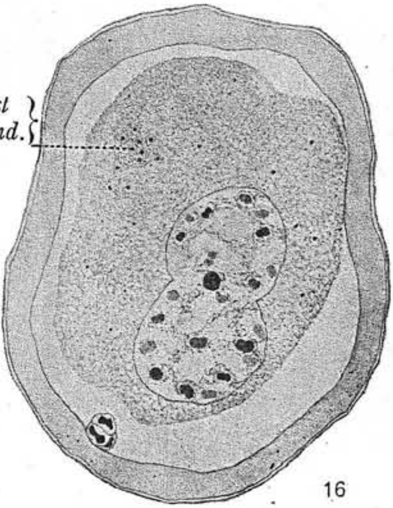

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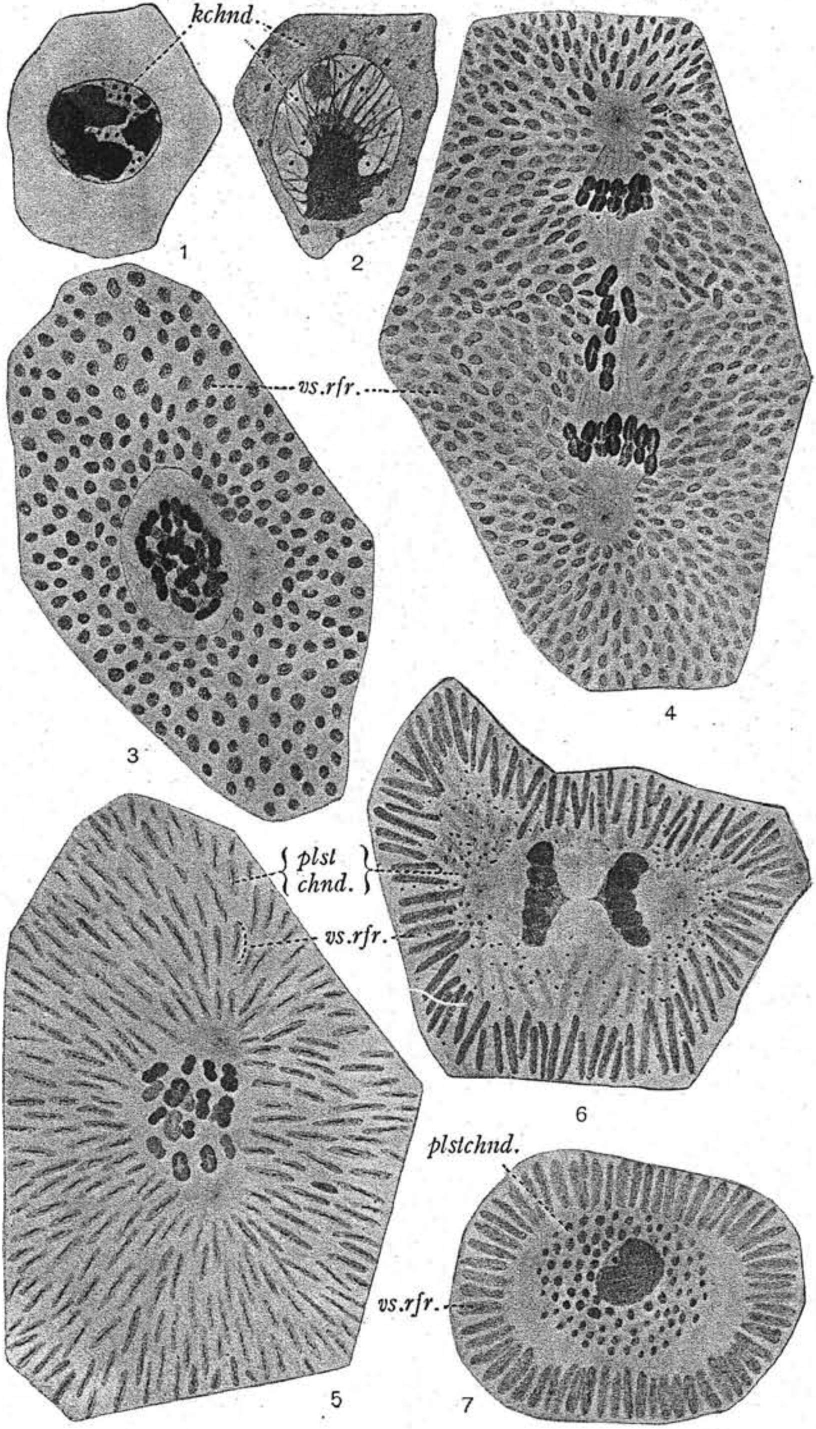
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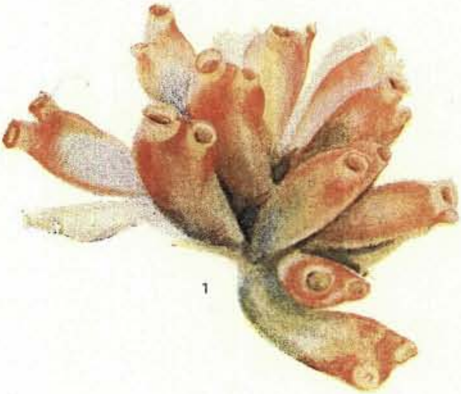

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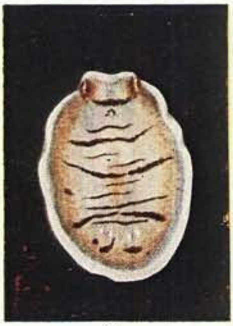

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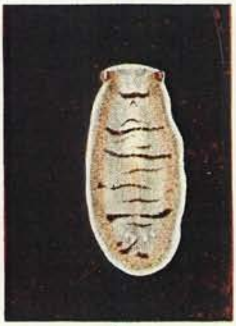

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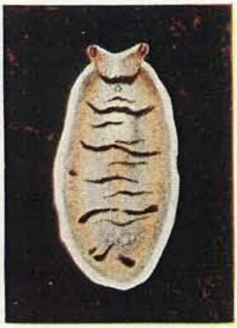

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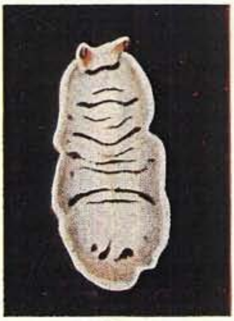

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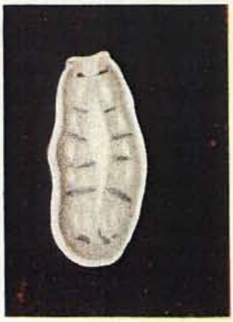

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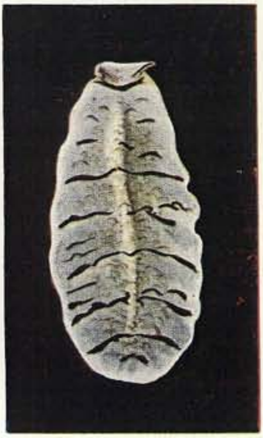

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